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Permutations realized by shifts

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Abstract. A permutation π is realized by the shift on N symbols if there is an infinite word on an N-letter alphabet whose successive left shifts by one position are lexicographically in the same relative order as π. The set of realized permutations is closed under consecutive pattern containment. Permutations that cannot be realized are called forbidden patterns. It was shown in [J.M. Amigó, S. Elizalde and M. Kennel, *J. Combin. Theory Ser. A* 115 (2008), 485–504] that the shortest forbidden patterns of the shift on N symbols have length $N+2$. In this paper we give a characterization of the set of permutations that are realized by the shift on N symbols, and we enumerate them with respect to their length.

Résumé. Une permutation π est réalisée par le *shift* avec N symboles s'il y a un mot infini sur un alphabet de N lettres dont les déplacements successifs d'une position à gauche sont lexicographiquement dans le même ordre relatif que π. Les permutations qui ne sont pas réalisées s'apellent des motifs interdits. On sait [J.M. Amigó, S. Elizalde and M. Kennel, *J. Combin. Theory Ser. A* 115 (2008), 485–504] que les motifs interdits les plus courts du *shift* avec N symboles ont longueur $N+2$. Dans cet article on donne une caractérisation des permutations réalisées par le *shift* avec N symboles, et on les dénombre selon leur longueur.

Keywords: shift, consecutive pattern, forbidden pattern

1 Introduction and definitions

This paper is motivated by an innovative application of pattern-avoiding permutations to dynamical systems (see (1; 2; 4)), which is based on the following idea. Given a piecewise monotone map on a one-dimensional interval, consider the finite sequences (orbits) that are obtained by iterating the map, starting from any point in the interval. It turns out that the relative order of the entries in these sequences cannot be arbitrary. This means that, for any given such map, there will be some order patterns that will never appear in any orbit. The set of such patterns, which we call forbidden patterns, is closed under consecutive pattern containment. These facts can be used to distinguish random from deterministic time series.

A natural question that arises is how to determine, for a given map, what its forbidden patterns are. While this problem is wide open in general, in the present paper we study it for a particular kind of maps, called (one-sided) shift systems. Shift systems are interesting for two reasons. One one hand, they exhibit all important features of low-dimensional chaos. On the other hand, they are natural maps from a combinatorial perspective, and the study of their forbidden patterns can be done in an elegant combinatorial way.

Forbidden patterns in shift systems were first considered in (1; 2). The authors prove that the smallest forbidden pattern of the shift on N symbols has length $N+2$. They also conjecture that, for any N, there are exactly six forbidden patterns of minimal length. In the present paper we give a complete characterization of forbidden patterns of shift systems, and enumerate them with respect to their length.

We will start with some background on consecutive pattern containment, forbidden patterns in maps, and shift systems. In Section 2 we give a formula for the parameter that determines how many symbols are needed in order for a permutation to be realized by a shift. This characterizes allowed and forbidden patterns of shift maps. In Section 3 we give another equivalent characterization involving a transformation on permutations, and we prove that the shift on N symbols has six forbidden patterns of minimal length $N+2$, as conjectured in (1). In Section 4 we give a formula for the number of patterns of a given length that are realized by the binary shift, and then we generalize it to the shift on N symbols, for arbitrary N. Many of the proofs are omitted in this extended abstract, but they can be found in the full version (5).

1.1 Permutations and consecutive patterns

We denote by $\mathcal{S}_n$ the set of permutations of $\{1,2,\dots,n\}$. If $\pi\in\mathcal{S}_n$, we will write its one-line notation as $\pi=[\pi(1),\pi(2),\dots,\pi(n)]$ (or $\pi=\pi(1)\pi(2)\dots\pi(n)$ if it creates no confusion). The use of square brackets is to distinguish it from the cycle notation, where π is written as a product of cycles of the form $(i,\pi(i),\pi^2(i),\dots,\pi^{k-1}(i))$, with $\pi^k(i)=i$. For example, $\pi=[2,5,1,7,3,6,4]=(1,2,5,3)(4,7)(6)$.

Given a permutation $\pi=\pi(1)\pi(2)\dots\pi(n)$, let $D(\pi)$ denote the *descent set* of π, that is, $D(\pi)=\{i\ :\ 1\le i\le n-1,\ \pi(i)>\pi(i+1)\}$. Let $\mathrm{des}(\pi)=|D(\pi)|$ be the number of descents. The Eulerian polynomials are defined by $A_n(x)=\sum_{\pi\in\mathcal{S}_n}x^{\mathrm{des}(\pi)+1}$. Its coefficients are called the Eulerian numbers. The descent set and the number of descents can be defined for any sequence of integers $a=a_1a_2\dots a_n$ by letting $D(a)=\{i\ :\ 1\le i\le n-1,\ a_i>a_{i+1}\}$.

Let X be a totally ordered set, and let $x_1,\dots,x_n\in X$ with $x_1<x_2<\dots<x_n$. Any permutation of these values can be expressed as $[x_{\pi(1)},x_{\pi(2)},\dots,x_{\pi(n)}]$, where $\pi\in\mathcal{S}_n$. We define its *reduction* to be $\rho([x_{\pi(1)},x_{\pi(2)},\dots,x_{\pi(n)}])=[\pi(1),\pi(2),\dots,\pi(n)]=\pi$. Note that the reduction is just a relabeling of the entries with the numbers from 1 to n, keeping the order relationships among them. For example $\rho([4,7,1,6.2,\sqrt{2}])=[3,5,1,4,2]$. If the values $y_1,\dots,y_n$ are not all different, then $\rho([y_1,\dots,y_n])$ is not defined.

Given two permutations $\sigma\in\mathcal{S}_m$, $\pi\in\mathcal{S}_n$, with $m\ge n$, we say that σ *contains* π *as a consecutive pattern* is there is some i such that $\rho([\sigma(i),\sigma(i+1),\dots,\sigma(i+n-1)])=\pi$. Otherwise, we say that σ *avoids* π *as a consecutive pattern*. The set of permutations in $\mathcal{S}_n$ that avoid π as a consecutive pattern is denoted by $\mathrm{Av}_n(\pi)$. We let $\mathrm{Av}(\pi)=\bigcup_{n\ge1}\mathrm{Av}_n(\pi)$. Consecutive pattern containment was first studied in (6), where the sets $\mathrm{Av}_n(\pi)$ are enumerated for certain permutations π.

1.2 Allowed and forbidden patterns in maps

Let f be a map $f:X\to X$. Given $x\in X$ and $n\ge1$, we define

$$\mathrm{Pat}(x,f,n)=\rho([x,f(x),f^2(x),\dots,f^{n-1}(x)]),$$

provided that there is no pair $1\le i<j\le n$ such that $f^{i-1}(x)=f^{j-1}(x)$. If there is such a pair, then $\mathrm{Pat}(x,f,n)$ is not defined. When it is defined, we have $\mathrm{Pat}(x,f,n)\in\mathcal{S}_n$. If $\pi\in\mathcal{S}_n$ and there is some $x\in X$ such that $\mathrm{Pat}(x,f,n)=\pi$, we say that π is *realized* by f (at x), or that π is an *allowed pattern* of f. The set of all permutations realized by f is denoted by $\mathrm{Allow}(f)=\bigcup_{n\ge1}\mathrm{Allow}_n(f)$, where

$$\mathrm{Allow}_n(f)=\{\mathrm{Pat}(x,f,n):x\in X\}\subseteq\mathcal{S}_n.$$

The remaining permutations are called *forbidden patterns*, and denoted by $\mathrm{Forb}(f) = \bigcup_{n\geq 1} \mathrm{Forb}_n(f)$, where $\mathrm{Forb}_n(f) = \mathcal{S}_n \setminus \mathrm{Allow}_n(f)$.

We are introducing some variations to the notation and terminology used in (1; 2; 4). The main change is that our permutation $\pi = \mathrm{Pat}(x, f, n)$ is essentially the inverse of the permutation of $\{0, 1, \dots, n-1\}$ that the authors of (1) refer to as the *order pattern defined by* x. Our convention, aside from simplifying the notation, will be more convenient from a combinatorial point of view. The advantage is that now the set $\mathrm{Allow}(f)$ is closed under consecutive pattern containment, in the standard sense used in the combinatorics literature, and we no longer need to talk about *outgrowth forbidden patterns* like in (1). Indeed, if $\sigma \in \mathrm{Allow}(f)$ and σ contains τ as a consecutive pattern, then $\tau \in \mathrm{Allow}(f)$. An equivalent statement is that if $\pi \in \mathrm{Forb}(f)$, then $\mathrm{Allow}(f) \subseteq \mathrm{Av}(\pi)$. The minimal elements of $\mathrm{Forb}(f)$, i.e., those permutations in $\mathrm{Forb}(f)$ that avoid all other patterns in $\mathrm{Forb}(f)$, will be called *basic forbidden patterns* of f. The set of these patterns will be denoted $\mathrm{BF}(f)$. Note that basic patterns are the inverses of root patterns as defined in (1).

Let us consider now the case in which X is a closed interval in $\mathbb{R}$, with the usual total order on real numbers. An important incentive to study the set of forbidden patterns of a map comes from the following result, which is a consequence of (4).

Proposition 1.1 *If $I \subset \mathbb{R}$ is a closed interval and $f : I \to I$ is piecewise monotone, then* $\mathrm{Forb}(f) \neq \emptyset$.

Recall that piecewise monotone means that there is a finite partition of I into intervals such that f is continuous and strictly monotone on each of those intervals. It fact, it is shown in (4) that for such a map, the number of allowed patterns of f grows at most exponentially, i.e., there is a constant C such that $|\mathrm{Allow}_n(f)| < C^n$ for n large enough. The value of C is related to the *topological entropy* of f (see (4) for details). Since the growth of the total number of permutations of length n is super-exponential, the above proposition follows.

Proposition 1.1, together with the above observation that $\mathrm{Allow}(f)$ is closed under consecutive pattern containment, provides an interesting connection between dynamical systems on one-dimensional interval maps and pattern avoiding permutations. An important application is that forbidden patterns can be used to distinguish random from deterministic time series. Indeed, in a sequence $(x_1, x_2, x_3, \dots)$ where each x_i has been chosen independently at random from some continuous probability distribution, any pattern $\pi \in \mathcal{S}_n$ appears as $\pi = \rho([x_i, x_{i+1}, \dots, x_{i+n-1}])$ for some i with nonvanishing probability, and this probability approaches one as the length of the sequence increases. On the other hand, if the sequence has been generated by defining $x_{k+1} = f(x_k)$ for $k \geq 1$, where $f : I \to I$ is a piecewise monotone map, then Proposition 1.1 guarantees that some patterns (in fact, most of them) will never appear in the sequence. The practical aspect of these applications has been considered in (3).

A structural property of the set of allowed patterns of a map is that it is closed under consecutive pattern containment. A new and interesting direction of research is to study more properties of the sets $\mathrm{Allow}(f)$. Some natural problems that arise are the following.

1. Understand how $\mathrm{Allow}(f)$ and $\mathrm{BF}(f)$ depend on the map f.
2. Describe and/or enumerate (exactly or asymptotically) $\mathrm{Allow}(f)$ and $\mathrm{BF}(f)$ for a particular f.
3. Among the sets of patterns Σ such that $\mathrm{Av}_n(\Sigma)$ grows at most exponentially in n (this is a necessary condition), characterize those for which there exists a map f such that $\mathrm{BF}(f) = \Sigma$.
4. Given a map f, determine the length of its smallest forbidden pattern.

Most of this paper is devoted to solving problem 2 for a specific family of maps, that we describe next.

1.3 One-sided shifts

We will concentrate on the set of allowed patterns of certain maps called *one-sided shift maps*, or simply *shifts* for short. For a detailed definition of the associated dynamical system, called the *one-sided shift space*, we refer the reader to (1).

The totally ordered set X considered above will now be the set $\mathcal{W}_N = \{0,1,\dots,N-1\}^{\mathbb{N}}$ of infinite words on N symbols, equipped with the lexicographic order. Define the (one-sided) shift transformation

$$\begin{array}{cccc} \Sigma_N : & \mathcal{W}_N & \longrightarrow & \mathcal{W}_N \\ & w_1w_2w_3\dots & \mapsto & w_2w_3w_4\dots. \end{array}$$

We will use Σ instead of Σ_N when it creates no confusion.

Given $w \in \mathcal{W}_N$, $n \geq 1$, and $\pi \in \mathcal{S}_n$, we have from the above definition that $\mathrm{Pat}(w,\Sigma,n) = \pi$ if, for all indices $1 \leq i,j \leq n$, $\Sigma^{i-1}(w) < \Sigma^{j-1}(w)$ if and only if $\pi(i) < \pi(j)$. For example,

$$\mathrm{Pat}(2102212210\dots,\Sigma,7) = [4,2,1,7,5,3,6], \tag{1}$$

because the relative order of the successive shifts is

$$\begin{array}{rc} 2102212210\dots & 4 \\ 102212210\dots & 2 \\ 02212210\dots & 1 \\ 2212210\dots & 7 \\ 212210\dots & 5 \\ 12210\dots & 3 \\ 2210\dots & 6, \end{array}$$

regardless of the entries in place of the dots. The case $N = 1$ is trivial, since the only allowed pattern of Σ_1 is the permutation of length 1. In the rest of the paper, we will assume that $N \geq 2$.

If $x \in \{0,1,\dots,N-1\}$, we will use the notation $x^\infty = xxx\dots$. If $w \in \mathcal{W}_N$, then w_n denotes the n-th letter of w, and we write $w = w_1w_2w_3\dots$. We will also write $w_{[k,\ell]} = w_kw_{k+1}\dots w_\ell$, and $w_{[k,\infty)} = w_kw_{k+1}\dots$. Note that $w_{[k,\infty)} = \Sigma^{k-1}(w)$.

It is shown in (1) that Σ_N has the same set of forbidden patterns as the so-called *sawtooth map* defined by $x \mapsto Nx \mod 1$ for $x \in [0,1]$. This map is piecewise linear, and therefore has forbidden patterns by Proposition 1.1. Forbidden patterns of shift systems were first studied in (1), where the main result is the following.

Proposition 1.2 ((1)) *Let $N \geq 2$. We have that*

(a) $\mathrm{Forb}_n(\Sigma_N) = \emptyset$ *for every* $n \leq N+1$,
(b) $\mathrm{Forb}_n(\Sigma_N) \neq \emptyset$ *for every* $n \geq N+2$.

Example 1. It can be checked that the smallest forbidden patterns of Σ_4 are $615243, 324156, 342516, 162534, 453621, 435261$.

Recall that a word $w \in \{0,1,\dots,N-1\}^k$ is *primitive* if it cannot be written as a power of any proper subword, i.e., it is not of the form $w = u^m$ for any $m > 1$, where the exponent indicates concatenation of u with itself m times. Let $\psi_N(k)$ denote the number of primitive words of length k over an N-letter alphabet. It is well known that $\psi_N(k) = \sum_{d|k} \mu(d) N^{k/d}$, where μ denotes the Möbius function.

2 The number of symbols needed to realize a pattern

Given a permutation $\pi \in \mathcal{S}_n$, let $N(\pi)$ be the smallest number N such that $\pi \in \mathrm{Allow}(\Sigma_N)$. The value of $N(\pi)$ indicates what is the minimum number of symbols needed in the alphabet in order for π to be realized by a shift. For example, if $\pi = [4,2,1,7,5,3,6]$, then $N(\pi) \leq 3$ because of equation (1), and it is not hard to see that $N(\pi) = 3$. The main result in this section is a formula for $N(\pi)$.

Theorem 2.1 *Let $n \geq 2$. For any $\pi \in \mathcal{S}_n$, $N(\pi)$ is given by*

$$N(\pi) = 1 + |A(\pi)| + \Delta(\pi), \tag{2}$$

where

$$A(\pi) = \{a \,:\, 1 \leq a \leq n-1 \text{ such that if } i = \pi^{-1}(a),\ j = \pi^{-1}(a+1),\ \text{then } i,j < n \text{ and } \pi(i+1) > \pi(j+1)\},$$

and $\Delta(\pi) = 0$ except in the following three cases, in which $\Delta(\pi) = 1$:

(I) $\pi(n) \notin \{1, n\}$*, and if* $i = \pi^{-1}(\pi(n)-1)$, $j = \pi^{-1}(\pi(n)+1)$*, then* $\pi(i+1) > \pi(j+1)$*;*
(II) $\pi(n) = 1$ *and* $\pi(n-1) = 2$*; or*
(III) $\pi(n) = n$ *and* $\pi(n-1) = n-1$.

Note that $A(\pi)$ is the set of entries a in the one-line notation of π such that the entry following $a+1$ is smaller than the entry following a. For example, if $\pi = [4,3,6,1,5,2]$, then $A(\pi) = \{3,4,5\}$, so Theorem 2.1 says that $N(\pi) = 1+3+0 = 4$. The following lemma, whose proof is omitted here, will be useful in the proof.

Lemma 2.2 *Suppose that* $\mathrm{Pat}(w, \Sigma, n) = \pi$.

1. *If $1 \leq i, j < n$, $\pi(i) < \pi(j)$, and $\pi(i+1) > \pi(j+1)$, then $w_i < w_j$.*
2. *If $1 \leq i < k \leq n$ are such that $|\pi(i) - \pi(k)| = 1$, then the word $w_{[i,k-1]}$ is primitive.*

We will prove Theorem 2.1 in two parts. First we show that $N(\pi) \geq 1 + |A(\pi)| + \Delta(\pi)$ by proving that if $w \in \mathcal{W}_N$ is such that $\mathrm{Pat}(w, \Sigma, n) = \pi$, then necessarily $N \geq 1 + |A(\pi)| + \Delta(\pi)$. This fact is a consequence of the following lemma.

Lemma 2.3 *Suppose that* $\mathrm{Pat}(w, \Sigma, n) = \pi$*, and let $b = \pi(n)$. The entries of w satisfy*

$$w_{\pi^{-1}(1)} \leq w_{\pi^{-1}(2)} \leq \cdots \leq w_{\pi^{-1}(n)}, \tag{3}$$

with strict inequalities $w_{\pi^{-1}(a)} < w_{\pi^{-1}(a+1)}$ for each $a \in A(\pi)$. Additionally, if $\Delta(\pi) = 1$, then in each of the three cases from Theorem 2.1 we have, respectively, that

(I) one of the inequalities $w_{\pi^{-1}(b-1)} \leq w_n \leq w_{\pi^{-1}(b+1)}$ is strict;
(II) $\cdots \leq w_{n+2} \leq w_{n+1} \leq w_n \leq w_{n-1}$ and one of these inequalities is strict;
(III) $w_{n-1} \leq w_n \leq w_{n+1} \leq w_{n+2} \leq \cdots$ and one of these inequalities is strict.

In all cases, the entries of w must satisfy $|A(\pi)| + \Delta(\pi)$ strict inequalities.

Proof: The condition $\mathrm{Pat}(w, \Sigma, n) = \pi$ is equivalent to

$$w_{[\pi^{-1}(1),\infty)} < w_{[\pi^{-1}(2),\infty)} < \cdots < w_{[\pi^{-1}(n),\infty)}, \tag{4}$$

which clearly implies equation (3). If we remove the term w_n from it, we get

$$\begin{cases}\text{(a)} \quad w_{\pi^{-1}(1)} \le w_{\pi^{-1}(2)} \le \cdots \le w_{\pi^{-1}(b-1)} \le w_{\pi^{-1}(b+1)} \le \cdots \le w_{\pi^{-1}(n)} & \text{if } b \notin \{1,n\}, \\ \text{(b)} \quad w_{\pi^{-1}(2)} \le w_{\pi^{-1}(3)} \le \cdots \le w_{\pi^{-1}(n)} & \text{if } b = 1, \\ \text{(c)} \quad w_{\pi^{-1}(1)} \le w_{\pi^{-1}(2)} \le \cdots \le w_{\pi^{-1}(n-1)} & \text{if } b = n.\end{cases} \tag{5}$$

For every $a \in A(\pi)$, the inequality $w_{\pi^{-1}(a)} < w_{\pi^{-1}(a+1)}$ in (5) has to be strict, by Lemma 2.2 with $i = \pi^{-1}(a)$ and $j = \pi^{-1}(a+1)$. Let us now see that in the three cases when $\Delta(\pi) = 1$, an additional strict inequality must be satisfied.

Consider first case (I). Let $i = \pi^{-1}(b-1)$ and $j = \pi^{-1}(b+1)$. Since $\pi(i+1) > \pi(j+1)$, Lemma 2.2 implies that $w_i < w_j$, so the inequality $w_{\pi^{-1}(b-1)} < w_{\pi^{-1}(b+1)}$ (equivalently, $w_i < w_j$) in (5a) has to be strict. In case (II), the leftmost inequality in (4) is $w_{[n,\infty)} < w_{[n-1,\infty)}$. For this to hold, we need $\cdots \le w_{n+2} \le w_{n+1} \le w_n \le w_{n-1}$ and at least one of these inequalities must be strict. Similarly, in case (III), the rightmost inequality in (4) is $w_{[n-1,\infty)} < w_{[n,\infty)}$. This forces $w_{n-1} \le w_n \le w_{n+1} \le w_{n+2} \le \cdots$ with at least one strict inequality. □

We will refer to the $|A(\pi)| + \Delta(\pi)$ strict inequalities in Lemma 2.3 as the *required strict inequalities*. Combined with the weak inequalities from the lemma, they force the number of symbols used in w to be at least $1 + |A(\pi)| + \Delta(\pi)$. Examples 2 and 3 illustrate how this lemma is used.

Now we show that $N(\pi) \le 1 + |A(\pi)| + \Delta(\pi)$. We will show how for any given $\pi \in \mathcal{S}_n$ one can construct a word $w \in \mathcal{W}_N$ with $\mathrm{Pat}(w, \Sigma, n) = \pi$, where $N = 1 + |A(\pi)| + \Delta(\pi)$. We need w to satisfy condition (4). Again, let $b = \pi(n)$.

The first important observation is that, if we can only use N different symbols, then the $|A(\pi)|+\Delta(\pi) = N-1$ required strict inequalities from Lemma 2.3 determine the values of the entries $w_1 w_2 \ldots w_{n-1}$. This fact is restated as Corollary 2.9. Consequently, we are forced to assign values to these entries as follows:

(a) If $b \notin \{1, n\}$, assign values to the variables in equation (5a) from left to right, starting with $w_{\pi^{-1}(1)} = 0$ and increasing the value by 1 at each required strict inequality.
(b) If $b = 1$, assign values to the variables in equation (5b) from left to right, starting with $w_{\pi^{-1}(2)} = 0$ if $\pi(n-1) \neq 2$, or with $w_{\pi^{-1}(2)} = 1$ if $\pi(n-1) = 2$ (this is needed in order for condition (II) in Lemma 2.3 to hold), and increasing the value by 1 at each required strict inequality.
(c) If $b = n$, assign values to the variables in equation (5c) from left to right, starting with $w_{\pi^{-1}(1)} = 0$ and increasing the value by 1 at each required strict inequality. (Note that when $\Delta(\pi) = 1$, the last assigned value is $w_{\pi^{-1}(n-1)} = w_{n-1} = |A(\pi)| = N-2$.)

It remains to assign the values to w_m for $m \ge n$. Before we do this, let us prove some facts about the entries $w_1 \ldots w_{n-1}$. In the following two lemmas, whose proof can be found in (5), π is any permutation in $\mathcal{S}_n$ with $N(\pi) = N$ and $w_1 \ldots w_{n-1}$ are the values in $\{0, 1, \ldots, N-1\}$ assigned above in order to satisfy the required strict inequalities.

Lemma 2.4 *Let $i < n$. If $\pi(i) > \pi(i+1)$, then $w_i \ge 1$. If $\pi(i) < \pi(i+1)$, then $w_i \le N-2$.*

Lemma 2.5 *If $1 \le i, j < n$ are such that $\pi(i) < \pi(j)$ and $\pi(i+1) > \pi(j+1)$, then $w_i < w_j$.*

Once the values $w_1 \ldots w_{n-1}$ have been determined, there are several ways to assign values to w_m for $m \ge n$. Two possibilities are the following.

A. Assume that $b \neq n$. Let $k = \pi^{-1}(b+1)$. Let $u = w_1w_2 \dots w_{k-1}$ and $p = w_kw_{k+1} \dots w_{n-1}$. Let m be any integer satisfying $m \geq 1 + \frac{n-2}{n-k}$ (for definiteness, we can pick $m = n-1$). Let $w_A(\pi) = up^m0^\infty$.
B. Assume that $b \neq 1$. Let $k = \pi^{-1}(b-1)$. Let $u = w_1w_2 \dots w_{k-1}$ and $p = w_kw_{k+1} \dots w_{n-1}$. Again, let m be such that $m \geq 1 + \frac{n-2}{n-k}$ (for definiteness, we can pick $m = n-1$). Let $w_B(\pi) = up^m(N{-}1)^\infty$.

Clearly, $w_A(\pi)$ and $w_B(\pi)$ use N different symbols. It remains to prove that if w is any of these two words, $\mathrm{Pat}(w, \Sigma, n) = \pi$, which is equivalent to showing that w satisfies condition (4). Let us now prove that this is the case for $w = w_A(\pi)$, when $b \neq n$.

In the following two lemmas (see the proof in (5)) and in Proposition 2.8, π is any permutation in $\mathcal{S}_n$ with $\pi(n) \neq n$, and $w = w_A(\pi)$. Also, k, u, p and m are as defined in case A above.

Lemma 2.6 *The word $p = w_kw_{k+1} \dots w_{n-1}$ is primitive and has some nonzero entry.*

Lemma 2.7 *We have that $w_{[n,\infty)} < w_{[k,\infty)}$. Moreover, there is no $1 \leq s \leq n$ such that $w_{[n,\infty)} < w_{[s,\infty)} < w_{[k,\infty)}$.*

Proposition 2.8 *If $1 \leq i, j \leq n$ are such that $\pi(i) < \pi(j)$, then $w_{[i,\infty)} < w_{[j,\infty)}$.*

The above proposition proves that $\mathrm{Pat}(w_A(\pi), \Sigma, n) = \pi$. If $b \neq 1$, proving that $\mathrm{Pat}(w_B(\pi), \Sigma, n) = \pi$ is analogous. We can complete the proof of the upper bound on $N(\pi)$ as follows. Let $\pi \in \mathcal{S}_n$ be given, and let $N = 1 + |A(\pi)| + \Delta(\pi)$. If $\pi(n-1) > \pi(n)$, let $w = w_A(\pi)$. If $\pi(n-1) < \pi(n)$, let $w = w_B(\pi)$. Since $\mathrm{Pat}(w, \Sigma, n) = \pi$ and $w \in \mathcal{W}_N$, the theorem is proved.

Example 2. Let $\pi = [4, 3, 6, 1, 5, 2]$. By Theorem 2.1, $N(\pi) = 4$. If $\mathrm{Pat}(w, \Sigma, n) = \pi$, then Lemma 2.3 implies that $w_4 \leq w_6 \leq w_2 < w_1 < w_5 < w_3$, and there are no more required strict inequalities. We assign $w_4 = w_2 = 0$, $w_1 = 1$, $w_5 = 2$, $w_3 = 3$. Since $\pi(5) > \pi(6)$ and $b = \pi(6) = 2$, we can take $w = w_A(\pi)$ (with $m = 2$), so $k = \pi^{-1}(3) = 2$, $u = w_1 = 1$, and $p = w_2w_3w_4w_5 = 0302$. We get $w = up^20^\infty = 1030203020^\infty$.

The following consequence of the proof of Theorem 2.1 will be used in Section 4.

Corollary 2.9 *Let $\pi \in \mathcal{S}_n$, $N = N(\pi)$, and let $w \in \mathcal{W}_N$ be such that $\mathrm{Pat}(w, \Sigma, n) = \pi$. Then the entries $w_1w_2 \dots w_{n-1}$ are uniquely determined by π.*

Note that, however, with the conditions of Corollary 2.9, w_n is not always determined. In the case that $\pi(n) \notin \{1, n\}$ and $\Delta(\pi) = 1$, we have two choices for w_n. In general, there is a lot of flexibility in the choice of w_m for $m \geq n$. The choices $w = w_A(\pi)$ and $w = w_B(\pi)$ in the proof of Theorem 2.1 were made to simplify the proof of Proposition 2.8 for all cases at once.

3 An equivalent characterization

We start this section by giving an expression for $N(\pi)$ that is sometimes more convenient to work with than the one in Theorem 2.1. We denote by $\mathcal{C}_n$ the set of permutations in $\mathcal{S}_n$ whose cycle decomposition consists of a unique cycle of length n. Let $\mathcal{T}_n$ be the set of permutations $\pi \in \mathcal{C}_n$ with one distinguished entry $\pi(i)$, for some $1 \leq i \leq n$. We call the elements of $\mathcal{T}_n$ *marked cycles*. We will use the symbol $\star$ to denote the distinguished entry, both in one-line and in cycle notation. Note that it is not necessary to keep track of its value, since it is determined once we know all the remaining entries. For example, $\mathcal{T}_3 = \{[\star, 3, 1], [2, \star, 1], [2, 3, \star], [\star, 1, 2], [3, \star, 2], [3, 1, \star]\}$. Clearly, $|\mathcal{T}_n| = (n-1)! \cdot n = n!$, since there are n ways to choose the distinguished entry.

Define a map $\theta : \mathcal{S}_n \to \mathcal{T}_n$ sending $\pi \mapsto \hat{\pi}$ as follows. For each $1 \le i \le n$ with $i \ne \pi(n)$, let $\hat{\pi}(i)$ be the entry immediately to the right of i in the one-line notation of π. For $i = \pi(n)$, let $\hat{\pi}(i) = \star$ be the distinguished entry.

We can also give the following equivalent definition of $\hat{\pi}$. If $\pi = [\pi(1), \pi(2), \dots, \pi(n)]$, then $\hat{\pi}$ is the permutation with cycle decomposition $(\pi(1), \pi(2), \dots, \pi(n))$ with the entry $\pi(1)$ distinguished. We write $\hat{\pi} = (\star, \pi(2), \dots, \pi(n))$. For example, if $\pi = [8, 9, 2, 3, 6, 4, 1, 5, 7]$, then $\hat{\pi} = (\star, 9, 2, 3, 6, 4, 1, 5, 7)$, or in one-line notation, $\hat{\pi} = [5, 3, 6, 1, 7, 4, \star, 9, 2]$.

The map θ is a bijection between $\mathcal{S}_n$ and $\mathcal{T}_n$, since it is clearly invertible. Indeed, to recover π from $\hat{\pi} \in \mathcal{T}_n$, write $\hat{\pi}$ in cycle notation, replace the $\star$ with the entry in $\{1, \dots, n\}$ that is missing, and turn the parentheses into brackets, thus recovering the one-line notation of π.

For $\hat{\pi} \in \mathcal{T}_n$, let $\operatorname{des}(\hat{\pi})$ denote the number of descents of the sequence that we get by deleting the $\star$ from the one-line notation of $\hat{\pi}$. That is, if $\hat{\pi} = [a_1, \dots, a_j, \star, a_{j+1}, \dots, a_{n-1}]$, then $\operatorname{des}(\hat{\pi}) = |\{i : 1 \le i \le n-2,\ a_i > a_{i+1}\}|$. We can now state a simpler formula for $N(\pi)$.

Proposition 3.1 *Let $\pi \in \mathcal{S}_n$, $\hat{\pi} = \theta(\pi)$. Then $N(\pi)$ is given by*

$$N(\pi) = 1 + \operatorname{des}(\hat{\pi}) + \epsilon(\hat{\pi}),$$

where

$$\epsilon(\hat{\pi}) = \begin{cases} 1 & \text{if } \hat{\pi} = [\star, 1, \dots] \text{ or } \hat{\pi} = [\dots, n, \star], \\ 0 & \text{otherwise.} \end{cases}$$

For example, if $\pi = [8, 9, 2, 3, 6, 4, 1, 5, 7]$, then $\hat{\pi} = [5, 3, 6, 1, 7, 4, \star, 9, 2]$ has 4 descents, so $N(\pi) = 1 + 4 + 0 = 5$. If $\pi = [8, 9, 3, 1, 4, 6, 2, 7, 5]$, then $\hat{\pi} = [4, 7, 1, 6, \star, 2, 5, 9, 3]$ has 3 descents, so $N(\pi) = 1 + 3 + 0 = 4$. If $\pi = [3, 4, 2, 1]$, then $\hat{\pi} = [\star, 1, 4, 2]$ has 1 descent, so $N(\pi) = 1 + 1 + 1 = 3$.

If $\pi \in \mathcal{S}_n$, we have by definition that $N(\pi) = \min\{N : \pi \notin \operatorname{Forb}_n(\Sigma_N)\} = \min\{N : \pi \in \operatorname{Allow}_n(\Sigma_N)\}$. As a consequence of Proposition 3.1 we recover Proposition 1.2, which in terms of the statistic $N(\pi)$ can be reformulated as follows.

Corollary 3.2 *Let $n \ge 3$. We have that*

(a) for every $\pi \in \mathcal{S}_n$, $N(\pi) \le n - 1$;
(b) there is some $\pi \in \mathcal{S}_n$ such that $N(\pi) = n - 1$.

We define $\mathcal{S}_{n,N} = \{\pi \in \mathcal{S}_n : N(\pi) = N\}$. We are interested in the numbers $a_{n,N} = |\mathcal{S}_{n,N}|$. To avoid the trivial cases, we will assume that $n, N \ge 2$. From the definitions, $\operatorname{Allow}_n(\Sigma_M) = \bigcup_{N=2}^{M} \mathcal{S}_{n,N}$, $\operatorname{Forb}_n(\Sigma_M) = \bigcup_{N=M+1}^{n-1} \mathcal{S}_{n,N}$. Since the sets $\mathcal{S}_{n,N}$ are disjoint, we have that

$$|\operatorname{Allow}_n(\Sigma_M)| = \sum_{N=2}^{M} a_{n,N}, \qquad |\operatorname{Forb}_n(\Sigma_M)| = \sum_{N=M+1}^{n-1} a_{n,N}.$$

The first few values of $a_{n,N}$ are given in Table 1. By symmetry considerations (5) it follows easily that all the $a_{n,N}$ are even.

$n\backslash N$	2	3	4	5	6	7
2	2					
3	6					
4	18	6				
5	48	66	6			
6	126	402	186	6		
7	306	2028	2232	468	6	
8	738	8790	19426	10212	1098	6

Tab. 1: The numbers $a_{n,N} = |\{\pi \in \mathcal{S}_n : N(\pi) = N\}|$ for $n \leq 8$.

The next result shows that, independently of n, there are exactly six permutations of length n that require the maximum number of symbols (i.e., $n-1$) in order to be realized. This settles a conjecture from (1). Given a permutation $\pi \in \mathcal{S}_n$, we will use π^{rc} to denote the permutation such that $\pi^{rc}(i) = n+1-\pi(n+1-i)$ for $1 \leq i \leq n$. If σ is a marked cycle, then σ^{rc} is defined similarly, where if $\sigma(i)$ is the marked entry of π, then $\sigma^{rc}(n+1-i)$ is the marked entry of σ^{rc}. It will be convenient to visualize $\pi \in \mathcal{S}_n$ as an $n \times n$ array with dots in positions $(i, \pi(i))$, for $1 \leq i \leq n$. The first coordinate refers to the row number, which increases from left to right, and the second coordinate is the column number, which increases from bottom to top. Then, the array of π^{rc} is obtained from the array of π by a 180-degree rotation. Of course, the array of π^{-1} is obtained from the one of π by reflecting it along the diagonal $y = x$. Notice also that the cycle structure of π is preserved in π^{-1} and in π^{rc}. A marked cycle can be visualized in the same way, replacing the dot corresponding to the distinguished element with a $\star$.

Proposition 3.3 *For every* $n \geq 3$, $a_{n,n-1} = 6$.

Proof: First we show that $a_{n,n-1} \geq 6$ by giving six permutations in $\mathcal{S}_{n,n-1}$. Let $m = \lceil n/2 \rceil$, and let

$$\sigma = [n, n-1, \ldots, m+1, \star, m, m-1, \ldots, 2], \quad \tau = [\star, 1, n, n-1 \ldots, m+2, m, m-1, \ldots, 2] \in \mathcal{T}_n$$

(see Figure 1). Using Proposition 3.1, it is easy to check that if $\hat{\pi} \in \{\sigma, \sigma^{rc}, \sigma^{-1}, (\sigma^{-1})^{rc}, \tau, \tau^{rc}\}$, then $N(\pi) = n-1$, and that the six permutations in the set are different.

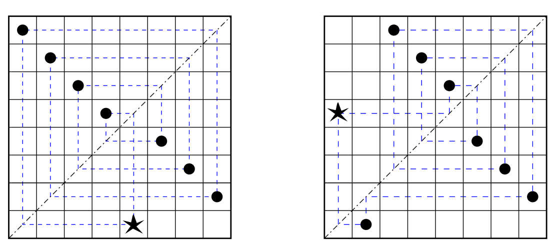

Fig. 1: The arrays of σ and τ for $n = 8$, with dotted lines indicating the cycle structure.

Let us now show that there are no other permutations with $N(\pi) = n-1$. We know by Proposition 3.1 that $N(\pi) = n-1$ can only happen if $\operatorname{des}(\hat{\pi}) = n-2$, or if $\operatorname{des}(\hat{\pi}) = n-3$ and $\epsilon(\hat{\pi}) = 1$.

Case 1: $\mathrm{des}(\hat{\pi}) = n-2$. In this case, all the entries in $\hat{\pi}$ other that the $\star$ must be in decreasing order. If the distinguished entry is neither $\hat{\pi}(1)$ nor $\hat{\pi}(n)$, then the $\star$ must be replacing either 1 or n; otherwise we would have that $\hat{\pi}(1) = n$ and $\hat{\pi}(n) = 1$, so $\hat{\pi}$ would not be an n-cycle. It follows that in the array of $\hat{\pi}$, the entry corresponding to the $\star$ is either in the top or bottom row, or in the leftmost or rightmost column.

If the $\star$ is replacing 1 (i.e, it is is the bottom row of the array), we claim that the only possible n-cycle in which the other entries are in decreasing order is $\hat{\pi} = \sigma$. Indeed, if we consider the cycle structure of $\hat{\pi} = (1, \hat{\pi}(1), \hat{\pi}^2(1), \ldots, \hat{\pi}^{n-1}(1))$, we see that $\hat{\pi}(1) = n$ and $\hat{\pi}^2(1) = \hat{\pi}(n) = 2$. Now, $\hat{\pi}^i(1) \neq 1$ for $3 \leq i \leq n-1$, so the decreasing condition on the remaining entries forces $\hat{\pi}^3(1) = \hat{\pi}(2) = n-1$, $\hat{\pi}^4(1) = \hat{\pi}(n-1) = 3$, and so on. A similar argument, considering that rotating the array 180 degrees preserves the cycle structure, shows that if the $\star$ is replacing n (i.e, it is in the top row of the array), then necessarily $\hat{\pi} = \sigma^{rc}$.

If the distinguished entry is $\hat{\pi}(1)$ (i.e, it is in the leftmost column of the array), then a symmetric argument, reflecting the array along $y = x$), shows that $\hat{\pi} = \sigma^{-1}$. Similarly, if the distinguished entry is $\hat{\pi}(n)$ (i.e, it is is the rightmost column of the array), then necessarily $\hat{\pi} = (\sigma^{-1})^{rc}$.

Case 2: $\mathrm{des}(\hat{\pi}) = n-3$ *and* $\epsilon(\hat{\pi}) = 1$. The second condition forces $\hat{\pi} = [\star, 1, \ldots]$ or $\hat{\pi} = [\ldots, n, \star]$. Let us restrict to the first case (the second one can be argued in a similar way if we rotate the array 180 degrees). We must have $\hat{\pi}(3) > \hat{\pi}(4) > \cdots > \hat{\pi}(n)$. We claim that the only such $\hat{\pi}$ that is also an n-cycle is $\hat{\pi} = \tau$. Indeed, looking at the cycle structure $\hat{\pi} = (\hat{\pi}^{-(n-1)}(1), \ldots, \hat{\pi}^{-1}(1), 1)$, we see that $\hat{\pi}^{-1}(1) = 2$. Now, $\hat{\pi}^{-i}(1) \neq 1$ for $2 \leq i \leq n-1$, so the decreasing condition on the remaining entries forces $\hat{\pi}^{-2}(1) = \hat{\pi}^{-1}(2) = n$, $\hat{\pi}^{-3}(1) = \hat{\pi}^{-1}(n) = 3$, $\hat{\pi}^{-4}(1) = \hat{\pi}^{-1}(3) = n-1$, and so on. □

4 The number of allowed patterns of a shift

In the rest of the paper, we will assume for simplicity that $w_A(\pi)$ and $w_B(\pi)$ are defined taking $m = n-1$, so they are of the form $up^{n-1}x^{\infty}$, with $x = 0$ or $x = N-1$ respectively. The following variation of Lemma 2.7 will be useful later.

Lemma 4.1 *Let* $w = up^{n-1}0^{\infty} \in \mathcal{W}_N$*, where* $|u| = k-1$ *and* $|p| = n-k$ *for some* $1 \leq k \leq n-1$*, and* p *is primitive. If* $\pi = \mathrm{Pat}(w, \Sigma, n)$ *is defined, then* $\pi(n) = \pi(k) - 1$.

For $n \geq 2$, the set of patterns of length n that are realized by the shift on two symbols is $\mathrm{Allow}_n(\Sigma_2) = \mathcal{S}_{n,2}$. The next result gives the number of these permutations. Recall that $a_{n,2} = |\mathcal{S}_{n,2}|$ and that $\psi_2(t)$ is the number of primitive binary words of length t.

Theorem 4.2 *For* $n \geq 2$*,*

$$a_{n,2} = \sum_{t=1}^{n-1} \psi_2(t) 2^{n-t-1}.$$

Proof: Fix $n \geq 2$. We will construct a set $W \subset \mathcal{W}_2$ with the following four properties:

(i) for all $w \in W$, $\mathrm{Pat}(w, \Sigma_2, n)$ is defined,
(ii) for all $w, w' \in W$ with $w \neq w'$, we have that $\mathrm{Pat}(w, \Sigma_2, n) \neq \mathrm{Pat}(w', \Sigma_2, n)$,
(iii) for all $\pi \in \mathrm{Allow}_n(\Sigma_2)$, there is a word $w \in W$ such that $\mathrm{Pat}(w, \Sigma_2, n) = \pi$,
(iv) $|W| = \sum_{t=1}^{n-1} \psi_2(t) 2^{n-t-1}$.

Properties (i)-(iii) imply that the map from W to $\mathcal{S}_{n,2}$ sending w to $\mathrm{Pat}(w,\Sigma_2,n)$ is a bijection. Thus, $a_{n,2} = |W|$ and the result will follow from property (iv).

Let

$$W = \bigcup_{t=1}^{n-1} \{up^{n-1}x^{\infty} : u \in \{0,1\}^{n-t-1}, p \in \{0,1\}^t \text{ is a primitive word, and } x = \bar{p_t}\},$$

where we use the notation $\bar{0} = 1$, $\bar{1} = 0$. Given binary words u, p of lengths $n-t-1$ and t respectively, where p is primitive, and $x = \bar{p_t}$, we will denote $v(u,p) = up^{n-1}x^{\infty}$.

To see that W satisfies (i), we have to show that for any $w \in W$ and any $1 \le i < j \le n$, we have $w_{[i,\infty)} \neq w_{[j,\infty)}$. This is clear because if $x = 0$ (resp. $x = 1$) both $w_{[i,\infty)}$ and $w_{[j,\infty)}$ end with 10^{∞} (resp. 01^{∞}), with the last 1 (resp. 0) being in different positions in $w_{[i,\infty)}$ and $w_{[j,\infty)}$.

Now we prove that W satisfies (ii). Let u, u' be binary words of lengths $n-t-1$, $n-t'-1$, respectively, and let p, p' be primitive binary words of lengths t, t', respectively. Let $w = v(u,p)$ and $w' = v(u',p')$, and let $\pi = \mathrm{Pat}(w,\Sigma_2,n)$, $\pi' = \mathrm{Pat}(w',\Sigma_2,n)$. We assume that $w \neq w'$, and want to show that $\pi \neq \pi'$. From $w \neq w'$ it follows that $u \neq u'$ or $p \neq p'$.

Corollary 2.9 for $N = 2$ implies that if $w_1w_2\dots w_{n-1} \neq w'_1w'_2\dots w'_{n-1}$, then $\mathrm{Pat}(w,\Sigma_2,n) \neq \mathrm{Pat}(w',\Sigma_2,n)$. In particular, if $t = t'$, then $up \neq u'p'$, so $\pi \neq \pi'$.

We are left with the case that $t \neq t'$and $up = u'p' = w_1w_2\dots w_{n-1}$. Let us first assume that $w_{n-1} = 1$ (and so $p_t = p'_{t'} = 1$). By Lemma 4.1 with $k = n-t$, we have that $\pi(n) = \pi(n-t) - 1$, and similarly $\pi'(n) = \pi'(n-t') - 1$. If we had that $\pi = \pi'$, then $\pi(n) = \pi'(n)$ and so $\pi(n-t) = \pi'(n-t') = \pi(n-t')$. But $t \neq t'$, so this is a contradiction. In the case $w_{n-1} = 0$, an analogous argument to the proof of Lemma 4.1 implies that $w_{[n-t,\infty)} = p^{n-1}1^{\infty} < p^{n-2}1^{\infty} = w_{[n,\infty)}$ and there is no s such that $w_{[s,\infty)}$ is strictly in between the two. Thus, $\pi(n) = \pi(n-t) + 1$, and similarly $\pi'(n) = \pi'(n-t') + 1$, so again $\pi \neq \pi'$.

To see that W satisfies (iii) we use the construction from the proof of the upper bound in Theorem 2.1. Let $\pi \in \mathrm{Allow}_n(\Sigma_2)$. If $\pi(n-1) > \pi(n)$, let $w = w_A(\pi) = up^{n-1}0^{\infty}$. By Lemma 2.4, $w_{n-1} = 1$, so $w \in W$. Similarly, if $\pi(n-1) < \pi(n)$, let $w = w_B(\pi) = up^{n-1}1^{\infty}$. Then $w_{n-1} = 0$, so $w \in W$. In both cases, $\mathrm{Pat}(w,\Sigma_2,n) = \pi$, so this construction is the inverse of the map $w \mapsto \mathrm{Pat}(w,\Sigma_2,n)$.

To prove (iv), observe that the union in the definition of W is a disjoint union. This is because the value of t determines the position of the last entry in w that is not equal to x. For fixed t, there are 2^{n-t-1} choices for u and $\psi_2(t)$ choices for t, so the formula follows. □

Example 3. For $n = 4$, we have

$$\begin{aligned} W = \quad & \{\underline{00}\,0\,0\,0\,1^{\infty}, \underline{00}\,1\,1\,1\,0^{\infty}, \underline{01}\,0\,0\,0\,1^{\infty}, \underline{01}\,1\,1\,1\,0^{\infty}, \underline{10}\,0\,0\,0\,1^{\infty}, \underline{10}\,1\,1\,1\,0^{\infty}, \underline{11}\,0\,0\,0\,1^{\infty}, \underline{11}\,1\,1\,1\,0^{\infty}, \\ & \underline{0}\,01\,01\,01\,0^{\infty}, \underline{0}\,10\,10\,10\,1^{\infty}, \underline{1}\,01\,01\,01\,0^{\infty}, \underline{1}\,10\,10\,10\,1^{\infty}, \\ & 001\,001\,001\,0^{\infty}, 010\,010\,010\,1^{\infty}, 011\,011\,011\,0^{\infty}, 100\,100\,100\,1^{\infty}, 101\,101\,101\,0^{\infty}, 110\,110\,110\,1^{\infty}\}, \end{aligned}$$

where each word is written as $w = \underline{u}\,p\,p\,p\,x^{\infty}$. The permutations corresponding to these words are

$$\begin{aligned} \mathrm{Allow}_4(\Sigma_2) = \quad & \{1234, 1243, 3412, 1432, 4123, 2143, 4312, 4321, \\ & 1342, 1324, 4231, 4213, \\ & 2341, 2413, 2431, 3124, 3142, 3214\}. \end{aligned}$$

Theorem 4.2 can be generalized to find a formula for the numbers $a_{n,N}$, which count permutations that can be realized by the shift on N symbols but not by the shift on $N-1$ symbols. The proof of the next result is more involved and is omitted here due to lack of space, but it can be found in (5).

Theorem 4.3 *For any $n, N \geq 2$,*

$$a_{n,N} = \sum_{i=0}^{N-2} (-1)^i \binom{n}{i} \left((N-i-2)(N-i)^{n-2} + \sum_{t=1}^{n-1} \psi_{N-i}(t)(N-i)^{n-t-1} \right). \tag{6}$$

We finish with two curious conjectures that came up while studying forbidden patterns of shift systems. They are derived from experimental evidence, and it would be interesting to find combinatorial proofs.

For the first one, let $\mathcal{T}_n^0$ be the set of n-cycles where one entry has been replaced with 0. The set $\mathcal{T}_n^0$ is essentially the same as $\mathcal{T}_n$, with the only difference that the $\star$ symbol in each element is replaced with a 0, so that it produces a descent if there is an entry to its left. We have checked this conjecture by computer for n up to 9.

Conjecture 4.4 *For any n and any subset $D \subseteq \{1, 2, \ldots, n-1\}$,*

$$|\{\sigma \in \mathcal{T}_n^0 \,:\, D(\sigma) = D\}| = |\{\pi \in \mathcal{S}_n \,:\, D(\pi) = D\}|.$$

In particular, the statistic des *has the same distribution in $\mathcal{T}_n^0$ as in $\mathcal{S}_n$, i.e,*

$$\sum_{\sigma \in \mathcal{T}_n^0} x^{\mathrm{des}(\sigma)+1} = A_n(x).$$

Our last conjecture concerns a divisibility property of the numbers $a_{n,N}$ which is not apparent from Theorem 4.3.

Conjecture 4.5 *For every $n, N \geq 3$, $a_{n,N}$ is divisible by 6.*

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