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Perfectness of Kirillov–Reshetikhin crystals for nonexceptional types

Ghislain Fourier[1]†, Masato Okado[2] ‡ and Anne Schilling[3] §

[1] *Mathematisches Institut der Universität zu Köln, Weyertal 86-90, 50931 Köln, Germany*
[2] *Department of Mathematical Science, Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka 560-8531, Japan*
[3] *Department of Mathematics, University of California, One Shields Avenue, Davis, CA 95616-8633, U.S.A.*

Abstract. For nonexceptional types, we prove a conjecture of Hatayama et al. about the prefectness of Kirillov–Reshetikhin crystals.

Résumé. Pour les types non-exceptionnels, on démontre une conjecture de Hatayama et al. concernant la perfection des cristaux de Kirillov–Reshetikhin.

Keywords: Crystal bases, combinatorial models for Kirillov–Reshetikhin crystals, perfectness

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1 Introduction

Kirillov–Reshetikhin (KR) crystals $B^{r,s}$ are crystals corresponding to finite-dimensional $U_q'(\mathfrak{g})$-modules [3, 4], where $\mathfrak{g}$ is an affine Kac–Moody algebra. Recently, a lot of progress has been made regarding long outstanding problems concerning these crystals which appear in mathematical physics and the path realization of affine highest weight crystals [13]. In [20, 21] the existence of KR crystals was shown. In [5] a major step in understanding these crystals was provided by giving explicit combinatorial realizations for all nonexceptional types. This abstract is based on [5, 6]. We prove a conjecture of Hatayama, Kuniba, Okado, Takagu, and Tsuboi [8, Conjecture 2.1] about the perfectness of these KR crystals.

Conjecture 1.1 *[8, Conjecture 2.1] The Kirillov-Reshetikhin crystal $B^{r,s}$ is perfect if and only if $\frac{s}{c_r}$ is an integer with c_r as in Table 1. If $B^{r,s}$ is perfect, its level is $\frac{s}{c_r}$.*

In [14], this conjecture was proven for all $B^{r,s}$ for type $A_n^{(1)}$, for $B^{1,s}$ for nonexceptional types (except for type $C_n^{(1)}$), for $B^{n-1,s}$, $B^{n,s}$ of type $D_n^{(1)}$, and $B^{n,s}$ for types $C_n^{(1)}$ and $D_{n+1}^{(2)}$. When the highest weight is given by the highest root, level-1 perfect crystals were constructed in [1]. For $1 \le r \le n-2$

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	$(c_1,\ldots,c_n)$
$A_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$	$(1,\ldots,1)$
$B_n^{(1)}$	$(1,\ldots,1,2)$
$C_n^{(1)}$	$(2,\ldots,2,1)$

Tab. 1: List of c_r

for type $D_n^{(1)}$, $1 \le r \le n-1$ for type $B_n^{(1)}$, and $1 \le r \le n$ for type $A_{2n-1}^{(2)}$, the conjecture was proved in [22]. The case $G_2^{(1)}$ and $r=1$ was treated in [24] and the case $D_4^{(3)}$ and $r=1$ was treated in [16]. Naito and Sagaki [18] showed that the conjecture holds for twisted algebras, if it is true for the untwisted simply-laced cases.

In this paper we prove Conjecture 1.1 in general for nonexceptional types.

Theorem 1.2 *If $\mathfrak{g}$ is of nonexceptional type, Conjecture 1.1 is true.*

The paper is organized as follows. In Section 2 we give basic notation and the definition of perfectness in Definition 2.1. In Section 3 we review the realizations of the KR crystals of nonexceptional types as recently provided in [5]. Section 4 is reserved for the proof of Theorem 1.2 and an explicit description of the minimal elements $B_{\min}^{r,c_r s}$ of the perfect crystals. A long version of this article containing further details and examples is available at [6].

2 Definitions and perfectness

We follow the notation of [12, 5]. Let $\mathcal{B}$ be a $U_q'(\mathfrak{g})$-crystal [15]. Denote by α_i and Λ_i for $i \in I$ the simple roots and fundamental weights and by c the canoncial central element associated to $\mathfrak{g}$, where I is the index set of the Dynkin diagram of $\mathfrak{g}$ (see Table 2). Let $P = \oplus_{i\in I}\mathbb{Z}\Lambda_i$ be the weight lattice of $\mathfrak{g}$ and P^+ the set of dominant weights. For a positive integer ℓ, the set of level-ℓ weights is

$$P_\ell^+ = \{\Lambda \in P^+ \mid \mathrm{lev}(\Lambda) = \ell\}.$$

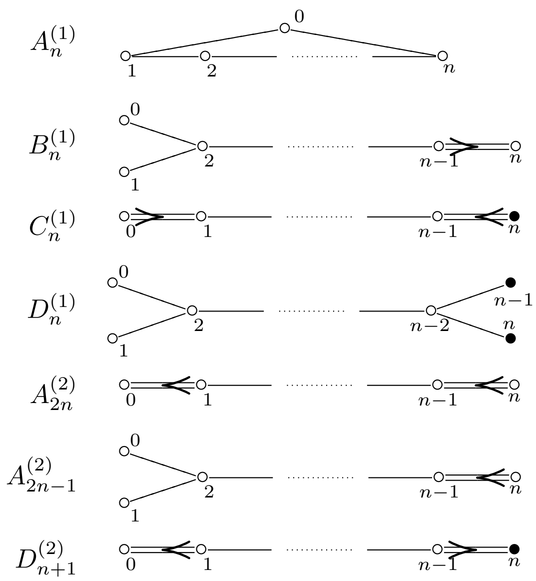
where $\mathrm{lev}(\Lambda) := \Lambda(c)$. The set of level-0 weights is denoted by P_0. We identify dominant weights with partitions; each Λ_i yields a column of height i (except for spin nodes). For more details, please consult [11].

We denote by $f_i, e_i : \mathcal{B} \to \mathcal{B} \cup \{\emptyset\}$ for $i \in I$ the Kashiwara operators and by $\mathrm{wt} : \mathcal{B} \to P$ the weight function on the crystal. For $b \in \mathcal{B}$ we define $\varepsilon_i(b) = \max\{k \mid e_i^k(b) \neq \emptyset\}$, $\varphi_i(b) = \max\{k \mid f_i^k(b) \neq \emptyset\}$, and

$$\varepsilon(b) = \sum_{i\in I} \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i\in I} \varphi_i(b)\Lambda_i.$$

Next we define perfect crystals, see for example [11].

Definition 2.1 *For a positive integer $\ell > 0$, a crystal $\mathcal{B}$ is called perfect crystal of level ℓ, if the following conditions are satisfied:*



Tab. 2: Dynkin diagrams

1. *$\mathcal{B}$ is isomorphic to the crystal graph of a finite-dimensional $U_q^{'}(\mathfrak{g})$-module.*
2. *$\mathcal{B} \otimes \mathcal{B}$ is connected.*
3. *There exists a $\lambda \in P_0$, such that* $\mathrm{wt}\,(\mathcal{B}) \subset \lambda + \sum_{i \in I \setminus \{0\}} \mathbb{Z}_{\leq 0} \alpha_i$ *and there is a unique element in $\mathcal{B}$ of classical weight λ.*
4. *$\forall\, b \in \mathcal{B},\ \ \mathrm{lev}(\varepsilon(b)) \geq \ell$.*
5. *$\forall\ \Lambda \in P_\ell^+$, there exist unique elements $b_\Lambda, b^\Lambda \in \mathcal{B}$, such that*

$$\varepsilon(b_\Lambda) = \Lambda = \varphi(b^\Lambda).$$

We denote by $\mathcal{B}_{\min}$ the set of minimal elements in $\mathcal{B}$, namely

$$\mathcal{B}_{\min} = \{b \in \mathcal{B} \mid \mathrm{lev}(\varepsilon(b)) = \ell\}.$$

Note that condition (5) of Definition 2.1 ensures that $\varepsilon, \varphi : \mathcal{B}_{\min} \to P_\ell^+$ are bijections. They induce an automorphism $\tau = \varepsilon \circ \varphi^{-1}$ on P_ℓ^+.

In [22, 5] $\pm$-diagrams were introduced, which describe the branching $X_n \to X_{n-1}$ where $X_n = B_n, C_n, D_n$. A $\pm$-diagram P of shape Λ/λ is a sequence of partitions $\lambda \subset \mu \subset \Lambda$ such that Λ/μ and μ/λ are horizontal strips (i.e. every column contains at most one box). We depict this $\pm$-diagram by the skew tableau of shape Λ/λ in which the cells of μ/λ are filled with the symbol $+$ and those of Λ/μ are filled with the symbol $-$. There are further type specific rules which can be found in [5, Section 3.2]. There exists a bijection Φ between $\pm$-diagrams and the X_{n-1}-highest weight vectors inside the X_n crystal of highest weight Λ.

3 Realization of KR-crystals

Throughout the paper we use the realization of $B^{r,s}$ as given in [5, 21, 22]. In this section we briefly recall the main constructions.

3.1 KR crystals of type $A_n^{(1)}$

Let $\Lambda = \ell_0\Lambda_0 + \ell_1\Lambda_1 + \cdots + \ell_n\Lambda_n$ be a dominant weight. Then the level is given by

$$\mathrm{lev}(\Lambda) = \ell_0 + \cdots + \ell_n.$$

A combinatorial description of $B^{r,s}$ of type $A_n^{(1)}$ was provided by Shimozono [23]. As a $\{1, 2, \ldots, n\}$-crystal

$$B^{r,s} \cong B(s\Lambda_r).$$

The Dynkin diagram of $A_n^{(1)}$ has a cyclic automorphism $\sigma(i) = i + 1 \pmod{n+1}$ which extends to the crystal in form of the promotion operator. The action of the affine crystal operators f_0 and e_0 is given by

$$f_0 = \sigma^{-1} \circ f_1 \circ \sigma \qquad \text{and} \qquad e_0 = \sigma^{-1} \circ e_1 \circ \sigma.$$

3.2 KR crystals of type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$

Let $\Lambda = \ell_0\Lambda_0 + \ell_1\Lambda_1 + \cdots + \ell_n\Lambda_n$ be a dominant weight. Then the level is given by

$$\begin{aligned}
&\mathrm{lev}(\Lambda) = \ell_0 + \ell_1 + 2\ell_2 + 2\ell_3 + \cdots + 2\ell_{n-2} + \ell_{n-1} + \ell_n && \text{for type } D_n^{(1)}\\
&\mathrm{lev}(\Lambda) = \ell_0 + \ell_1 + 2\ell_2 + 2\ell_3 + \cdots + 2\ell_{n-2} + 2\ell_{n-1} + \ell_n && \text{for type } B_n^{(1)}\\
&\mathrm{lev}(\Lambda) = \ell_0 + \ell_1 + 2\ell_2 + 2\ell_3 + \cdots + 2\ell_{n-2} + 2\ell_{n-1} + 2\ell_n && \text{for type } A_{2n-1}^{(2)}.
\end{aligned} \tag{3.1}$$

We have the following realization of $B^{r,s}$. Let $X_n = D_n, B_n, C_n$ be the classical subalgebra for $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$, respectively.

Definition 3.1 *Let $1 \le r \le n-2$ for type $D_n^{(1)}$, $1 \le r \le n-1$ for type $B_n^{(1)}$, and $1 \le r \le n$ for type $A_{2n-1}^{(2)}$. Then $B^{r,s}$ is defined as follows. As an X_n-crystal*

$$B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda), \tag{3.2}$$

where the sum runs over all dominant weights Λ that can be obtained from $s\Lambda_r$ by the removal of vertical dominoes. The affine crystal operators e_0 and f_0 are defined as

$$f_0 = \sigma^{-1} \circ f_1 \circ \sigma \quad \textit{and} \quad e_0 = \sigma^{-1} \circ e_1 \circ \sigma, \tag{3.3}$$

where σ is the crystal automorphism defined in [22, Definition 4.2].

Definition 3.2 *Let $B^{n,s}_{A^{(2)}_{2n-1}}$ be the $A^{(2)}_{2n-1}$-KR crystal. Then $B^{n,s}$ of type $B^{(1)}_n$ is defined through the unique injective map $S : B^{n,s} \to B^{n,s}_{A^{(2)}_{2n-1}}$ such that*

$$S(e_i b) = e_i^{m_i} S(b), \quad S(f_i b) = f_i^{m_i} S(b) \quad \textit{for } i \in I,$$

where $(m_i)_{0 \le i \le n} = (2, 2, \ldots, 2, 1)$.

In addition, the $\pm$-diagrams of $A^{(2)}_{2n-1}$ that occur in the image are precisely those which can be obtained by doubling a $\pm$-diagram of $B^{n,s}$ (see [5, Lemma 3.5]). S induces an embedding of dominant weights of $B^{(1)}_n$ into dominant weights of $A^{(2)}_{2n-1}$, namely $S(\Lambda_i) = m_i \Lambda_i$. It is easy to see that for any $\Lambda \in P^+$ we have $\mathrm{lev}(S(\Lambda)) = 2\,\mathrm{lev}(\Lambda)$ using (3.1).

For the definition of $B^{n,s}$ and $B^{n-1,s}$ of type $D^{(1)}_n$, see for example [5, Section 6.2].

3.3 KR crystal of type $C^{(1)}_n$

The level of a dominant $C^{(1)}_n$ weight $\Lambda = \ell_0 \Lambda_0 + \cdots + \ell_n \Lambda_n$ is given by

$$\mathrm{lev}(\Lambda) = \ell_0 + \cdots + \ell_n.$$

We use the realization of $B^{r,s}$ as the fixed point set of the automorphism σ [22, Definition 4.2] (see Definition 3.1) inside $B^{r,s}_{A^{(2)}_{2n+1}}$ of [5, Theorem 5.7].

Definition 3.3 *For $1 \le r < n$, the KR crystal $B^{r,s}$ of type $C^{(1)}_n$ is defined to be the fixed point set under σ inside $B^{r,s}_{A^{(2)}_{2n+1}}$ with the operators*

$$e_i = \begin{cases} e_0 e_1 & \textit{for } i = 0, \\ e_{i+1} & \textit{for } 1 \le i \le n, \end{cases}$$

where the Kashiwara operators on the right act in $B^{r,s}_{A^{(2)}_{2n+1}}$. Under the crystal embedding $S : B^{r,s} \to B^{r,s}_{A^{(2)}_{2n+1}}$ we have

$$\Lambda_i \mapsto \begin{cases} \Lambda_0 + \Lambda_1 & \textit{for } i = 0, \\ \Lambda_{i+1} & \textit{for } 1 \le i \le n. \end{cases}$$

Under the embedding S, the level of $\Lambda \in P^+$ doubles, that is $\mathrm{lev}(S(\Lambda)) = 2\,\mathrm{lev}(\Lambda)$.

For $B^{n,s}$ of type $C^{(1)}_n$ we refer to [5, Section 6.1].

3.4 KR crystals of type $A^{(2)}_{2n}$, $D^{(2)}_{n+1}$

Let $\Lambda = \ell_0 \Lambda_0 + \ell_1 \Lambda_1 + \cdots + \ell_n \Lambda_n$ be a dominant weight. The level is given by

$$\mathrm{lev}(\Lambda) = \ell_0 + 2\ell_1 + 2\ell_2 + \cdots + 2\ell_{n-2} + 2\ell_{n-1} + 2\ell_n \qquad \text{for type } A^{(2)}_{2n}$$

$$\mathrm{lev}(\Lambda) = \ell_0 + 2\ell_1 + 2\ell_2 + \cdots + 2\ell_{n-2} + 2\ell_{n-1} + \ell_n \qquad \text{for type } D^{(2)}_{n+1}.$$

Define positive integers m_i for $i \in I$ as follows:

$$(m_0, m_1, \ldots, m_{n-1}, m_n) = \begin{cases} (1,2,\ldots,2,2) & \text{for } A_{2n}^{(2)}, \\ (1,2,\ldots,2,1) & \text{for } D_{n+1}^{(2)}. \end{cases} \tag{3.4}$$

Then $B^{r,s}$ can be realized as follows.

Definition 3.4 *For $1 \le r \le n$ for $\mathfrak{g} = A_{2n}^{(2)}$, $1 \le r < n$ for $\mathfrak{g} = D_{n+1}^{(2)}$ and $s \ge 1$, there exists a unique injective map $S : B_{\mathfrak{g}}^{r,s} \longrightarrow B_{C_n^{(1)}}^{r,2s}$ such that*

$$S(e_i b) = e_i^{m_i} S(b), \quad S(f_i b) = f_i^{m_i} S(b) \qquad \textit{for } i \in I.$$

The $\pm$-diagrams of $C_n^{(1)}$ that occur in the image of S are precisely those which can be obtained by doubling a $\pm$-diagram of $B^{r,s}$ (see [5, Lemma 3.5]). S induces an embedding of dominant weights for $A_{2n}^{(2)}, D_{n+1}^{(2)}$ into dominant weights of type $C_n^{(1)}$, with $S(\Lambda_i) = m_i \Lambda_i$. This map preserves the level of a weight, that is $\mathrm{lev}(S(\Lambda)) = \mathrm{lev}(\Lambda)$.

For the case $r = n$ of type $D_{n+1}^{(2)}$ we refer to [5, Definition 6.2].

4 Proof of Theorem 1.2

For type $A_n^{(1)}$, perfectness of $B^{r,s}$ was proven in [14]. For all other types, in the case that $\frac{s}{c_r}$ is an integer, we need to show that the 5 defining conditions in Definition 2.1 are satisfied:

1. This was recently shown in [21].

2. This follows from [7, Corollary 6.1] under [7, Assumption 1]. Assumption 1 is satisfied except for type $A_{2n}^{(2)}$: The regularity of $B^{r,s}$ is ensured by (1), the existence of an automorphism σ was proven in [5, Section 7], and the unique element $u \in B^{r,s}$ such that $\varepsilon(u) = s\Lambda_0$ and $\varphi(u) = s\Lambda_\nu$ (where $\nu = 1$ for r odd for types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, $\nu = r$ for $A_n^{(1)}$, and $\nu = 0$ otherwise) is given by the classically highest weight element in the component $B(0)$ for $\nu = 0$, $B(s\Lambda_1)$ for $\nu = 1$, and $B(s\Lambda_r)$ for $\nu = r$. Note that $\Lambda_0 = \tau(\Lambda_\nu)$, where $\tau = \varepsilon \circ \varphi^{-1}$. For type $A_{2n}^{(2)}$, perfectness follows from [18].

3. The statement is true for $\lambda = s(\Lambda_r - \Lambda_r(c)\Lambda_0)$, which follows from the decomposition formulas [2, 9, 10, 19].

Conditions (4) and (5) will be shown in the following subsections using case by case considerations: Section 4.1 for type $A_n^{(1)}$, Sections 4.2, 4.3, and 4.4 for types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, Sections 4.5 and 4.6 for type $C_n^{(1)}$, Section 4.7 for type $A_{2n}^{(2)}$, and Sections 4.8 and 4.9 for type $D_{n+1}^{(2)}$.

When $\frac{s}{c_r}$ is not an integer, we show in the subsequent sections that the minimum of the level of $\varepsilon(b)$ is the smallest integer exceeding $\frac{s}{c_r}$, and provide examples that contradict condition (5) of Definition 2.1 for each crystal, thereby proving that $B^{r,s}$ is not perfect. In the case that $\frac{s}{c_r}$ is an integer, we provide an explicit construction of the minimal elements of $B^{r,s}$.

4.1 Type $A_n^{(1)}$

It was already proven in [14] that $B^{r,s}$ is perfect. We give below its associated automorphism τ and minimal elements. τ on P is defined by

$$\tau(\sum_{i=0}^{n} k_i\Lambda_i) = \sum_{i=0}^{n} k_i\Lambda_{i-r \bmod n+1}.$$

Recall that $B^{r,s}$ is identified with the set of semistandard tableaux of $r \times s$ rectangular shape over the alphabet $\{1, 2, \dots, n+1\}$. For $b \in B^{r,s}$ let $x_{ij} = x_{ij}(b)$ denote the number of letters j in the i-th row of b for $1 \le i \le r, 1 \le j \le n+1$. Set $r' = n + 1 - r$, then

$$x_{ij} = 0 \quad \text{unless} \quad i \le j \le i + r'.$$

Let $\Lambda = \sum_{i=0}^{n} \ell_i\Lambda_i$ be in P_s^+, that is, $\ell_0, \ell_1, \dots, \ell_n \in \mathbb{Z}_{\ge 0}, \sum_{i=0}^{n} \ell_i = s$. Then $x_{ij}(b)$ of the minimal element b such that $\varepsilon(b) = \Lambda$ is given by

$$\begin{aligned} x_{ii} &= \ell_0 + \sum_{\alpha=i}^{r-1} \ell_{\alpha+r'}, \\ x_{ij} &= \ell_{j-i} \quad (i < j < i + r'), \\ x_{i,i+r'} &= \sum_{\alpha=0}^{i-1} \ell_{\alpha+r'} \end{aligned} \tag{4.1}$$

for $1 \le i \le r$.

4.2 Types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$

Conditions (4) and (5) of Definition 2.1 for $1 \le r \le n-2$ for type $D_n^{(1)}$, $1 \le r \le n-1$ for type $B_n^{(1)}$, and $1 \le r \le n$ for type $A_{2n-1}^{(2)}$ were shown in [22, Section 6]. To a given fundamental weight Λ_k a $\pm$-diagram $\mathrm{diagram}(\Lambda_k)$ was associated. This map can be extended to any dominant weight $\Lambda = \ell_0\Lambda_0 + \cdots + \ell_n\Lambda_n$ by concatenating the columns of the $\pm$-diagrams of each piece. To every fundamental weight Λ_k a string of operators $f(\Lambda_k)$ can be associated as in [22, Section 6].

The minimal element b in $B^{r,s}$ that satisfies $\varepsilon(b) = \Lambda$ can now be constructed as follows

$$b = f(\Lambda_n)^{\ell_n} \cdots f(\Lambda_2)^{\ell_2}\ \Phi(\mathrm{diagram}(\Lambda)).$$

For $\Lambda = \sum_{i=0}^{n} \ell_i\Lambda_i \in P_s^+$, we have

$$\tau(\Lambda) = \begin{cases} \Lambda & \text{if } r \text{ is even,} \\ \ell_0\Lambda_1 + \ell_1\Lambda_0 + \sum_{i=2}^{n} \ell_i\Lambda_i & \text{if } r \text{ is odd,} \\ & \quad \text{types } B_n^{(1)}, A_{2n-1}^{(2)}, \\ \ell_0\Lambda_1 + \ell_1\Lambda_0 + \sum_{i=2}^{n-2} \ell_i\Lambda_i + \ell_{n-1}\Lambda_n + \ell_n\Lambda_{n-1} & \text{if } r \text{ is odd, type } D_n^{(1)}. \end{cases}$$

4.3 Type $D_n^{(1)}$ for $r = n-1, n$

The cases when $r = n, n-1$ for type $D_n^{(1)}$ were treated in [14]. We refer to [14] or [6, Section 4.3] for an explicit description of the minimal elements.

The automorphism τ is given by

$$\tau\Big(\sum_{i=0}^{n} \ell_i\Lambda_i\Big) = \ell_0\Lambda_{n-1} + \ell_1\Lambda_n + \sum_{i=2}^{n-2} \ell_i\Lambda_{n-i} + \begin{cases} \ell_{n-1}\Lambda_0 + \ell_n\Lambda_1 & n \text{ even,} \\ \ell_{n-1}\Lambda_1 + \ell_n\Lambda_0 & n \text{ odd.} \end{cases}$$

4.4 Type $B_n^{(1)}$ for $r = n$

In this section we consider the perfectness of $B^{n,s}$ of type $B_n^{(1)}$.

Proposition 4.1 *We have*

$$\min\{\mathrm{lev}(\varepsilon(b)) \mid b \in B^{n,2s+1}\} \geq s+1,$$
$$\min\{\mathrm{lev}(\varepsilon(b)) \mid b \in B^{n,2s}\} \geq s.$$

Proof: Suppose, there exists an element $b \in B^{n,2s+1}$ with $\mathrm{lev}(\varepsilon(b)) = p < s+1$. Since $B^{n,2s+1}$ is embedded into $B^{n,2s+1}_{A^{(2)}_{2n-1}}$ by Definition 3.2, this would yield an element $\tilde{b} \in B^{n,2s+1}_{A^{(2)}_{2n-1}}$ with $\mathrm{lev}(\tilde{b}) < 2s+1$. But this is not possible, since $B^{n,2s+1}_{A^{(2)}_{2n-1}}$ is a perfect crystal of level $2s+1$.

Suppose there exists an element $b \in B^{n,2s}$ with $\mathrm{lev}(\varepsilon(b)) = p < s$. By the same argument one obtains a contradiction to the level of $B^{n,2s}_{A^{(2)}_{2n-1}}$. □

Hence to show that $B^{n,2s+1}$ is not perfect, it is enough to provide two elements $b_1, b_2 \in B^{n,2s+1}_{A^{(2)}_{2n-1}}$ which are in the realization of $B^{r,s}$ under S and satisfy $\varepsilon(b_1) = \varepsilon(b_2) = \Lambda$, where $\mathrm{lev}(\Lambda) = 2s+2$. We use the notation $f_{\vec{a}} = f_{a_1}^{m_1} \cdots f_{a_k}^{m_k}$ for $\vec{a} = (a_1^{m_1}, \dots, a_k^{m_k})$.

Proposition 4.2 *Define the following elements $b_1, b_2 \in B^{n,2s+1}_{A^{(2)}_{2n-1}}$: For n odd, let P_1 be the $\pm$-diagram corresponding to one column of height n containing one $+$, and $2s$ columns of height 1 each containing a $-$ sign, and P_2 the analogous $\pm$-diagram but with a $-$ in the column of height n. Set $\vec{a} = (n, (n-1)^2, n, (n-2)^2, (n-1)^2, n, \dots, 2^2, \dots, (n-1)^2, n)$ and*

$$b_1 = f_{\vec{a}}(\Phi(P_1)) \quad \textit{and} \quad b_2 = f_{\vec{a}}(\Phi(P_2)).$$

For n even, replace the columns of height 1 with columns of height 2 and fill them with $\pm$-pairs. Then $b_1, b_2 \in S(B^{n,2s+1})$ and $\varepsilon(b_1) = \varepsilon(b_2) = 2s\Lambda_1 + \Lambda_n$, which is of level $2s+2$.

Proof: It is clear from the construction that the $\pm$-diagrams corresponding to b_1 and b_2 can be obtained by doubling a $B_n^{(1)}$ $\pm$-diagram (see [5, Lemma 3.5]). Hence $\Phi(P_1), \Phi(P_2) \in S(B^{n,2s+1})$. The sequence $\vec{a}$ can be obtained by doubling a type $B_n^{(1)}$ sequence using $(m_1, m_2, \dots, m_n) = (2, \dots, 2, 1)$, so by Definition 3.2 b_1 and b_2 are in the image of the embedding S that realizes $B^{n,2s+1}$. The claim that $\varepsilon(b_1) = \varepsilon(b_2) = 2s\Lambda_1 + \Lambda_n$ can be checked explicitly. □

Corollary 4.3 *The KR crystal $B^{n,2s+1}$ of type $B_n^{(1)}$ is not perfect.*

Proof: This follows directly from Proposition 4.2 using the embedding S of Definition 3.2. □

Proposition 4.4 *There exists a bijection, induced by ε, from $B^{n,2s}_{\min}$ to P_s^+. Hence $B^{n,2s}$ is perfect of level s.*

Proof: Let S be the embedding from Definition 3.2. Then we have an induced embedding of dominant weights Λ of $B_n^{(1)}$ into dominant weights of $A_{2n-1}^{(2)}$ via the map S, that sends $\Lambda_i \mapsto m_i\Lambda_i$.

In [22, Section 6] (see Section 4.2) the minimal elements for $A_{2n-1}^{(2)}$ were constructed by giving a $\pm$-diagram and a sequence from the $\{2,\dots,n\}$-highest weight to the minimal element. Since $(m_0,\dots,m_n) = (2,\dots,2,1)$ and columns of height n for type $A_{2n-1}^{(2)}$ are doubled, it is clear from the construction that the $\pm$-diagrams corresponding to weights $S(\Lambda)$ are in the image of S of $\pm$-diagrams for $B_n^{(1)}$ (see [5, Lemma 3.5]). Also, since under S all weights Λ_i for $1 \le i < n$ are doubled, it follows that the sequences are "doubled" using the m_i. Hence a minimal element of $B^{n,2s}$ of level s is in one-to-one correspondence with those minimal elements in $B^{n,2s}_{A_{2n-1}^{(2)}}$ that can be obtained from doubling a $\pm$-diagram of $B^{n,2s}$. This implies that ε defines a bijection between $B^{n,2s}_{\min}$ and P_s^+. □

The automorphism τ of the perfect KR crystal $B^{n,2s}$ is given by

$$\tau\Big(\sum_{i=0}^{n} \ell_i\Lambda_i\Big) = \begin{cases} \sum_{i=0}^{n} \ell_i\Lambda_i & \text{if } n \text{ is even,} \\ \ell_0\Lambda_1 + \ell_1\Lambda_0 + \sum_{i=2}^{n} \ell_i\Lambda_i & \text{if } n \text{ is odd.} \end{cases}$$

4.5 Type $C_n^{(1)}$

In this section we consider $B^{r,s}$ of type $C_n^{(1)}$ for $r < n$.

Proposition 4.5 *Let $r < n$. Then*

$$\min\{\mathrm{lev}(\varepsilon(b)) \mid b \in B^{r,2s+1}\} \ge s+1,$$
$$\min\{\mathrm{lev}(\varepsilon(b)) \mid b \in B^{r,2s}\} \ge s.$$

Proof: By Definition 3.3, the crystal $B^{r,s}$ is realized inside $B^{r,s}_{A_{2n+1}^{(2)}}$. The proof is similar to the proof of Proposition 4.1 for type $B_n^{(1)}$. □

Hence to show that $B^{r,2s+1}$ is not perfect, it is suffices to give two elements $b_1, b_2 \in B^{r,2s+1}_{A_{2n+1}^{(2)}}$ that are fixed points under σ with $\varepsilon(b_1) = \varepsilon(b_2) = \Lambda$, where $\mathrm{lev}(\Lambda) = 2s+2$.

Proposition 4.6 *Let $b_1, b_2 \in B^{r,2s+1}_{A_{2n+1}^{(2)}}$, where b_1 consists of s columns of the form read from bottom to top $(1,2,\dots,r)$, s columns of the form $(\overline{r},\overline{r-1},\dots,\overline{1})$, and a column $(\overline{r+1},\dots,\overline{2})$. In b_2 the last column is replaced by $(r+2,\dots,2r+2)$ if $2r+2 \le n$ and $(r+2,\dots,n,\overline{n},\dots,\overline{k})$ of height n otherwise. Then*

$$\varepsilon(b_1) = \varepsilon(b_2) = \begin{cases} s\Lambda_r + \Lambda_{r+1} & \text{if } r > 1, \\ s(\Lambda_0+\Lambda_1) + \Lambda_2 & \text{if } r = 1, \end{cases}$$

which is of level $2s+2$.

Proof: The claim is easy to check explicitly. □

Corollary 4.7 *The KR crystal $B^{n,2s+1}$ of type $C_n^{(1)}$ is not perfect.*

Proof: The $\{2,\dots,n\}$-highest weight elements in the same component as b_1 and b_2 of Proposition 4.6 correspond to $\pm$-diagrams that are invariant under σ. Hence, by Definition 3.3, b_1 and b_2 are fixed points under σ. Combining this result with Proposition 4.5 proves that $B^{r,2s+1}$ is not perfect. □

Proposition 4.8 *There exists a bijection, induced by ε, from $B^{r,2s}_{\min}$ to P_s^+. Hence $B^{r,2s}$ is perfect of level s.*

Proof: By Definition 3.3, $B^{r,s}$ of type $C_n^{(1)}$ is realized inside $B^{r,s}_{A^{(2)}_{2n+1}}$ as the fixed points under σ. Under the embedding S, it is clear that a dominant weight $\Lambda = \ell_0\Lambda_0 + \ell_1\Lambda_1 + \cdots + \ell_{n+1}\Lambda_{n+1}$ of type $A^{(2)}_{2n+1}$ is in the image if and only if $\ell_0 = \ell_1$. Hence it is clear from the construction of the minimal elements for $A^{(2)}_{2n+1}$ as described in Section 4.2 that the minimal elements corresponding to Λ with $\ell_0 = \ell_1$ are invariant under σ. By [22, Theorem 6.1] there is a bijection between all dominant weights Λ of type $A^{(2)}_{2n+1}$ with $\ell_0 = \ell_1$ and $\mathrm{lev}(\Lambda) = 2s$ and minimal elements in $B^{r,2s}_{A^{(2)}_{2n+1}}$ that are invariant under σ. Hence using S, there is a bijection between dominant weights in P_s^+ of type $C_n^{(1)}$ and $B^{r,2s}_{\min}$. □

The automorphism τ of the perfect KR crystal $B^{r,2s}$ is given by the identity.

4.6 Type $C_n^{(1)}$ for $r = n$

This case is treated in [14]. For the minimal elements, we follow the construction in Section 4.2. To every fundamental weight Λ_k we associate a column tableau $T(\Lambda_k)$ of height n whose entries are $k+1, k+2, \dots, n, \overline{n}, \dots, \overline{n-k+1}$ $(1,2,\dots,n$ for $k=0)$ reading from bottom to top. Let $f(\Lambda_k)$ be defined such that $T(\Lambda_k) = f(\Lambda_k)b_1$, where b_k is the highest weight tableau in $B(k\Lambda_n)$. Then the minimal element b in $B^{n,s}$ such that $\varepsilon(b) = \Lambda = \sum_{i=0}^n \ell_i\Lambda_i \in P_s^+$ is constructed as

$$b = f(\Lambda_n)^{\ell_n} \cdots f(\Lambda_1)^{\ell_1} b_s.$$

The automorphism τ is given by

$$\tau(\sum_{i=0}^{n} \ell_i\Lambda_i) = \sum_{i=0}^{n} \ell_i\Lambda_{n-i}.$$

4.7 Type $A_{2n}^{(2)}$

For type $A_{2n}^{(2)}$ one may use the result of Naito and Sagaki [18, Theorem 2.4.1] which states that under their [18, Assumption 2.3.1] (which requires that $B^{r,s}$ for $A_{2n}^{(1)}$ is perfect) all $B^{r,s}$ for $A_{2n}^{(2)}$ are perfect. Here we provide a description of the minimal elements via the emebdding S into $B^{r,2s}_{C_n^{(1)}}$.

Proposition 4.9 *The minimal elements of $B^{r,s}$ of level s are precisely those that corresponding to doubled $\pm$-diagrams in $B^{r,2s}_{C_n^{(1)}}$.*

Proof: In Proposition 4.8 a description of the minimal elements of $B^{r,2s}_{C_n^{(1)}}$ is given. We have the realization of $B^{r,s}$ via the map S from Definition 3.4. In the same way as in the proof of Proposition 4.4 one can show, that the minimal elements of $B^{r,2s}_{C_n^{(1)}}$ that correspond to doubled dominant weights are precisely those in the realization of $B^{r,s}$, hence ε defines a bijection between $B^{r,s}_{\min}$ and P_s^+. □

The automorphism τ is given by the identity.

4.8 Type $D_{n+1}^{(2)}$ for $r < n$

Proposition 4.10 *Let $r < n$. There exists a bijection $B^{r,s}_{\min}$ to P_s^+, defined by ε. Hence $B^{r,s}$ is perfect.*

Proof: This proof is analogous to the proof of Proposition 4.9. □

The automorphism τ is given by the identity.

4.9 Type $D_{n+1}^{(2)}$ for $r = n$

This case is already treated in [14], which we summarize below. As a B_n-crystal it is isomorphic to $B(s\Lambda_n)$. There is a description of its elements in terms of semistandard tableaux of $n \times s$ rectangular shape with letters from the alphabet $\mathcal{A} = \{1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{1}\}$. Moreover, each column does not contain both k and $\overline{k}$. Let c_i be the ith column, then the action of e_i, f_i $(i = 1, \ldots, n)$ is calculated through that of $c_s \otimes \cdots \otimes c_1$ of $B(\Lambda_n)^{\otimes s}$. With this realization the minimal element b_Λ such that $\varepsilon(b_\Lambda) = \Lambda = \sum_{i=0}^n \ell_i \Lambda_i \in P_s^+$ is given as follows. Let x_{ij} $(1 \le i \le n, j \in \mathcal{A})$ be the number of j in the ith row. Note that $x_{ij} = 0$ unless $i \le j \le \overline{n-i+1}$. The table (x_{ij}) of b_Λ is then given by $x_{ii} = \ell_0 + \cdots + \ell_{n-i}$ $(1 \le i \le n)$, $x_{ij} = \ell_{j-i}$ $(i+1 \le j \le n)$, $x_{i\overline{j}} = \ell_j + \cdots + \ell_n$ $(n-i+1 \le j \le n)$. The automorphism τ is given by

$$\tau(\sum_{i=0}^n \ell_i \Lambda_i) = \sum_{i=0}^n \ell_i \Lambda_{n-i}.$$

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