The maximum of Brownian motion with parabolic drift (Extended abstract)

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We study the maximum of a Brownian motion with a parabolic drift; this is a random variable that often occurs as a limit of the maximum of discrete processes whose expectations have a maximum at an interior point. This has some applications in algorithmic and data structures analysis. We give series expansions and integral formulas for the distribution and the first two moments, together with numerical values to high precision.

Keywords: Brownian motion, parabolic drift, Airy functions.

1 Introduction

Let $W(t)$ be a two-sided Brownian motion with $W(0) = 0$; i.e., $(W(t))_{t\geq 0}$ and $(W(-t))_{t\geq 0}$ are two independent standard Brownian motions. We are interested in the process

$$W_\gamma(t) := W(t) - \gamma t^2 \tag{1.1}$$

for a given $\gamma > 0$, and in particular in its maximum

$$M_\gamma := \max_{-\infty<t<\infty} W_\gamma(t) = \max_{-\infty<t<\infty} \bigl(W(t) - \gamma t^2\bigr). \tag{1.2}$$

We also consider the corresponding one-sided maximum

$$N_\gamma := \max_{0\leq t<\infty} W_\gamma(t) = \max_{0\leq t<\infty} \bigl(W(t) - \gamma t^2\bigr). \tag{1.3}$$

Since the restrictions of W to the positive and negative half-axes are independent, we have the relation

$$M_\gamma \overset{\mathrm{d}}{=} \max(N_\gamma, N'_\gamma) \tag{1.4}$$

where N'_γ is an independent copy of N_γ.

Note that (a.s.) $W_\gamma(t) \to -\infty$ as $t \to \pm\infty$, so the maxima in (1.2) and (1.3) exist and are finite; moreover, they are attained at unique points and $M_\gamma, N_\gamma > 0$. It is easily seen (e.g., by Cameron–Martin) that M_γ and N_γ have absolutely continuous distributions.

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1.1 Scaling

For any $a > 0$, $W(at) \overset{\mathrm{d}}{=} a^{1/2}W(t)$ (as processes on $(-\infty, \infty)$), and thus

$$M_\gamma = \max_{-\infty<t<\infty} \bigl(W(at) - \gamma(at)^2\bigr) \overset{\mathrm{d}}{=} \max_{-\infty<t<\infty} \bigl(a^{1/2}W(t) - a^2\gamma t^2\bigr) = a^{1/2}M_{a^{3/2}\gamma}, \tag{1.5}$$

and similarly

$$N_\gamma \overset{\mathrm{d}}{=} a^{1/2}N_{a^{3/2}\gamma}. \tag{1.6}$$

The parameter γ is thus just a scale parameter, and it suffices to consider a single choice of γ. We choose the normalization $\gamma = 1/2$ as the standard case, and write $M := M_{1/2}$, $N := N_{1/2}$. In general, (1.5)–(1.6) with $a = (2\gamma)^{-2/3}$ yield

$$M_\gamma \overset{\mathrm{d}}{=} (2\gamma)^{-1/3}M, \qquad N_\gamma \overset{\mathrm{d}}{=} (2\gamma)^{-1/3}N. \tag{1.7}$$

Remark 1.1 *More generally, if $a, b > 0$, then*

$$\max_{-\infty<t<\infty} \bigl(aW(t) - bt^2\bigr) = aM_{b/a} \overset{\mathrm{d}}{=} \Bigl(\frac{a^4}{2b}\Bigr)^{1/3} M. \tag{1.8}$$

The purpose of this extended abstract is to provide formulas for the distribution function of M and, in particular, its moments. Detailed proofs (based on delicate complex analysis) and many complementary results can be found in the full version of this paper, [13].

1.2 Background

The random variable M is studied by Barbour [4], Daniels and Skyrme [8] and Groeneboom [9] (using the Cameron–Martin–Girsanov formula). (Further results on N are given by Lachal [14].) M arises as a natural limit distribution in many different problems, and in many related problems its expectation $\mathbb{E}\, M$ enters in a second order term for the asymptotics of means or in improved normal approximations. For various examples and general results, see for example Daniels [6; 7], Daniels and Skyrme [8], Barbour [4; 5], Smith [19], Louchard, Kenyon and Schott [17], Steinsaltz [20], Janson [12]. As discussed in several of these papers, the appearance of M in these limit results can be explained as follows, ignoring technical conditions: Consider the maximum over time t of a random process $X_n(t)$, defined on a compact interval I, for example $[0, 1]$, such that as $n \to \infty$, the mean $\mathbb{E}\, X_n(t)$, after scaling, converges to deterministic function $f(t)$, and that the fluctuations $X_n(t) - \mathbb{E}\, X_n(t)$ are of smaller order and, after a different scaling, converge to a gaussian process $G(t)$. If we assume that f is continuous on I and has a unique maximum at a point $t_0 \in I$, then the maximum of the process $X_n(t)$ is attained close to t_0. Assuming that t_0 is an interior point of I and that f is twice differentiable at t_0 with $f''(t_0) \neq 0$, we can locally at t_0 approximate f by a parabola and $G(t) - G(t_0)$ by a two-sided Brownian motion (with some scaling), and thus $\max_t X_n(t) - X_n(t_0)$ is approximated by a scaling constant times the variable M. In the typical case where the mean of $X_n(t)$ is of order n and the Gaussian fluctuations are of order $n^{1/2}$, it is easily seen that the correct scaling is that $n^{-1/3}(\max_t X_n(t) - X_n(t_0)) \xrightarrow{\mathrm{d}} cM$, for some $c > 0$, which for the mean gives $\mathbb{E} \max_t X_n(t) = nf(t_0) + n^{1/3}c\,\mathbb{E}\, M + o(n^{1/3})$, see [4; 6; 7]. As examples of applications in algorithmic and data structures analysis, this type of asymptotics appears in the analysis of linear lists, priority queues and dictionaries [16; 17] and in a sorting algorithm [12].

2 Main results

The mean of M can be expressed as integrals involving the Airy functions Ai and Bi, for example as follows. (For general definitions and properties of Airy functions, see [1, Section 10.4]. We remind the reader that $\mathrm{Ai}(t)$ and $\mathrm{Bi}(t)$ are linearly independent solutions of $f''(t) = tf(t)$, and that $\mathrm{Ai}(t) \to 0$ as $t \to +\infty$.)

Theorem 2.1 (Daniels and Skyrme [8])

$$\mathbb{E}\, M = -\frac{2^{-1/3}}{2\pi i}\int_{-i\infty}^{i\infty}\frac{z\,\mathrm{d}z}{\mathrm{Ai}(z)^2} = \frac{2^{-1/3}}{2\pi i}\int_{-\infty}^{\infty}\frac{y\,\mathrm{d}y}{\mathrm{Ai}(iy)^2} \tag{2.1}$$

$$= 2^{2/3}\int_0^\infty \frac{\mathrm{Ai}(t)^2+\sqrt{3}\mathrm{Ai}(t)\mathrm{Bi}(t)}{\mathrm{Ai}(t)^2+\mathrm{Bi}(t)^2}\,\mathrm{d}t \tag{2.2}$$

$$= 2^{2/3}\,\mathrm{Re}\left(\big(1+i\sqrt{3}\big)\int_0^\infty \frac{\mathrm{Ai}(t)}{\mathrm{Ai}(t)+i\mathrm{Bi}(t)}\,\mathrm{d}t\right) \tag{2.3}$$

$$= \frac{2^{2/3}}{\pi}\int_0^\infty \frac{\sqrt{3}\mathrm{Bi}(t)^2-\sqrt{3}\mathrm{Ai}(t)^2+2\mathrm{Ai}(t)\mathrm{Bi}(t)}{\big(\mathrm{Ai}(t)^2+\mathrm{Bi}(t)^2\big)^2}\,t\,\mathrm{d}t. \tag{2.4}$$

The expressions (2.1) and (2.2) (unfortunately with typos in the latter) are given by Daniels and Skyrme [8]. Since detailed proofs of the formulas are not given there, we for completeness give a complete proof in the full paper. (The proof includes a direct analytical verification of the equivalence of (2.1) and (2.2), which was left open in [8].)

$|\mathrm{Ai}(iy)|$ increases superexponentially as $y \to \pm\infty$, while $\mathrm{Ai}(t)$ decreases superexponentially and $\mathrm{Bi}(t)$ increases superexponentially as $t \to \infty$; hence, the integrands in the integrals in Theorem 2.1 all decrease superexponentially and the integrals converge rapidly, so they are suited for numerical calculations. We obtain by numerical integration (using `Maple`), improving the numerical values in [4; 5; 8; 7],

$$\mathbb{E}\, M = 0.99619\,30199\,28363\,11660\,37766\ldots \tag{2.5}$$

We do not know any similar integral formulas for the second moment of M (or higher moments). Instead we give expressions using infinite series, summing over the zeros a_k, $k \ge 1$, of the Airy function.

We first introduce more notation. Let $F_N(x)$ be the distribution function of N, i.e., $F_N(x) := \mathbb{P}(N \le x)$, and let $F_M(x)$ be the distribution function of M; further, let $G_N(x) = 1 - F_N(x) = \mathbb{P}(N > x)$ and $G_M(x) = 1 - F_M(x) = \mathbb{P}(M > x)$ be the corresponding tail probabilities. Then, by (1.4),

$$F_M(x) := \mathbb{P}(M \le x) = \mathbb{P}(N \le x)^2 = F_N(x)^2. \tag{2.6}$$

and, equivalently,

$$G_M(x) := 1 - (1 - G_N(x))^2 = 2G_N(x) - G_N(x)^2. \tag{2.7}$$

If we know $G_N(x)$, we thus know the distribution of both N and M, and we can compute moments by

$$\mathbb{E}\, N^p = p\int_0^\infty x^{p-1}G_N(x)\,\mathrm{d}x, \tag{2.8}$$

$$\mathbb{E}\, M^p = p\int_0^\infty x^{p-1}G_M(x)\,\mathrm{d}x = p\int_0^\infty x^{p-1}\big(2G_N(x)-G_N(x)^2\big)\,\mathrm{d}x. \tag{2.9}$$

Two formulas for the distribution function are given in the following theorem. (For definition of the function Hi, related to the Airy functions, see [1, (10.4.44)].)

Theorem 2.2 *The distribution functions of M and N are $F_M(x) = (1-G_N(x))^2$ and $F_N(x) = 1 - G_N(x)$, where*

$$G_N(x) = \pi\sum_{k=1}^\infty \frac{\mathrm{Hi}(a_k)}{\mathrm{Ai}'(a_k)}\mathrm{Ai}(a_k + 2^{1/3}x), \qquad x > 0. \tag{2.10}$$

The sum converges conditionally but not absolutely for every $x > 0$. Alternatively, with an absolutely convergent sum, for $x \geq 0$,

$$G_N(x) = \frac{\mathrm{Ai}(2^{1/3}x)}{\mathrm{Ai}(0)} + \sum_{k=1}^{\infty} \frac{\pi \mathrm{Hi}(a_k) + a_k^{-1}}{\mathrm{Ai}'(a_k)} \mathrm{Ai}(a_k + 2^{1/3}x). \tag{2.11}$$

The sum in (2.11) can be differentiated termwise and we have the following result.

Theorem 2.3 *N and M have absolutely continuous distributions with infinitely differentiable density functions, for $x > 0$,*

$$f_N(x) = -2^{1/3} \frac{\mathrm{Ai}'(2^{1/3}x)}{\mathrm{Ai}(0)} - 2^{1/3} \sum_{k=1}^{\infty} \frac{\pi \mathrm{Hi}(a_k) + a_k^{-1}}{\mathrm{Ai}'(a_k)} \mathrm{Ai}'(a_k + 2^{1/3}x), \tag{2.12}$$

$$f_M(x) = 2(1 - G_N(x)) f_N(x). \tag{2.13}$$

Integral formulas for $f_N(x)$ are given in the full report.

Remark 2.4 *In contrast, the sum*

$$2^{1/3}\pi \sum_{k=1}^{\infty} \frac{\mathrm{Hi}(a_k)}{\mathrm{Ai}'(a_k)} \mathrm{Ai}'(a_k + 2^{1/3}x) \tag{2.14}$$

obtained by termwise differentiation of (2.10) *is* not *convergent for any $x \geq 0$.*

Moments of M and N now can be obtained from (2.8) and (2.9) by integrating (2.11) termwise. We define for convenience

$$\varphi(k) := \pi \mathrm{Hi}(a_k) + a_k^{-1}, \qquad \varphi(k) = O(|a_k|^{-4}) = O(k^{-8/3}). \tag{2.15}$$

Recall also the function $\mathrm{Gi}(z) = \mathrm{Bi}(z) - \mathrm{Hi}(z)$, see [1, (10.4.42)–(10.4.46)].

Theorem 2.5 *The means and second moments of M and N are given by the absolutely convergent sums*

$$\mathbb{E}\, N = \frac{1}{2^{1/3} 3 \mathrm{Ai}(0)} - \frac{\pi}{2^{1/3}} \sum_{k=1}^{\infty} \varphi(k) \mathrm{Gi}(a_k) \tag{2.16}$$

$$= \frac{1}{2^{1/3} 3 \mathrm{Ai}(0)} - \frac{\pi}{2^{1/3}} \sum_{k=1}^{\infty} [\varphi(k)\mathrm{Bi}(a_k) - \varphi(k)\mathrm{Hi}(a_k)], \tag{2.17}$$

$$\mathbb{E}\, M = \frac{2^{2/3}}{3\mathrm{Ai}(0)} - \frac{\mathrm{Ai}'(0)^2}{2^{1/3}\mathrm{Ai}(0)^2} - \frac{1}{2^{1/3}} \sum_{k=1}^{\infty} \left[2\pi\varphi(k)\mathrm{Bi}(a_k) - \varphi(k)^2\right], \tag{2.18}$$

$$\mathbb{E}\, N^2 = -\frac{2^{1/3}\mathrm{Ai}'(0)}{\mathrm{Ai}(0)} + 2^{1/3} \sum_{k=1}^{\infty} \varphi(k) \big[\pi a_k \mathrm{Gi}(a_k) - 1\big] \tag{2.19}$$

$$= -\frac{2^{1/3}\mathrm{Ai}'(0)}{\mathrm{Ai}(0)} + 2^{1/3} \sum_{k=1}^{\infty} \big[\pi a_k \varphi(k) \mathrm{Bi}(a_k) - a_k \varphi(k)^2\big]; \tag{2.20}$$

$$\mathbb{E}\, M^2 = -\frac{2^{1/3} 5\, \mathrm{Ai}'(0)}{3\mathrm{Ai}(0)} + 2^{4/3} \sum_{k=1}^{\infty} \varphi(k) \left[\pi a_k \mathrm{Bi}(a_k) - \frac{2}{3} a_k \varphi(k) + \frac{2}{a_k^3} + \frac{2\mathrm{Ai}'(0)}{\mathrm{Ai}(0) a_k^2}\right]$$

$$+ 2^{7/3} \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} \frac{\varphi(k)\varphi(j)}{(a_k - a_j)^2}. \tag{2.21}$$

Numerically we have

$$\mathbb{E}(N) = 0.6955289995\dots, \qquad \mathbb{E}(M) = 0.9961930199\dots,$$
$$\mathbb{E}(N^2) = 1.1027982645\dots, \qquad \mathbb{E}(M^2) = 1.8032957042\dots,$$

and thus

$$\operatorname{Var}(N) = \mathbb{E}(N^2) - \mathbb{E}(N)^2 = 0.6190376754\dots$$
$$\operatorname{Var}(M) = \mathbb{E}(M^2) - \mathbb{E}(M)^2 = 0.8108951713\dots.$$

The numerical value for $\mathbb{E}\, M$ agrees with the one in (2.5).

3 Distributions

Salminen [18, Example 3.2] studied the hitting time

$$\tau := \inf\{t \geq 0 : x + W(t) = -\beta t^2\}, \tag{3.1}$$

and gave the formula [18, (3.10)], for $x, \beta > 0$, (with $\alpha = -\beta$ in his notation),

$$f_\tau(t) = 2^{1/3}\beta^{2/3} \sum_{k=1}^{\infty} \exp\bigl(2^{1/3}\beta^{2/3} a_k t - \tfrac{2}{3}\beta^2 t^3\bigr) \frac{\operatorname{Ai}(a_k + 2^{2/3}\beta^{1/3} x)}{\operatorname{Ai}'(a_k)} \tag{3.2}$$

for the density function of τ. Note that τ is a defect random variable, and that $\tau = \infty$ if and only if $\min_{t\geq 0}(x + W(t) + \beta t^2) > 0$. By symmetry, $W \overset{\mathrm{d}}{=} -W$, and thus

$$\mathbb{P}(\tau = \infty) = \mathbb{P}\bigl(\max_{t\geq 0}(W(t) - \beta t^2 - x) < 0\bigr) = \mathbb{P}(N_\beta < x).$$

Hence, choosing $\beta = 1/2$,

$$\begin{aligned} G_N(x) = 1 - F_N(x) = \mathbb{P}(N \geq x) = \mathbb{P}(\tau < \infty) &= \int_0^\infty f_\tau(t)\,\mathrm{d}t \\ &= \int_0^\infty 2^{-1/3} \sum_{k=1}^{\infty} \frac{\operatorname{Ai}(a_k + 2^{1/3}x)}{\operatorname{Ai}'(a_k)} \exp\bigl(2^{-1/3} a_k t - \tfrac{1}{6}t^3\bigr)\,\mathrm{d}t \\ &= \int_0^\infty \sum_{k=1}^{\infty} \frac{\operatorname{Ai}(a_k + 2^{1/3}x)}{\operatorname{Ai}'(a_k)} \exp\bigl(a_k t - \tfrac{1}{3}t^3\bigr)\,\mathrm{d}t. \end{aligned} \tag{3.3}$$

If we formally integrate termwise we obtain (2.10). However, the sum is not absolutely convergent, so we cannot use e.g. Fubini's theorem, and we have to justify the termwise integration by a more complicated argument, given in the full paper. This argument also gives (2.11) and (2.12). The result for M and (2.13) follow from $F_M(x) = F_N(x)^2 = (1 - G_N(x))^2$.

4 Proof of Theorem 2.5 and further moment formulas

Let us define

$$c(0) := \frac{1}{\mathrm{Ai}(0)}, \qquad c(k) := \frac{\pi \mathrm{Hi}(a_k) + a_k^{-1}}{\mathrm{Ai}'(a_k)} = \frac{\varphi(k)}{\mathrm{Ai}'(a_k)}, \qquad k \geq 1.$$

Then

$$\mathbb{E}\, N = \int_0^\infty G_N(x)\,\mathrm{d}x = 2^{-1/3} \int_0^\infty G_N(2^{-1/3}x)\,\mathrm{d}x = 2^{-1/3} \sum_{k=0}^\infty c(k)\mathrm{AI}(a_k),$$

where, using [1, 10.4.82 and 10.4.47],

$$\mathrm{AI}(z) := \int_z^{+\infty} \mathrm{Ai}(t)\,\mathrm{d}t = \frac{1}{3} - \int_0^z \mathrm{Ai}(t)\,\mathrm{d}t = \pi\big(\mathrm{Ai}(z)\mathrm{Gi}'(z) - \mathrm{Ai}'(z)\mathrm{Gi}(z)\big). \tag{4.1}$$

We have $\mathrm{AI}(0) = \int_0^\infty \mathrm{Ai}(x)\,\mathrm{d}x = 1/3$ [1, 10.4.82], see (4.1), and, for $k \geq 1$, $\mathrm{AI}(a_k) = -\pi \mathrm{Ai}'(a_k)\mathrm{Gi}(a_k)$ since $\mathrm{Ai}(a_k) = 0$. Thus (2.16) follows, and so does (2.17).

Further,

$$\begin{aligned}
&\int_0^\infty G_N(x)^2\,\mathrm{d}x = 2^{-1/3} \int_0^\infty G_N(2^{-1/3}x)^2\,\mathrm{d}x \\
&= 2^{-1/3} \sum_{k=0}^\infty \sum_{\ell=0}^\infty c(k)c(\ell) \int_0^\infty \mathrm{Ai}(x+a_k)\mathrm{Ai}(x+a_\ell)\,\mathrm{d}x \\
&= 2^{-1/3} c(0)^2 (\mathrm{Ai}'(0))^2 - 2^{-1/3} 2 \sum_{k=1}^\infty c(0)c(k) \frac{\mathrm{Ai}(0)\mathrm{Ai}'(a_k)}{a_k} + 2^{-1/3} \sum_{k=1}^\infty c(k)^2 (\mathrm{Ai}'(a_k))^2 \\
&= 2^{-1/3} \left(\frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)}\right)^2 - 2^{2/3} \sum_{k=1}^\infty \frac{\varphi(k)}{a_k} + 2^{-1/3} \sum_{k=1}^\infty \varphi(k)^2.
\end{aligned}$$

Thus, by (2.9) and (2.17),

$$\begin{aligned}
\mathbb{E}\, M &= 2 \int_0^\infty G_N(x)\,\mathrm{d}x - \int_0^\infty G_N(x)^2\,\mathrm{d}x = 2\,\mathbb{E}\, N - \int_0^\infty G_N(x)^2\,\mathrm{d}x \\
&= \frac{2^{2/3}}{3\mathrm{Ai}(0)} - 2^{-1/3} \left(\frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)}\right)^2 - \frac{1}{2^{1/3}} \sum_{k=1}^\infty \varphi(k) \left[2\pi \mathrm{Bi}(a_k) - 2\pi \mathrm{Hi}(a_k) - \frac{2}{a_k} + \varphi(k)\right],
\end{aligned}$$

and (2.18) follows by the definition of $\varphi(k)$.

For the second moments we similarly compute

$$\begin{aligned}
&\int_0^\infty G_N(x)x\,\mathrm{d}x = 2^{-2/3} \int_0^\infty G_N(2^{-1/3}x)x\,\mathrm{d}x = 2^{-2/3} \sum_{k=0}^\infty \int_0^\infty c(k)\mathrm{Ai}(x+a_k)x\,\mathrm{d}x \\
&= 2^{-2/3} \sum_{k=0}^\infty c(k)\big(-\mathrm{Ai}'(a_k) - a_k \mathrm{AI}(a_k)\big) = -\frac{\mathrm{Ai}'(0)}{2^{2/3}\mathrm{Ai}(0)} + 2^{-2/3} \sum_{k=1}^\infty \varphi(k)\big(-1 + \pi a_k \mathrm{Gi}(a_k)\big); \\
&\int_0^\infty G_N(x)^2 x\,\mathrm{d}x = 2^{-2/3} \int_0^\infty G_N(2^{-1/3}x)^2 x\,\mathrm{d}x
\end{aligned}$$

$$= 2^{-2/3} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c(k)c(j) \int_0^{\infty} x\mathrm{Ai}(x+a_k)\mathrm{Ai}(x+a_j)\,\mathrm{d}x$$

$$= -\frac{\mathrm{Ai}'(0)}{2^{2/3}3\mathrm{Ai}(0)} + \frac{2}{2^{2/3}} \sum_{k=1}^{\infty} \varphi(k) \left[-\frac{2}{a_k^3} - \frac{2\mathrm{Ai}'(0)}{\mathrm{Ai}(0)a_k^2} \right]$$

$$+ \frac{1}{2^{2/3}} \left[-\sum_{k=1}^{\infty} \varphi(k)^2 \frac{2}{3} a_k - 2 \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} [\![k \neq j]\!] \frac{\varphi(k)\varphi(j)}{(a_k - a_j)^2} \right].$$

By (2.8), $\mathbb{E}\, N^2 = 2\int_0^\infty G_N(x)x\,\mathrm{d}x$, and (2.19)–(2.20) follow. Similarly, $\mathbb{E}\, M^2 = 4\int_0^\infty G_N(x)x\,\mathrm{d}x - 2\int_0^\infty G_N(x)^2 x\,\mathrm{d}x = 2\,\mathbb{E}\, N^2 - 2\int_0^\infty G_N(x)^2 x\,\mathrm{d}x$, and (2.21) follows. The numerical evaluation is done by `Maple`, using the method discussed in Section 5.

Remark 4.1 *Higher moments can be computed by the same method, with Airy integrals evaluated as shown in Appendix A, but in order to get convergence, one may have to use a version of* (2.11) *with more terms taken out of the expansion of* Hi*.*

We also obtain the following alternative formulas.

Theorem 4.2

$$\mathbb{E}\, N = 2^{-1/3}\pi^2 \sum_{k=1}^{\infty} \mathrm{Hi}(a_k)(\mathrm{Hi}(a_k) - \mathrm{Bi}(a_k)), \tag{4.2}$$

$$\mathbb{E}\, M = 2^{-1/3}\pi^2 \sum_{k=1}^{\infty} \mathrm{Hi}(a_k)(\mathrm{Hi}(a_k) - 2\mathrm{Bi}(a_k)). \tag{4.3}$$

These formulas are closely related to (2.17)–(2.18). They are simpler, but less suitable for numerical calculations since they do not even converge absolutely. (However, they alternate in sign, and the sums converge.) The formulas (4.2) and (4.3) are what we obtain if we substitute (2.10) in (2.8) and (2.9) (with $p = 1$) and integrate termwise; however, since the resulting sums are not absolutely convergent, termwise integration has to be justified carefully, and we use a detour via complex integration.

We can also give integral formulas, using the contour Γ is defined as follows: Let a_1 be the first zero of $\mathrm{Ai}(x)$. Fix $\theta_0 \in (0, \pi/2)$ and $x_0 \in (a_1, 0)$, and let $\Gamma = \Gamma(\theta_0, x_0)$ be the contour consisting of the ray $\{re^{i(\pi+\theta_0)}\}$ for r from ∞ to $r_0 := |x_0|/\cos\theta_0$, the line segment $\{x_0 + iy\}$ for $y \in [-r_0\sin\theta_0, r_0\sin\theta_0]$ and the ray $\{re^{i(\pi-\theta_0)}\}$ for $r \in (r_0, \infty)$.

Theorem 4.3 *The moments of M and N are given by, for any real $p > 0$,*

$$\mathbb{E}\, N^p = -p2^{-p/3-1} i \int_\Gamma \int_0^\infty x^{p-1}\mathrm{Ai}(z+x)\,\mathrm{d}x\, \frac{\mathrm{Hi}(z)}{\mathrm{Ai}(z)}\,\mathrm{d}z,$$

$$\mathbb{E}\, M^p = -p2^{-p/3} i \int_\Gamma \int_0^\infty x^{p-1}\mathrm{Ai}(z+x)\,\mathrm{d}x\, \frac{\mathrm{Hi}(z)}{\mathrm{Ai}(z)}\,\mathrm{d}z$$

$$+ p2^{-p/3-2} \int_\Gamma \int_\Gamma \int_0^\infty x^{p-1}\mathrm{Ai}(z+x)\mathrm{Ai}(w+x)\,\mathrm{d}x\, \frac{\mathrm{Hi}(z)}{\mathrm{Ai}(z)} \frac{\mathrm{Hi}(w)}{\mathrm{Ai}(w)}\,\mathrm{d}z\,\mathrm{d}w.$$

5 Numerical computation

The sums in Theorem 2.5 converge, but rather slowly. For example, in (2.16), $|\varphi(k)\mathrm{Gi}(k)| = |\varphi(k)|\,|\mathrm{Bi}(k) - \mathrm{Hi}(k)| \sim ck^{-17/6}$ for some $c > 0$, see the asymptotic expansions below.

To obtain numerical values with high accuracy of the sums in (2.16)–(2.21), we therefore use asymptotic expansions of the summands. More precisely, for each sum, we first compute $\sum_{k=1}^{199}$ numerically. (All computations are done by `Maple`.) We then evaluate $\sum_{200}^{\infty}$ by replacing the summands by the first terms in the following asymptotic expansions. (We provide here only one or two terms in each asymptotic expansions, but we use at least 5 terms in our numerical computations.) We note that $\mathrm{Bi}(a_k)$ alternates in sign, so for terms containing it, we group the terms with $k = 2j$ and $k = 2j + 1$ together, for $j \geq 100$. The resulting infinite sums are readily computed numerically. (They can be expressed in the Riemann zeta function at some points.)

We use the expansions, see [1, (10.4.94), (10.4.105)], (2.15),

$$
\begin{aligned}
|a_k| &\sim \frac{3^{2/3}\pi^{2/3}}{2^{2/3}}\Big(k^{2/3} - \frac{1}{6}k^{-1/3} + \dots\Big)\\
\mathrm{Hi}(a_k) &\sim -\frac{1}{\pi}a_k^{-1} - \frac{2}{\pi}a_k^{-4} + \dots = \frac{2^{2/3}}{3^{2/3}\pi^{5/3}}\Big(k^{-2/3} + \frac{1}{6}k^{-5/3} + \dots\Big)\\
\varphi(k) &\sim -2a_k^{-4} + \dots = -\frac{2^{11/3}}{3^{8/3}\pi^{8/3}}k^{-8/3} + \dots,\\
\mathrm{Bi}(a_k) &\sim (-1)^k\frac{2^{1/6}}{3^{1/6}\pi^{2/3}}k^{-1/6} + \dots,\\
g_1(j) &:= \varphi(2j)\mathrm{Bi}(a_{2j}) + \varphi(2j+1)\mathrm{Bi}(a_{2j+1}) \sim -\frac{17\cdot 3^{1/6}}{162\,\pi^{10/3}}j^{-23/6} + \dots,\\
g_2(j) &:= a_{2j}\varphi(2j)\mathrm{Bi}(a_{2j}) + a_{2j+1}\varphi(2j+1)\mathrm{Bi}(a_{2j+1}) \sim \frac{13\cdot 3^{5/6}}{162\,\pi^{8/3}}j^{-19/6} + \dots.
\end{aligned}
$$

Thus, for example, $\mathbb{E}\,N = S_0^N + S_1^N + S_2^N$, where

$$
\begin{aligned}
S_0^N &= 0.6955290109\dots,\\
S_1^N &:= -\frac{\pi}{2^{1/3}}\sum_{k=200}^{\infty}\varphi(k)\mathrm{Bi}(a_k) = -\frac{\pi}{2^{1/3}}\sum_{j=100}^{\infty}g_1(j) = 0.5317\dots\cdot 10^{-8},\\
S_2^N &:= \frac{\pi}{2^{1/3}}\sum_{k=200}^{\infty}\varphi(k)\mathrm{Hi}(a_k) = -0.16722\dots\cdot 10^{-7},
\end{aligned}
$$

yielding $\mathbb{E}(N) = 0.6955289995\dots$. Similar techniques are used for the computation of $\mathbb{E}(M)$, of the second moments and of $G_N(x)$ and $f_N(x)$.

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A Some Airy function estimates and Airy integrals

Some detailed Airy function estimates are given in the full paper. Integrals of the Airy functions (and their derivatives) times powers of x are easily reduced using the relations $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$ and $\mathrm{Bi}''(x) = x\mathrm{Bi}(x)$ and integration by parts. We have, for example, using also the definition (4.1), the recursion

$$\int x^n \mathrm{Ai}(x)\,\mathrm{d}x = \int x^{n-1}\mathrm{Ai}''(x)\,\mathrm{d}x = x^{n-1}\mathrm{Ai}'(x) - (n-1)\int x^{n-2}\mathrm{Ai}'(x)\,\mathrm{d}x$$
$$= x^{n-1}\mathrm{Ai}'(x) - (n-1)x^{n-2}\mathrm{Ai}(x) + (n-1)(n-2)\int x^{n-3}\mathrm{Ai}(x)\,\mathrm{d}x. \quad \text{(A.1)}$$

Integrals of products of two Airy functions and powers of x can be treated similarly, see [2]; we quote the following recursion

$$\int x^n \mathrm{Ai}(x)^2\,\mathrm{d}x = \tfrac{1}{2n+1}\Big(x^{n+1}\mathrm{Ai}(x)^2 - x^n(\mathrm{Ai}'(x))^2 + nx^{n-1}\mathrm{Ai}(x)\mathrm{Ai}'(x)$$
$$- \frac{n(n-1)}{2}x^{n-2}(\mathrm{Ai}(x))^2 + \frac{n(n-1)(n-2)}{2}\int x^{n-3}(\mathrm{Ai}(x))^2\,\mathrm{d}x\Big). \quad \text{(A.2)}$$

By the same method, we can also treat products involving two different translates of Airy functions; this gives for example a recursion for the integral $\int (x+c)^n \mathrm{Ai}(x+a)\mathrm{Ai}(x+b)\,\mathrm{d}x$, with $a \neq b$ and $c = (a+b)/2$.

A remarkable formula is proved in the full paper:

Lemma A.1 *If* $\operatorname{Re} z > 0$*, then*
$$\int_{-\infty}^{\infty} e^{zt}\mathrm{Ai}(t)\,\mathrm{d}t = e^{z^3/3}.$$

By Fourier inversion we find, for any $\sigma > 0$,

$$\mathrm{Ai}(t) = \frac{1}{2\pi i}\int_{\sigma-\infty i}^{\sigma+\infty i} e^{-zt+z^3/3}\,\mathrm{d}z. \quad \text{(A.3)}$$

By analytic extension, this holds for any complex t.

B An integral equation for $f_\tau(t)$

We give here another approach, based on Daniels [personal communication, 1993], to find the density function f_τ of the defect stopping time $\tau = \tau_x$, which was the basis of our development in Section 3. Unfortunately, we have not succeeded to make this approach rigorous, but we find it intriguing that it nevertheless yields the right result, so we present it here as an inspiration for further research.

Let as in (3.1) be the first passage time of $W(t)$ to the barrier $b(t) := -x - t^2/2$, where $x > 0$ is fixed. Let $f_\tau(t)$ be the (defect) density of τ, and $\phi(y;t) = e^{-y^2/2t}/\sqrt{2\pi t}$ the density of $W(t)$.

The first entrance decomposition of $W(t)$ to the region $w < b(t)$ gives the integral equation (using the strong Markov property)

$$\phi(w;t) = \int_0^t f_\tau(u)\phi\big(w - b(u); t-u\big)\,\mathrm{d}u$$

for $w < b(t)$. Letting $w \nearrow b(t)$ we get the equation

$$\phi(b(t);t) = \int_0^t f_\tau(u)\phi\big(b(t) - b(u); t-u\big)\,\mathrm{d}u \quad \text{(B.1)}$$

for $t > 0$. (Similar arguments using the last exit decomposition, which leads to another functional equation involving also another unknown function, are used by Daniels [6] and Daniels and Skyrme [8].) Since

$$b(t) - b(u) = (u^2 - t^2)/2 = -(t-u)(t+u)/2,$$

we have by (B.1)

$$\frac{e^{-(x+t^2/2)^2/2t}}{\sqrt{2\pi t}} = \int_0^t f_\tau(u) \frac{e^{-(t-u)(t+u)^2/8}}{\sqrt{2\pi(t-u)}} \,\mathrm{d}u. \tag{B.2}$$

The exponents can be written as $-(t-u)(t+u)^2/8 = -t^3/6 + u^3/6 + (t-u)^3/24$ and $-(x+t^2/2)^2/2t = -x^2/2t - xt/2 + t^3/24 - t^3/6$, so the integral equation (B.2) can be transformed into

$$\frac{e^{-x^2/2t-xt/2+t^3/24}}{\sqrt{2\pi t}} = \int_0^t f_\tau(u) e^{u^3/6} \frac{e^{(t-u)^3/24}}{\sqrt{2\pi(t-u)}} \,\mathrm{d}u. \tag{B.3}$$

This is a convolution equation of the form

$$h(t) = \int_0^t g(u)k(t-u)\,\mathrm{d}u$$

with

$$g(t) := f_\tau(t)e^{t^3/6}, h(t) := \frac{e^{-x^2/2t-xt/2+t^3/24}}{\sqrt{2\pi t}} = \frac{e^{-b(t)^2/2t+t^3/6}}{\sqrt{2\pi t}}, k(t) := \frac{e^{t^3/24}}{\sqrt{2\pi t}}.$$

If the Laplace transforms $\widetilde{g}(s) := \int_0^\infty e^{-st}g(t)\,\mathrm{d}t$ etc. were finite for $\operatorname{Re} s$ large enough, we could get the solution from $\widetilde{g}(s) = \widetilde{h}(s)/\widetilde{k}(s)$. However, the factors $e^{t^3/24}$ in $h(t)$ and $k(t)$ grow too fast, so $\widetilde{h}(s)$ and $\widetilde{k}(s)$ are not finite for any $s > 0$ and this method does not work. Nevertheless, if we instead define $\widehat{h}(s)$ and $\widehat{k}(s)$ by $\widehat{h}(s) := \int_{\sigma-\infty i}^{\sigma+\infty i} e^{-st}h(t)\,\mathrm{d}t$ and $\widehat{k}(s) := \int_{\sigma-\infty i}^{\sigma+\infty i} e^{-st}k(t)\,\mathrm{d}t$, integrating along vertical lines in the complex plane with real part $\sigma > 0$, then the formula $\widetilde{g}(s) = \widehat{h}(s)/\widehat{k}(s)$ yields the correct formula for $\widetilde{g}(s)$ and thus for $g(t)$. (Note that $\widehat{h}$ and $\widehat{k}$ can be seen as Fourier transform of h and k restricted to vertical lines. The value of $\sigma > 0$ is arbitrary and does not affect $\widehat{h}$ and $\widehat{k}$.) Let us show this remarkable fact by calculating $\widehat{h}(s)$ and $\widehat{k}(s)$.

Consider first $h(t)$ and express the Gaussian factor by Fourier inversion:

$$\frac{e^{-b(t)^2/2t}}{\sqrt{2\pi t}} = \int_{\sigma-\infty i}^{\sigma+\infty i} e^{b(t)u+tu^2/2} \frac{\mathrm{d}u}{2\pi i}, \qquad \operatorname{Re} t > 0.$$

The exponent $b(t)u + tu^2/2$ can then be written as $-ux + u^3/6 + (t-u)^3/6 - t^3/6$, so that

$$h(t) = \int_{\sigma-\infty i}^{\sigma+\infty i} e^{-ux+u^3/6+(t-u)^3/6} \frac{\mathrm{d}u}{2\pi i},$$

and, choosing $\sigma_1 > \sigma > 0$ and letting $\sigma_2 := \sigma_1 - \sigma$,

$$\begin{aligned}
\widehat{h}(s) &= \int_{\sigma_1-\infty i}^{\sigma_1+\infty i} \int_{\sigma-\infty i}^{\sigma+\infty i} e^{-st-ux+u^3/6+(t-u)^3/6} \frac{\mathrm{d}u\,\mathrm{d}t}{2\pi i} \\
&= \int_{\sigma-\infty i}^{\sigma+\infty i} \int_{\sigma_2-\infty i}^{\sigma_2+\infty i} e^{-s(u+v)-ux+u^3/6+v^3/6} \frac{\mathrm{d}v\,\mathrm{d}u}{2\pi i} \\
&= \int_{\sigma-\infty i}^{\sigma+\infty i} \int_{\sigma_2-\infty i}^{\sigma_2+\infty i} e^{-sv+v^3/6} e^{-(s+x)u+u^3/6} \frac{\mathrm{d}v\,\mathrm{d}u}{2\pi i} \\
&= 2\pi i \operatorname{Ai}\bigl(c(s+x)\bigr)\operatorname{Ai}(cs)c^2,
\end{aligned}$$

with $c := 2^{1/3}$, using (A.3).

Since $k(t)$ is obtained by putting $x = 0$ in $h(t)$, we get directly

$$\widehat{k}(s) = 2\pi i \operatorname{Ai}(cs)^2 c^2$$

and hence

$$\widehat{h}(s)/\widehat{k}(s) = \operatorname{Ai}\big(2^{1/3}(s+x)\big)/\operatorname{Ai}(2^{1/3}s).$$

This is indeed the Laplace transform of $g(t) = f_\tau(t)e^{t^3/6}$ which by inversion yields some formulas given in the full report.

It seems reasonable that it should be possible to verify the crucial formula $\widetilde{g}(s)\widehat{k}(s) = \widehat{h}(s)$ by suitable manipulations of integrals. However, we have to leave that as an open problem. Note that, by Lefebvre [15], see also Groeneboom [9, Theorem 2.1],

$$\mathbb{E}_x\left(e^{-s\tau_0 - \int_0^{\tau_0} W(t)dt}\right) = \operatorname{Ai}\big(2^{1/3}(s+x)\big)/\operatorname{Ai}(2^{1/3}s)$$

where τ_0 is the first hitting time of $W(t)$ to 0, with $W(0) = x$; This could be a hint for proving $\widetilde{g}(s)\widehat{k}(s) = \widehat{h}(s)$.

References

[1] M. Abramowitz & I. A. Stegun, eds., *Handbook of Mathematical Functions*. Dover, New York, 1972.

[2] J. R. Albright, Integrals of products of Airy functions. *J. Phys. A* **10** (1977), no. 4, 485–490.

[3] J. R. Albright & E. P. Gavathas, Integrals involving Airy functions. Comment on: "Evaluation of an integral involving Airy functions" by L. T. Wille and J. Vennik. *J. Phys. A* **19** (1986), no. 13, 2663–2665.

[4] A. D. Barbour, A note on the maximum size of a closed epidemic. *J. Roy. Statist. Soc. Ser. B* **37** (1975), no. 3, 459–460.

[5] A. D. Barbour, Brownian motion and a sharply curved boundary. *Adv. Appl. Probab.* **13** (1981), no. 4, 736–750.

[6] H. E. Daniels, The maximum size of a closed epidemic. *Adv. Appl. Probab.* **6** (1974), 607–621.

[7] H. E. Daniels, The maximum of a Gaussian process whose mean path has a maximum, with an application to the strength of bundles of fibres. *Adv. Appl. Probab.* **21** (1989), no. 2, 315–333.

[8] H. E. Daniels & T. H. R. Skyrme, The maximum of a random walk whose mean path has a maximum. *Adv. Appl. Probab.* **17** (1985), no. 1, 85–99.

[9] P. Groeneboom, Brownian motion with a parabolic drift and Airy functions. *Probab. Theory Related Fields* **81** (1989), no. 1, 79–109.

[10] L. Hörmander, *The Analysis of Linear Partial Differential Operators. I. Distribution Theory and Fourier Analysis*. Springer-Verlag, Berlin, 1983.

[11] S. Janson, Brownian excursion area, Wright's constants in graph enumeration, and other Brownian areas. *Probability Surveys* **3** (2007), 80–145.

[12] S. Janson, Sorting using complete subintervals and the maximum number of runs in a randomly evolving sequence. *Ann. Comb.* **12** (2009), no. 4, 417–447.

[13] S. Janson, G. Louchard, and A. Martin-Löf. The maximum of Brownian motion with parabolic drift. Technical report, 2010. arXiv:1002.0497 [math.PR].

[14] A. Lachal. Sur la distribution de certaines fonctionnelles de l'intégrale du mouvement brownien avec dérives parabolique et cubique. *Communications on Pure and Applied Mathematics*, 49(12):1299–1338, 1996.

[15] M. Lefebvre. First-passage densities of a two-dimensional process. *SIAM Journal on Applied Mathematics*, 49(5):1514–1523, 1989.

[16] G. Louchard, Random walks, Gaussian processes and list structures. *Theoret. Comput. Sci.* **53** (1987), no. 1, 99–124.

[17] G. Louchard, C. Kenyon & R. Schott, Data structures' maxima. *SIAM J. Comput.* **26** (1997), no. 4, 1006–1042.

[18] P. Salminen, On the first hitting time and the last exit time for a Brownian motion to/from a moving boundary. *Adv. Appl. Probab.* **20** (1988), no. 2, 411–426.

[19] R. L. Smith, The asymptotic distribution of the strength of a series-parallel system with equal load-sharing. *Ann. Probab.* **10** (1982), no. 1, 137–171.

[20] D. Steinsaltz, Random time changes for sock-sorting and other stochastic process limit theorems. *Electron. J. Probab.* **4** (1999), no. 14, 25 pp.