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Counting Quiver Representations over Finite Fields Via Graph Enumeration

Geir Helleloid[1] and Fernando Rodriguez-Villegas[2†]

[1]*Department of Mathematics, The University of Texas at Austin, 1 University Station C1200, Austin, TX 78712-0257*
[2]*Department of Mathematics, The University of Texas at Austin, 1 University Station C1200, Austin, TX 78712-0257*

Abstract. Let Γ be a quiver on n vertices $v_1, v_2, \ldots, v_n$ with g_{ij} edges between v_i and v_j, and let $\boldsymbol{\alpha} \in \mathbb{N}^n$. Hua gave a formula for $A_\Gamma(\boldsymbol{\alpha}, q)$, the number of isomorphism classes of absolutely indecomposable representations of Γ over the finite field $\mathbb{F}_q$ with dimension vector $\boldsymbol{\alpha}$. We use Hua's formula to show that the derivatives of $A_\Gamma(\boldsymbol{\alpha}, q)$ with respect to q, when evaluated at $q = 1$, are polynomials in the variables g_{ij}, and we can compute the highest degree terms in these polynomials. The formulas for these coefficients depend on the enumeration of certain families of connected graphs. This note simply gives an overview of these results; a complete account of this research is available on the arXiv and has been suboldsymbolitted for publication.

Résumé. Soit Γ un carquois sur n sommets $v_1, v_2, \ldots, v_n$ avec g_{ij} arêtes entre v_i et v_j, et soit $\boldsymbol{\alpha} \in \mathbb{N}^n$. Hua a donné une formule pour $A_\Gamma(\boldsymbol{\alpha}, q)$, le nombre de classes d'isomorphisme absolument indécomposables de représentations de Γ sur le corps fini $\mathbb{F}_q$ avec vecteur de dimension $\boldsymbol{\alpha}$. Nous utilisons la formule de Hua pour montrer que les dérivées de $A_\Gamma(\boldsymbol{\alpha}, q)$ par rapport à q, alors évaluée à $q = 1$, sont des polynômes dans les variables g_{ij}, et on peut calculer les termes de plus haut degré de ces polynômes. Les formules pour ces coefficients dépendent de l'énumération de certaines familles de graphes connectés. Cette note donne simplement un aperçu de ces résultats, un compte rendu complet de cette recherche est disponible sur arXiv et a été soumis pour publication.

Keywords: quiver representation, finite field, graph enumeration, absolutely indecomposable representation

1 Introduction

Let Γ be a quiver on n vertices $v_1, v_2, \ldots, v_n$ with g_{ij} edges between vertices v_i and v_j for $1 \leq i \leq j \leq n$. All of the following results are independent of the orientation of these edges. Let $\mathbf{0} \neq \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ (throughout the paper, vectors will be represented by boldface symbols, and $\mathbb{N}$ denotes the set of non-negative integers). We are interested in $A_\Gamma(\boldsymbol{\alpha}, q)$, the number of isomorphism classes of absolutely indecomposable representations of Γ over the finite field $\mathbb{F}_q$ with dimension vector $\boldsymbol{\alpha}$. Kac [6] proved that $A_\Gamma(\boldsymbol{\alpha}, q)$ is a polynomial in q with integer coefficients and that it is independent of the orientation of Γ. He conjectured that the coefficients of $A_\Gamma(\boldsymbol{\alpha}, q)$ are non-negative and that if Γ has no loops, then the constant term of $A_\Gamma(\boldsymbol{\alpha}, q)$ is equal to the multiplicity of $\boldsymbol{\alpha}$ in the Kac-Moody algebra

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defined by Γ. Both conjectures are true for quivers of finite and tame type and remain open for quivers of wild type (see Crawley-Boevey and Van den Bergh [1]). A proof of the multiplicity statement in Kac's conjectures for general quivers was recently announced by Hausel [3].

Our goal is to understand $A_\Gamma(\boldsymbol{\alpha}, 1)$, and more generally $\left(\frac{d^s}{dq^s} A_\Gamma(\boldsymbol{\alpha}, q)\right)\Big|_{q=1}$, as a function of the variables g_{ij}. The paper [4] offers complete descriptions and proofs of our results; this note is an extended abstract of that paper, content with stating the main theorems. Our primary impetus for studying $A_\Gamma(\boldsymbol{\alpha}, 1)$ comes from the work of Hausel and Rodriguez-Villegas [2]. They show that when Γ is the quiver S_g consisting of one vertex v with g self-loops, $A_{S_g}(\alpha, 1)$ (where $\boldsymbol{\alpha} = \alpha \in \mathbb{N}$) is (conjecturally) the dimension of the middle cohomology group of a character variety parameterizing certain representations of the fundamental group of a closed genus-g Riemann surface to $\mathrm{GL}_n(\mathbb{C})$.

One can imagine that specializing to $q = 1$ will relate $A_\Gamma(\boldsymbol{\alpha}, q)$ to counting representations of Γ in the category of finite sets; this hope follows a well-known philosophy about the significance of letting $q \to 1$ in formulas that depend on a finite field $\mathbb{F}_q$, although it seems hard to make this philosophy precise. In this paper we show in Theorems 4.3 and 5.1 that $\left(\frac{d^s}{dq^s} A_\Gamma(\boldsymbol{\alpha}, q)\right)\Big|_{q=1}$ is a polynomial in the variables g_{ij}, and we give a formula for its leading coefficients. This formula relies on the number of connected graphs in a family determined by Γ and on Stirling numbers of the second kind, which arise from derivatives of q-binomial coefficients. The description of the graphs in question is given prior to Theorem 3.1. Unfortunately, our proofs of Theorems 4.3 and 5.1 do not give any conceptual indication as to why our results should involve the enumeration of connected graphs.

To illustrate the type of result found in this paper, consider $\Gamma = S_g$. Using a formula of Hua [5, Theorem 4.6] for $A_\Gamma(\boldsymbol{\alpha}, q)$, which we will present in Section 2 and which is the starting point for our results, we can compute the polynomial $A_{S_g}(\alpha, q)$ for small α and g. These computations are displayed in the following table:

$A_{S_g}(\alpha, q)$	$g=1$	$g=2$	$g=3$	$g=4$
$\alpha=1$	q	q^2	q^3	q^4
$\alpha=2$	q	q^5+q^3	$q^9+q^7+q^5$	$q^{13}+q^{11}+\cdots$
$\alpha=3$	q	$q^{10}+q^8+\cdots$	$q^{19}+q^{17}+\cdots$	$q^{28}+q^{26}+\cdots$
$\alpha=4$	q	$q^{17}+q^{15}+\cdots$	$q^{33}+q^{31}+\cdots$	$q^{49}+q^{47}+\cdots$

Evaluating each polynomial at $q = 1$ gives the following values for $A_{S_g}(\alpha, 1)$:

$A_{S_g}(\alpha, 1)$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$	$g=6$
$\alpha=1$	1	1	1	1	1	1
$\alpha=2$	1	2	3	4	5	6
$\alpha=3$	1	6	15	28	45	66
$\alpha=4$	1	22	95	252	525	946

Fitting each row of the above table to a polynomial gives empirical evidence that the next table is correct:

	$A_{S_g}(\alpha, 1)$
$\alpha = 1$	1
$\alpha = 2$	$\binom{g}{1}$
$\alpha = 3$	$4\binom{g}{2} + \binom{g}{1}$
$\alpha = 4$	$32\binom{g}{3} + 20\binom{g}{2} + \binom{g}{1}$

This suggests that $A_{S_g}(\alpha, 1)$ is a polynomial in g of degree $\alpha - 1$ with leading coefficient $2^{\alpha-1}\alpha^{\alpha-2}/\alpha!$. We prove this and a generalization to all quivers in Theorem 4.3 below. Theorem 5.1 offers a similar result for any derivative (with respect to q) of $A_\Gamma(\boldsymbol{\alpha}, q)$ evaluated at $q = 1$.

The fact that the leading coefficient of $A_{S_g}(\alpha, 1)$ equals $2^{\alpha-1}\alpha^{\alpha-2}/\alpha!$ was mentioned (without proof) in [2, Remark 4.4.6]. As mentioned above, in the context of that paper, S_g corresponds to a closed Riemann surface of genus g and it seems more appropriate to use its Euler characteristic $2g - 2$ instead of g as a variable. Then $A_{S_g}(\alpha, 1)$ is a polynomial in $2g - 2$ of degree $\alpha - 1$ with leading coefficient $\alpha^{\alpha-2}/\alpha!$, and the disappearance of the factor $2^{\alpha-1}$ suggests that $2g - 2$ may be the "right" variable to use, though we do not know of a similar approach for the general case. Finally, we note that $\alpha^{\alpha-2}$ appears in the formula for the leading coefficient of $A_{S_g}(\alpha, 1)$ because $\alpha^{\alpha-2}$ is the number of trees on α labeled vertices by Cayley's Theorem. As indicated above, for other quivers, the leading coefficient formula involves the enumeration of other families of graphs.

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2 Hua's Formula

We begin with a presentation of Hua's formula for $A_\Gamma(\boldsymbol{\alpha}, q)$. Let $\boldsymbol{T} = (T_1, T_2, \ldots, T_n)$ be a vector of indeterminates. Let $\mathcal{P}$ denote the set of all integer partitions, including the unique partition of 0. If λ and μ are partitions with transposes λ' and μ' respectively, let

$$\langle \lambda, \mu \rangle := \sum_{1 \le i} \lambda'_i \mu'_i.$$

Also, let

$$b_\lambda(q) := \prod_{1 \le i} \prod_{1 \le j \le n_i} (1 - q^j),$$

where λ has n_i parts of size i for each i. As a notational convenience, we will write monomials as a vector with a vector exponent, as in $\boldsymbol{T}^{\boldsymbol{\alpha}} = T_1^{\alpha_1} \cdots T_n^{\alpha_n}$. If λ is a vector (say a partition or a weak composition), let $|\lambda|$ denote the sum of the parts of λ.

Finally, define the function $P_\Gamma(\boldsymbol{T}, q)$ by

$$P_\Gamma(\boldsymbol{T}, q) := \sum_{\lambda^1, \ldots, \lambda^n \in \mathcal{P}} \frac{\prod_{1 \le i \le j \le n} q^{g_{ij} \langle \lambda^i, \lambda^j \rangle}}{\prod_{1 \le i \le n} q^{\langle \lambda^i, \lambda^i \rangle} b_{\lambda^i}(q^{-1})} \, T_1^{|\lambda^1|} \cdots T_n^{|\lambda^n|} \tag{1}$$

and the function $H_\Gamma(\boldsymbol{\alpha}, q)$ implicitly by

$$\log P_\Gamma(\boldsymbol{T}, q) = \sum_{\mathbf{0}\neq\boldsymbol{\alpha}\in\mathbb{N}^n} \frac{H_\Gamma(\boldsymbol{\alpha}, q)}{\overline{\boldsymbol{\alpha}}} \boldsymbol{T}^{\boldsymbol{\alpha}}, \tag{2}$$

where $\overline{\boldsymbol{\alpha}} = \gcd(\alpha_1, \ldots, \alpha_n)$ (since $\boldsymbol{\alpha} \neq \mathbf{0}$, this is well-defined if we consider every integer to be a divisor of 0). Hua expresses $A_\Gamma(\boldsymbol{\alpha}, q)$ in terms of $H_\Gamma(\boldsymbol{\alpha}, q)$.

Theorem 2.1 (Hua [5, Theorem 4.6]).

$$A_\Gamma(\boldsymbol{\alpha}, q) = \frac{q-1}{\overline{\boldsymbol{\alpha}}} \sum_{d|\overline{\boldsymbol{\alpha}}} \mu(d) H_\Gamma(\boldsymbol{\alpha}/d, q^d). \tag{3}$$

Although we want to understand $A_\Gamma(\boldsymbol{\alpha}, 1)$, we cannot use Equations (1), (2), and (3) directly, since the summands in $P_\Gamma(\boldsymbol{T}, q)$ have poles at $q = 1$. We proceed instead by introducing extra variables, computing certain limits as q approaches 1, and then specializing the results. The remainder of this section analyzes $A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$, a generalization of $A_\Gamma(\boldsymbol{\alpha}, q)$, while Sections 4 and 5 apply the results to $A_\Gamma(\boldsymbol{\alpha}, q)$.

In what follows, vectors $\boldsymbol{u} \in \mathbb{N}^{n(n+1)/2}$ will have components u_{ij} for $1 \leq i \leq j \leq n$, and for $\boldsymbol{\ell} \in \mathbb{N}^n$ we let $\boldsymbol{u}^{\boldsymbol{\ell}} := \prod_{1\leq i\leq j\leq n} u_{ij}^{\ell_i \ell_j}$. Let $\boldsymbol{u} \in \mathbb{N}^{n(n+1)/2}$. Define functions $P_\Gamma(\boldsymbol{T}, \boldsymbol{u}, q)$, $H_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$, and $A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$ by the formulas

$$P_\Gamma(\boldsymbol{T}, \boldsymbol{u}, q) := \sum_{\lambda^1,\ldots,\lambda^n\in\mathcal{P}} \frac{\prod_{1\leq i\leq j\leq n} u_{ij}^{\langle\lambda^i,\lambda^j\rangle}}{\prod_{1\leq i\leq n} q^{\langle\lambda^i,\lambda^i\rangle} b_{\lambda^i}(q^{-1})} T_1^{|\lambda^1|} \cdots T_n^{|\lambda^n|}, \tag{4}$$

$$\log P_\Gamma(\boldsymbol{T}, \boldsymbol{u}, q) := \sum_{\mathbf{0}\neq\boldsymbol{\alpha}\in\mathbb{N}^n} \frac{H_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)}{\overline{\boldsymbol{\alpha}}} \boldsymbol{T}^{\boldsymbol{\alpha}}, \text{ and} \tag{5}$$

$$A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q) := \frac{q-1}{\overline{\boldsymbol{\alpha}}} \sum_{d|\overline{\boldsymbol{\alpha}}} \mu(d) H_\Gamma(\boldsymbol{\alpha}/d, \boldsymbol{u}^d, q^d). \tag{6}$$

Observe that $P_\Gamma(\boldsymbol{T}, \boldsymbol{u}, q)$, $H_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$, and $A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$ specialize to $P_\Gamma(\boldsymbol{T}, q)$, $H_\Gamma(\boldsymbol{\alpha}, q)$, and $A_\Gamma(\boldsymbol{\alpha}, q)$ respectively when $u_{ij} = q^{g_{ij}}$ for $1 \leq i \leq j \leq n$. However, $A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$ typically is not a polynomial in q even though $A_\Gamma(\boldsymbol{\alpha}, q)$ is. For $\boldsymbol{\ell} \in \mathbb{N}^n$ let $\boldsymbol{\ell}! := \ell_1! \cdots \ell_n!$ and for $\boldsymbol{u} \in \mathbb{N}^{n(n+1)/2}$ let $\boldsymbol{u}! := u_{11}! \cdots u_{ij}! \cdots u_{nn}!$. Our first result computes a limit involving $A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$.

Proposition 2.2.

$$\lim_{q\to 1} (q-1)^{|\boldsymbol{\alpha}|-1} A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q) = [\textit{the coefficient of } \boldsymbol{T}^{\boldsymbol{\alpha}} \textit{ in }] \log \sum_{\boldsymbol{\ell}\in\mathbb{N}^n} \boldsymbol{u}^{\boldsymbol{\ell}} \frac{\boldsymbol{T}^{\ell}}{\boldsymbol{\ell}!}. \tag{7}$$

3 $A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$ and Connected Graphs

The limit in Proposition 2.2, namely Equation (7), can be rewritten using a multivariate version of the Exponential Formula applied to the enumeration of graphs. To describe this enumerative result, we must introduce some more notation. If $\boldsymbol{\ell} \in \mathbb{N}^n$, let $\mathcal{G}^{\boldsymbol{\ell}}$ be the set of graphs on the vertices $v_1, v_2, \ldots, v_{|\boldsymbol{\ell}|}$. Let

$V_1 := \{v_1, \ldots, v_{\ell_1}\}, V_2 := \{v_{\ell_1+1}, \ldots, v_{\ell_2}\}, \ldots, V_n := \{v_{|\boldsymbol{\ell}|-\ell_n+1}, \ldots, v_{|\boldsymbol{\ell}|}\}$. If $\boldsymbol{k} \in \mathbb{N}^{n(n+1)/2}$, then let $\mathcal{G}^{\boldsymbol{\ell}}_{\boldsymbol{k}}$ be the set of graphs in $\mathcal{G}^{\boldsymbol{\ell}}$ that have k_{ij} edges between V_i and V_j for $1 \le i \le j \le n$ and let $G^{\boldsymbol{\ell}}_{\boldsymbol{k}}$ be the number of connected graphs in $\mathcal{G}^{\boldsymbol{\ell}}_{\boldsymbol{k}}$. Now let $\boldsymbol{x} = (x_{11}, \ldots, x_{ij}, \ldots, x_{nn})$ be a vector of $n(n+1)/2$ indeterminates, where $1 \le i \le j \le n$, and define the weight of $G \in \mathcal{G}^{\boldsymbol{\ell}}_{\boldsymbol{k}}$ to be $\boldsymbol{x}^{\boldsymbol{k}} := \prod_{1 \le i \le j \le n} x_{ij}^{k_{ij}}$.

Theorem 3.1.

$$\log\left(\sum_{\boldsymbol{\ell}\in\mathbb{N}^n}\left(\prod_{1\le i<j\le n}(1+x_{ij})^{\ell_i\ell_j}\right)\left(\prod_{1\le i\le n}(1+x_{ii})^{\binom{\ell_i}{2}}\right)\frac{\boldsymbol{X}^{\boldsymbol{\ell}}}{\boldsymbol{\ell}!}\right) \tag{8}$$
$$= \sum_{0\ne\boldsymbol{\alpha}\in\mathbb{N}^n}\ \sum_{\boldsymbol{k}\in\mathbb{N}^{n(n+1)/2}} G^{\boldsymbol{\alpha}}_{\boldsymbol{k}}\boldsymbol{x}^{\boldsymbol{k}}\frac{\boldsymbol{X}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!},$$

where $G^{\boldsymbol{\alpha}}_{\boldsymbol{k}}$ is the number of connected graphs in $\mathcal{G}^{\boldsymbol{\alpha}}_{\boldsymbol{k}}$.

Perhaps Theorem 3.1 is known, but we do not know of any reference. Presumbly there is no explicit formula for $G^{\boldsymbol{\alpha}}_{\boldsymbol{k}}$ in general. However, when $|\boldsymbol{k}| = |\boldsymbol{\alpha}| - 1$ (that is, when the connected graphs in $\mathcal{G}^{\boldsymbol{\alpha}}_{\boldsymbol{k}}$ are trees), certain sums of the numbers $G^{\boldsymbol{\alpha}}_{\boldsymbol{k}}$ can be computed by the methods in Knuth [7].

To understand Equation (7) better, we obtain a corollary of Theorem 3.1 by rewriting Equation (8) with the substitutions $1 + x_{ij} = u_{ij}$ $(1 \le i < j \le n)$, $1 + x_{ii} = u_{ii}^2$ $(1 \le i \le n)$, and $X_i = u_{ii}T_i$ $(1 \le i \le n)$. This allows us to rewrite the result of Proposition 2.2. For each $\boldsymbol{k} \in \mathbb{N}^{n(n+1)/2}$, let

$$S_{\boldsymbol{k}} = \left\{\boldsymbol{p} \in \mathbb{N}^{n(n+1)/2} \ : \ \begin{array}{l} k_{ii} \ge p_{ii} \text{ for } 1 \le i \le n \\ k_{ij} = p_{ij} \text{ for } 1 \le i < j \le n \end{array}\right\}. \tag{9}$$

Proposition 3.2. *For each $\boldsymbol{k} \in \mathbb{N}^{n(n+1)/2}$,*

$$\big[\textit{the coefficient of } (\boldsymbol{u}-\boldsymbol{1})^{\boldsymbol{k}} \textit{ in}\,\big] \lim_{q\to1}(q-1)^{|\boldsymbol{\alpha}|-1}A_\Gamma(\boldsymbol{\alpha},\boldsymbol{u},q) \tag{10}$$
$$= \frac{1}{\boldsymbol{\alpha}!}\sum_{\boldsymbol{p}\in S_{\boldsymbol{k}}} c^{\boldsymbol{\alpha}}_{\boldsymbol{kp}}\, G^{\boldsymbol{\alpha}}_{\boldsymbol{p}},$$

where

$$c^{\boldsymbol{\alpha}}_{\boldsymbol{kp}} := \prod_{1\le i\le n}\left(\sum_{j=0}^{\infty}\binom{p_{ii}}{j}\binom{\alpha_i}{k_{ii}-p_{ii}-j}2^{p_{ii}-j}\right)$$

and

$$(\boldsymbol{u}-\boldsymbol{1})^{\boldsymbol{k}} := \prod_{1\le i\le j\le n}(u_{ij}-1)^{k_{ij}}.$$

In particular, if $|\boldsymbol{k}| = |\boldsymbol{\alpha}| - 1$, then

$$\big[\textit{the coefficient of } (\boldsymbol{u}-\boldsymbol{1})^{\boldsymbol{k}} \textit{ in}\,\big] \lim_{q\to1}(q-1)^{|\boldsymbol{\alpha}|-1}A_\Gamma(\boldsymbol{\alpha},\boldsymbol{u},q) \tag{11}$$
$$= \frac{1}{\boldsymbol{\alpha}!}2^{t(\boldsymbol{k})}\, G^{\boldsymbol{\alpha}}_{\boldsymbol{k}},$$

where

$$t(\boldsymbol{k}) := \sum_{1\le i\le n} k_{ii}.$$

Observe that the left- and right-hand sides of Equation (10) are nonzero for finitely many $\boldsymbol{k}$ and that the sum over j is actually finite by the definition of binomial coefficients. Also, the sum over j can be expressed in terms of a hypergeometric series as

$$\begin{aligned} & \sum_{j=0}^{\infty} \binom{p_{ii}}{j} \binom{\alpha_i}{k_{ii} - p_{ii} - j} 2^{p_{ii}-j} \\ = \quad & 2^{p_{ii}} \binom{\alpha_i}{k_{ii} - p_{ii}} {}_2F_1(-p_{ii}, -k_{ii} + p_{ii}; a_i - k_{ii} + p_{ii} + 1; 1/2), \end{aligned}$$

if desired.

4 A Mahler-type Expansion for $A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$

We can use Proposition 3.2 to understand $A_\Gamma(\boldsymbol{\alpha}, q)$ if we rewrite $A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$ using the Mahler-type expansion given in the following theorem.

Theorem 4.1. *If $f \in \mathbb{Q}(q)[x_1, \ldots, x_r]$ and $f(q^{b_1}, \ldots, q^{b_r}) \in \mathbb{Z}[q]$ for all non-negative integers $b_1, \ldots, b_r$, then there are polynomials $\{c_{\boldsymbol{\ell}}(q) \in \mathbb{Z}[q] \;:\; \boldsymbol{\ell} \in \mathbb{N}^r\}$ such that*

$$f \quad = \quad \sum_{\boldsymbol{\ell} \in \mathbb{N}^r} c_{\boldsymbol{\ell}}(q) \prod_{1 \le i \le r} \left\langle {x_i \atop \ell_i} \right\rangle_q, \tag{12}$$

where

$$\left\langle {x \atop \ell} \right\rangle_q \quad := \quad \prod_{1 \le i' \le \ell} \frac{(x/q^{i'-1} - 1)}{(q^{i'} - 1)} \tag{13}$$

and $c_{\boldsymbol{\ell}}(q) = 0$ for all but finitely many $\boldsymbol{\ell}$.

The proof of this theorem was communicated to us by Keith Conrad. Note that $\left\langle {x \atop \ell} \right\rangle_q = \left[{b \atop \ell} \right]_q$ when $x = q^b$ and that

$$\lim_{q \to 1} (q-1)^\ell \left\langle {x \atop \ell} \right\rangle_q \quad = \quad \frac{(x-1)^\ell}{\ell!}. \tag{14}$$

Here $\left[{b \atop \ell} \right]_q$ is a q-binomial coefficient. By Theorem 4.1 and the fact that $A_\Gamma(\boldsymbol{\alpha}, q) \in \mathbb{Z}[q]$, we can write

$$A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q) \quad = \quad \sum_{\boldsymbol{k} \in \mathbb{N}^{n(n+1)/2}} a_\Gamma(\boldsymbol{\alpha}, \boldsymbol{k}, q) \left\langle {\boldsymbol{u} \atop \boldsymbol{k}} \right\rangle_q, \tag{15}$$

for some $a_\Gamma(\boldsymbol{\alpha}, \boldsymbol{k}, q) \in \mathbb{Z}[q]$, where

$$\left\langle {\boldsymbol{u} \atop \boldsymbol{k}} \right\rangle_q := \prod_{1 \le i \le j \le n} \left\langle {u_{ij} \atop k_{ij}} \right\rangle_q.$$

Hence

$$A_\Gamma(\boldsymbol{\alpha}, q) \quad = \sum_{\boldsymbol{k}\in\mathbb{N}^{n(n+1)/2}} a_\Gamma(\boldsymbol{\alpha}, \boldsymbol{k}, q) \begin{bmatrix}\boldsymbol{g}\\ \boldsymbol{k}\end{bmatrix}_q, \tag{16}$$

where

$$\begin{bmatrix}\boldsymbol{g}\\ \boldsymbol{k}\end{bmatrix}_q := \prod_{1\le i\le j\le n} \begin{bmatrix}g_{ij}\\ k_{ij}\end{bmatrix}_q.$$

It turns out that Proposition 3.2 leads to a formula for the derivatives of $a_\Gamma(\boldsymbol{\alpha}, \boldsymbol{k}, q)$ evaluated at $q = 1$, given in Proposition 4.2, which in turn produces a formula for the derivatives of $A_\Gamma(\boldsymbol{\alpha}, q)$ evaluated at $q = 1$ (see Theorems 4.3 and 5.1).

Proposition 4.2. *For $\boldsymbol{k} \in \mathbb{N}^{n(n+1)/2}$ such that $|\boldsymbol{k}| > |\boldsymbol{\alpha}|$ we have*

$$\left.\frac{a_\Gamma(\boldsymbol{\alpha}, \boldsymbol{k}, q)}{(q-1)^{|\boldsymbol{k}|-|\boldsymbol{\alpha}|+1}}\right|_{q=1} \quad = \quad \frac{\boldsymbol{k}!}{\boldsymbol{\alpha}!} \sum_{\boldsymbol{p}\in S_{\boldsymbol{k}}} c^{\boldsymbol{\alpha}}_{\boldsymbol{k}\boldsymbol{p}}\, G^{\boldsymbol{\alpha}}_{\boldsymbol{p}}. \tag{17}$$

Note that if $|\boldsymbol{k}| \le |\boldsymbol{\alpha}|$, Proposition 4.2 says nothing about $a_\Gamma(\boldsymbol{\alpha}, \boldsymbol{k}, q)$ at $q = 1$. This is why Theorems 4.3 and 5.1 below only give information about leading coefficients. The first consequence of Proposition 4.2 appears when we evaluate $A_\Gamma(\boldsymbol{\alpha}, 1)$. By Equation (16),

$$A_\Gamma(\boldsymbol{\alpha}, 1) \quad = \sum_{\boldsymbol{k}\in\mathbb{N}^{n(n+1)/2}} a_\Gamma(\boldsymbol{\alpha}, \boldsymbol{k}, 1) \binom{\boldsymbol{g}}{\boldsymbol{k}}, \tag{18}$$

where

$$\binom{\boldsymbol{g}}{\boldsymbol{k}} := \prod_{1\le i\le j\le n} \binom{g_{ij}}{k_{ij}}.$$

Theorem 4.3. *The quantity $A_\Gamma(\boldsymbol{\alpha}, 1)$ is a polynomial in the variables g_{ij} whose homogeneous component of highest degree $A^*_\Gamma(\boldsymbol{\alpha}, 1)$ has total degree $|\boldsymbol{\alpha}| - 1$ and has the form*

$$A^*_\Gamma(\boldsymbol{\alpha}, 1) = \frac{1}{\boldsymbol{\alpha}!} \sum_{|\boldsymbol{k}|=|\boldsymbol{\alpha}|-1} C^{\boldsymbol{\alpha}}_{\Gamma,\boldsymbol{k}}\, \boldsymbol{g}^{\boldsymbol{k}}, \tag{19}$$

where

$$C^{\boldsymbol{\alpha}}_{\Gamma,\boldsymbol{k}} := 2^{t(\boldsymbol{k})}\, G^{\boldsymbol{\alpha}}_{\boldsymbol{k}} \quad \textit{and} \quad t(\boldsymbol{k}) := \sum_{1\le i\le n} k_{ii}.$$

As a special case of this theorem, we can consider the quiver S_g from Section 1, which has a single vertex (so $n = 1$) and g loops, and $\boldsymbol{\alpha} = \alpha$. In this case, $A_\Gamma(\alpha, 1)$ is a polynomial in g of degree $\alpha - 1$ and leading coefficient $2^{\alpha-1}G^{\alpha}_{\alpha-1}/\alpha!$ But $G^{\alpha}_{\alpha-1}$ is just the number of (spanning) trees on α labeled vertices, which is $\alpha^{\alpha-2}$ by Cayley's Theorem. So the leading coefficient is $2^{\alpha-1}\alpha^{\alpha-2}/\alpha!$ as claimed in the introduction.

5 The derivatives $\frac{d^s}{dq^s} A_\Gamma(\boldsymbol{\alpha}, q)$ at $q = 1$

We can proceed further by differentiating Equation (16) to obtain information about the highest order terms of the s-th derivative of $A_\Gamma(\boldsymbol{\alpha}, q)$ evaluated at $q = 1$. If $\boldsymbol{k}, \boldsymbol{\ell} \in \mathbb{N}^{n(n+1)/2}$, we write $\boldsymbol{k} \leq \boldsymbol{\ell}$ if $k_{ij} \leq \ell_{ij}$ for all $1 \leq i \leq j \leq n$. To simplify the notation we let

$$A_{\Gamma,s}(\boldsymbol{\alpha}, q) := \frac{d^s}{dq^s} A_\Gamma(\boldsymbol{\alpha}, q).$$

Theorem 5.1. *The quantity $A_{\Gamma,s}(\boldsymbol{\alpha}, 1)$ is a polynomial in the variables g_{ij} whose homogeneous component of highest degree $A^*_{\Gamma,s}(\boldsymbol{\alpha}, 1)$ has total degree $s + |\boldsymbol{\alpha}| - 1$ and is given by*

$$A^*_{\Gamma,s}(\boldsymbol{\alpha}, 1) = \frac{1}{\boldsymbol{\alpha}!} \sum_{|\boldsymbol{\ell}|=s+|\boldsymbol{\alpha}|-1} C^{\boldsymbol{\alpha}}_{\Gamma,s,\boldsymbol{\ell}}\, \boldsymbol{g}^{\boldsymbol{\ell}}, \tag{20}$$

where

$$C^{\boldsymbol{\alpha}}_{\Gamma,s,\boldsymbol{\ell}} := \frac{s!}{\boldsymbol{\ell}!} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^{n(n+1)/2} \\ \boldsymbol{k}\leq\boldsymbol{\ell}}} S(\boldsymbol{k}, \boldsymbol{\ell})\, \boldsymbol{k}! \sum_{\boldsymbol{p}\in S_{\boldsymbol{k}}} c^{\alpha}_{\boldsymbol{k}\boldsymbol{p}}\, G^{\boldsymbol{\alpha}}_{\boldsymbol{p}},$$

$$S(\boldsymbol{\ell}, \boldsymbol{k}) := \prod_{1\leq i\leq j\leq n} S(\ell_{ij}, k_{ij}),$$

and $S(\ell, k)$ is the Stirling number of the second kind.

Incidentally, one ingredient in the proof of Theorem 5.1 is an auxiliary theorem which shows that for each $t \geq 0$, the quantity $\left(\frac{d^t}{dq^t} \begin{bmatrix} b \\ k \end{bmatrix}_q\right)\Big|_{q=1}$ is a polynomial in b of degree $k + t$ with leading coefficient $\frac{t!}{(k+t)!} \cdot S(k + t, k)$.

As a special case of this theorem, we can consider the quiver S_g from Section 1. In this case, $A_{\Gamma,s}(\boldsymbol{\alpha}, 1)$ is a polynomial in g of degree $s + \alpha - 1$ and leading coefficient

$$\frac{s!}{\alpha!(s+\alpha-1)!} \sum_{k=\alpha-1}^{s+\alpha-1} S(s+\alpha-1, k)\, k! \sum_{p=\alpha-1}^{k} G^{\alpha}_p \sum_{j=0}^{\infty} \binom{p}{j} \binom{\alpha}{k-p-j} 2^{p-j}.$$

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