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Asymptotics for Walks in a Weyl chamber of Type B (extended abstract)[†]

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We consider lattice walks in $\mathbb{R}^k$ confined to the region $0 < x_1 < x_2 ... < x_k$ with fixed (but arbitrary) starting and end points. The walks are required to be "reflectable", that is, we assume that the number of paths can be counted using the reflection principle. The main result is an asymptotic formula for the total number of walks of length n with fixed but arbitrary starting and end point for a general class of walks as the number n of steps tends to infinity. As applications, we find the asymptotics for the number of k-non-crossing tangled diagrams on the set $\{1, 2, ..., n\}$ as n tends to infinity, and asymptotics for the number of k-vicious walkers subject to a wall restriction in the random turns model as well as in the lock step model. Asymptotics for all of these objects were either known only for certain special cases, or have only been partially determined.

Keywords: lattice walks, Weyl chambers, determinants, asymptotics, saddlepoint method

1 Introduction

Lattice paths are well-studied objects in combinatorics as well as in probability theory. A typical problem that is often encountered is the determination of the number of lattice paths that stay within a certain fixed region. In many situations, this region can be identified with a Weyl chamber corresponding to some reflection group. In this paper, the region is a Weyl chamber of type B, and, more precisely, it is given by $0 < x_1 < \cdots < x_k$. (Here, x_j refers to the j-th coordinate in $\mathbb{R}^k$.)

Under certain assumptions on the set of allowed steps and on the underlying lattice, the total number of paths as described above can be counted using the *reflection principle* as formulated by Gessel and Zeilberger [5]. This reflection principle is a generalisation of a reflection argument, which is often attributed to André [1], to the context of general finite reflection groups (for details on reflection groups, see [7]).

A necessary and sufficient condition on the set of steps for ensuring the applicability of the reflection principle as formulated by Gessel and Zeilberger [5] has been given by Grabiner and Magyar [6]. In their paper, Grabiner and Magyar also stated a precise list of steps that satisfy these conditions.

In a recent paper that attracted the author's interest, and that was also the main initial motivation for this work, Chen et al. [2, Obervations 1 and 2] gave lattice path descriptions for combinatorial objects

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called k*-non-crossing tangled diagrams*. In their work, they determined the order of asymptotic growth of these objects, but they did not succeed in determining precise asymptotics. Interestingly, the sets of steps appearing in this description do not satisfy Grabiner and Magyar's condition. Nevertheless, a slightly generalised reflection principle (see Lemma 1 below) turns out to be applicable because the steps can be interpreted as sequences of certain *atomic steps*, where these atomic steps satisfy Grabiner and Magyar's condition.

Our main result (see Theorem 1) is an asymptotic formula for the total number of walks of length n that stay within the region $0 < x_1 < \cdots < x_k$, starting and ending in some arbitrary but fixed points as the number n of steps tends to infinity. The proof essentially consists of an application of the saddlepoint method, but there are some technical problems in between that we have to overcome. The most significant comes from the fact that we have to determine asymptotics for a determinant. The problem here is the large number of cancellations of asymptotically leading terms. It is surmounted by means of a general technique that is sketched in Section 3. As corollaries to our main result, we obtain precise asymptotics for certain vicious walkers models for which asymptotics where known only for special configurations (for the lock step model, see [8, 9, 11]), as well as for k-non-crossing tangled diagrams, for which only the asymptotic growth order was established (see [2]). To the author's best knowledge, the asymptotics for the number of vicious walks in the random turns model seem to be new.

In some sense, one of the achievements of the present work is that it shows how to overcome a technical difficulty put to the fore in [12]. The key ingredient is the technique of Section 3. For details, we refer to the introduction of the full version of this paper [3], or the paragraph after [12, Theorem 8].

The paper is organised as follows. In the next section, we give the basic definitions and precise description of the lattice walk model underlying this work. We also state and prove a slightly generalised reflection principle (see Lemma 1 below) that can be used to count the number of lattice walks in our model. At the end of this section, we prove an exact integral formula for this number. Section 3 presents a factorisation technique for certain functions defined by determinants. These results are crucial to our proof since they enable us to determine precise asymptotics for these functions. The main result is the content of section 4. The last section presents applications of the main theorem to vicious walkers in the lock step model as well as in the random turns model and and also to k-non-crossing tangled diagrams.

2 Reflectable walks of type B

The intention of this section is twofold. First, we give a precise description of the lattice walk model underlying this work, and state some basic results. Second, we derive an *exact integral formula* (see Lemma 3 below) for the generating function of lattice walks in this model with respect to a given weight.

Let us start with the presentation of the lattice path model. We will have two kind of steps: atomic steps and composite steps. *Atomic steps* are elements of $\mathbb{R}^k$. The set of all atomic steps in our model will always be denoted by $\mathcal{A}$. *Composite steps* are finite sequences of atomic steps. The set of composite steps in our model will be always be denoted by $\mathcal{S}$. Both sets, $\mathcal{A}$ and $\mathcal{S}$, are assumed to be finite sets. By $\mathcal{L}$ we denote the $\mathbb{Z}$-lattice spanned by the atomic step set $\mathcal{A}$.

The walks in our model are walks on the lattice $\mathcal{L}$ consisting of steps from the composite step set $\mathcal{S}$ that are confined to the region

$$\mathcal{W}^0 = \left\{(x_1, \ldots, x_k) \in \mathbb{R}^k \;:\; 0 < x_1 < \cdots < x_k\right\}.$$

For a given function $w : \mathcal{S} \to \mathbb{R}_+$, called the *weight function*, we define the weight of a walk with step sequence $(\mathbf{s}_1, \ldots, \mathbf{s}_n) \in \mathcal{S}^n$ by $\prod_{j=1}^n w(\mathbf{s}_j)$.

The generating function for all n-step paths from $\mathbf{u} \in \mathcal{L}$ to $\mathbf{v} \in \mathcal{L}$ with respect to the weight w will be denoted by $P_n(\mathbf{u} \to \mathbf{v})$, that is,

$$P_n(\mathbf{u} \to \mathbf{v}) = \sum_{\substack{\mathbf{s}_1,\ldots,\mathbf{s}_n \in \mathcal{S} \\ \mathbf{u}+\mathbf{s}_1+\cdots+\mathbf{s}_n = \mathbf{v}}} \prod_{j=1}^n w(\mathbf{s}_j),$$

and the generating function of those paths of length n from $\mathbf{u}$ to $\mathbf{v}$ with respect to the weight w that stay within the region $\mathcal{W}^0$ will be denoted by $P_n^+(\mathbf{u} \to \mathbf{v})$.

The ultimate goal of this work is the derivation of an asymptotic formula for $P_n^+(\mathbf{u} \to \mathbf{v})$ as n tends to infinity for certain step sets $\mathcal{S}$ and certain weight functions w.

In the theory of reflection groups (or Coxeter groups), $\mathcal{W}^0$ is called a *Weyl chamber of type B_k*. By $\mathcal{W}$, we denote the closure of $\mathcal{W}^0$, viz.

$$\mathcal{W} = \left\{(x_1, \ldots, x_k) \in \mathbb{R}^k \ : \ 0 \le x_1 \le \cdots \le x_k\right\},$$

The boundary of $\mathcal{W}$ is contained in the union of the hyperplanes

$$x_i - x_j = 0 \quad \text{for } 1 \le i < j \le k, \qquad \text{and} \qquad x_1 = 0. \tag{1}$$

The set of reflections in these hyperplanes is a generating set for the finite reflection group of type B_k (see Humphreys [7]).

We would like to point out that all results presented in this section have analogues for all general finite or affine reflection groups. In order to keep this section as short and simple as possible, we restrict our presentation to the type B_k case. For the general results, we refer the interested reader to the corresponding literature. A good introduction to the theory of reflection groups can be found in the standard reference book by Humphreys [7].

The fundamental assumption underlying this manuscript is the applicability of a reflection principle argument to the problem of counting walks with n composite steps that stay within the region $\mathcal{W}^0$. Such a reflection principle has been proved by Gessel and Zeilberger [5] for lattice walks in Weyl chambers of arbitrary type that consist of steps from an atomic step set. We need to slightly extend their result for Weyl chambers of type B_k to walks consisting of steps from a composite step set. The precise result is stated in the following lemma. The proof is very similar to the original proof in [5] and is therefore omitted.

Lemma 1 (Reflection Principle) *Let $\mathcal{A}$ be an atomic step set that is invariant under the reflection group generated by the reflections* (1)*, and such that for all $\mathbf{a} \in \mathcal{A}$ and all $\mathbf{u} \in \mathcal{W}^0 \cap \mathcal{L}$ we have $\mathbf{u} + \mathbf{a} \in \mathcal{W}$. By $\mathcal{S}$ we denote a composite step set over $\mathcal{A}$ such that for all $(\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathcal{S}$ we also have $(\rho(\mathbf{a}_1), \ldots, \rho(\mathbf{a}_j), \mathbf{a}_{j+1}, \ldots, \mathbf{a}_m) \in \mathcal{S}$ for all $j = 1, 2, \ldots, m$ and all reflections ρ in the group generated by* (1)*. Finally, assume that the weight function $w : \mathcal{S} \to \mathbb{R}_+$ satisfies $w\left((\mathbf{a}_1, \ldots, \mathbf{a}_m)\right) = w\left((\rho(\mathbf{a}_1), \ldots, \rho(\mathbf{a}_j), \mathbf{a}_{j+1}, \ldots, \mathbf{a}_m)\right)$ for all j and ρ as before.*

Then, for all $\mathbf{u} = (u_1, \ldots, u_k) \in \mathcal{W}^0 \cap \mathcal{L}$ and all $\mathbf{v} \in \mathcal{W}^0 \cap \mathcal{L}$, the generating function for all n-step walks with steps from the composite step set $\mathcal{S}$ with respect to the weight w that stay within $\mathcal{W}^0$ satisfies

$$P_n^+(\mathbf{u} \to \mathbf{v}) = \sum_{\substack{\sigma \in \mathfrak{S}_k \\ \varepsilon_1,\ldots,\varepsilon_k \in \{-1,+1\}}} \left(\prod_{j=1}^k \varepsilon_j\right) \operatorname{sgn}(\sigma) P_n\Big((\varepsilon_1 u_{\sigma(1)}, \ldots, \varepsilon_k u_{\sigma(k)}) \to \mathbf{v}\Big), \tag{2}$$

where $\mathfrak{S}_k$ is the set of all permutations on $\{1,\dots,k\}$.

In view of this last lemma, the question that now arises is: what composite step sets $\mathcal{S}$ satisfy the conditions in Lemma 1? This question boils down the question: what atomic step sets $\mathcal{A}$ satisfy the conditions in Lemma 1? The answer to this latter question has been given by Grabiner and Magyar [6]. For type B, the result reads as follows.

Lemma 2 (Grabiner and Magyar [6]) *The atomic step set $\mathcal{A} \subset \mathbb{R}^k \setminus \{\mathbf{0}\}$ satisfies the conditions stated in Lemma 1 if and only if $\mathcal{A}$ is (up to rescaling) equal either to*

$$\left\{\pm\mathbf{e}^{(1)}, \pm\mathbf{e}^{(2)}, \dots, \pm\mathbf{e}^{(k)}\right\} \qquad \textit{or to} \qquad \left\{\sum_{j=1}^{k} \varepsilon_j \mathbf{e}^{(j)} \ :\ \varepsilon_1,\dots,\varepsilon_k \in \{-1,+1\}\right\},$$

where $\left\{\mathbf{e}^{(1)},\dots,\mathbf{e}^{(k)}\right\}$ is the canonical basis in $\mathbb{R}^k$.

In this manuscript we will always assume that our lattice walk model satisfies all the requirements of Lemma 1. Therefore, we make the following assumption.

Assumption 1 *From now on, we assume that the atomic step set $\mathcal{A}$ is equal to one of the two sets given in Lemma 2. Further, we assume that the composite step set $\mathcal{S}$ and the weight function $w : \mathcal{S} \to \mathbb{R}^k$ satisfy the conditions of Lemma 1.*

The final objective in this section is an integral formula for $P_n^+(\mathbf{u} \to \mathbf{v})$. The result is stated in Lemma 3 below. Its derivation is based on a generating function approach.

In order to simplify the presentation, we apply the standard multi-index notation: If $\mathbf{z} = (z_1,\dots,z_k)$ is a vector of indeterminates and $\mathbf{a} = (a_1,\dots,a_k) \in \mathbb{Z}^k$, then we set $\mathbf{z}^{\mathbf{a}} := z_1^{a_1} z_2^{a_2} \dots z_k^{a_k}$. Furthermore, if $F(\mathbf{z})$ is a series in $\mathbf{z}$, then we denote by $[\mathbf{z}^{\mathbf{a}}]F(\mathbf{z})$ the coefficient of the monomial $\mathbf{z}^{\mathbf{a}}$ in $F(\mathbf{z})$.

Now, we define the *atomic step generating function* $A(\mathbf{z}) = A(z_1,\dots,z_k)$ associated with the atomic step set $\mathcal{A}$ by

$$A(z_1,\dots,z_k) = A(\mathbf{z}) = \sum_{\mathbf{a}\in\mathcal{A}} \mathbf{z}^{\mathbf{a}}.$$

The *composite step generating function* associated with the composite step set $\mathcal{S}$ with respect to the weight w is defined by

$$S(z_1,\dots,z_k) = S(\mathbf{z}) = \sum_{\substack{m\geq 0\\ (\mathbf{a}_1,\dots,\mathbf{a}_m)\in\mathcal{S}}} w\Big((\mathbf{a}_1,\dots,\mathbf{a}_m)\Big)\mathbf{z}^{\mathbf{a}_1+\dots+\mathbf{a}_m}.$$

The generating function for the number of n-step paths with steps from the composite step set $\mathcal{S}$ that start in $\mathbf{u} \in \mathcal{L}$ and end in $\mathbf{v} \in \mathcal{L}$ with respect to the weight w can then be expressed as

$$P_n(\mathbf{u} \to \mathbf{v}) = \left[\mathbf{z}^{\mathbf{v}-\mathbf{u}}\right] S(\mathbf{z})^n. \tag{3}$$

We can now state and prove the main result of this section: the integral formula for $P_n^+(\mathbf{u} \to \mathbf{v})$.

Lemma 3 *Let $\mathcal{S}$ be a composite step set and let $w : \mathcal{S} \to \mathbb{R}_+$ be weight function, both satisfying Assumption 1. Furthermore, let $S(z_1, \ldots, z_k)$ be the associated composite step generating function.*

Then the generating function $P_n^+(\mathbf{u} \to \mathbf{v})$ for the number of n-step paths from $\mathbf{u} \in \mathcal{W}^0 \cap \mathcal{L}$ to $\mathbf{v} \in \mathcal{W}^0 \cap \mathcal{L}$ that stay within $\mathcal{W}^0$ with steps from the composite step set S satisfies

$$P_n^+(\mathbf{u} \to \mathbf{v}) = \frac{1}{(2\pi i)^k} \int\limits_{|z_1|=\cdots=|z_k|=\rho} \cdots \int \det_{1 \le j,m \le k} \left(z_j^{u_m} - z_j^{-u_m} \right) S(z_1, \ldots, z_k)^n \left(\prod_{j=1}^{k} \frac{dz_j}{z_j^{v_j+1}} \right), \tag{4}$$

where $\rho > 0$.

Proof: The proof of this lemma relies on the reflection principle (Lemma 1) and Cauchy's integral formula.

Lemma 1 and Equation (3) together give us

$$P_n^+(\mathbf{u} \to \mathbf{v}) = \sum_{\substack{\sigma \in \mathfrak{S}_k \\ (\varepsilon_1, \ldots, \varepsilon_k) \in \{-1,+1\}^k}} \left(\prod_{j=1}^{k} \varepsilon_j \right) \operatorname{sgn}(\sigma) \left[z_1^{v_1 - \varepsilon_1 u_{\sigma(1)}} \ldots z_k^{v_k - \varepsilon_k u_{\sigma(k)}} \right] S(z_1, \ldots, z_k)^n.$$

Now, Cauchy's formula followed by interchanging summation and integration gives us

$$\int\limits_{|z_1|=\cdots=|z_k|=1} \cdots \int \frac{S(z_1, \ldots, z_k)^n}{(2\pi i)^k} \left(\sum_{\substack{\sigma \in \mathfrak{S}_k \\ (\varepsilon_1, \ldots, \varepsilon_k) \in \{-1,+1\}^k}} \operatorname{sgn}(\sigma) \left(\prod_{j=1}^{k} \varepsilon_j z_j^{\varepsilon_j u_{\sigma(j)}} \right) \right) \left(\prod_{j=1}^{k} \frac{dz_j}{z_j^{v_j+1}} \right),$$

which proves the theorem. □

3 Determinants and asymptotics

Asymptotics for determinants are often hard to obtain, the reason being a typical large number of cancellations of asymptotically leading terms. In this section, we sketch a factorisation technique that allows one to represent certain functions in several complex variables defined by determinants as a product of two factors. One of these factors will always be a symmetric (Laurent) polynomial (this accounts for the cancellations of asymptotically leading terms mentioned before). The second factor is a determinant, the entries of which are certain contour integrals. In many cases, asymptotics for this second factor can be established by geometric series expansions, coefficient extraction and known determinant evaluations. The fundamental technique is illustrated in Lemma 4 below.

We want to stress that Lemma 4 should be seen as a general technique, not as a particular result. The main intention of this lemma is to give the reader an unblurred view at the technique. An application of Lemma 4 together with some remarks on asymptotics can be found right after the proof. For a more detailed presentation of this technique, we refer the reader to full version of this manuscript [3].

Lemma 4 *Let $A_m(x, y)$, $1 \le m \le k$, be analytic and one-valued for $(x, y) \in \mathcal{R} \times \mathcal{D} \subset \mathbb{C}^2$, where $\mathcal{D} \subset \mathbb{C}$ is some non empty set and $\mathcal{R} = \{x \in \mathbb{C} : |x| < R^*\}$ for some $0 < R^*$.*

Then, the function

$$\det_{1\le j,m\le k}\left(A_m(x_j,y_m)\right)$$

is analytic for $(x_1,\dots,x_k,y_1,\dots,y_k)\in\mathcal{R}^k\times\mathcal{D}^k$*, and it satisfies*

$$\det_{1\le j,m\le k}\left(A_m(x_j,y_m)\right)=\left(\prod_{1\le j<m\le k}(x_m-x_j)\right)\det_{1\le j,m\le k}\left(\frac{1}{2\pi i}\int_{|\xi|=R}\frac{A_m(\xi,y_m)d\xi}{\prod\limits_{\ell=1}^{j}(\xi-x_\ell)}\right),$$

where $\max_j|x_j|<R<R^*$.

Proof: By Cauchy's theorem, we have

$$\det_{1\le j,m\le k}\left(A_m(x_j,y_m)\right)=\det_{1\le j,m\le k}\left(\frac{1}{2\pi i}\int_{|\xi|=R}\frac{A_m(\xi,y_m)d\xi}{\xi-x_j}\right). \tag{5}$$

Now, short calculations show that for any $L\ge 0$ and all $n_1,\dots,n_L\in\{1,2,\dots,k\}$ we have

$$\begin{aligned}\int_{|\xi|=\rho_1}\frac{A_m(\xi,y_m)d\xi}{(\xi-x_j)\prod_{\ell=1}^{L}(\xi-x_{n_\ell})}-\int_{|\xi|=\rho_1}\frac{A_m(\xi,y_m)d\xi}{(\xi-x_j)\prod_{\ell=1}^{L}(\xi-x_{n_\ell})}\\ =(x_m-x_j)\int_{|\xi|=\rho_1}\frac{A(\xi,y)d\xi}{(\xi-x_j)(\xi-x_m)\prod_{\ell=1}^{L}(\xi-x_{n_\ell})}.\end{aligned}$$

Consequently, we can prove the claimed factorisation as follows. First, we subtract the first row of the determinant in (5) from all other rows. By the computations above we can then take the factor (x_j-x_1) out of the j-th row of the determinant. In a second run, we subtract the second row from the rows $3,4,\dots,k$, and so on. In general, after subtracting row j from row ℓ we take the factor $(x_\ell-x_j)$ out of the determinant. □

Example 1 *Consider the function*

$$\det_{1\le j,m\le k}\left(e^{x_jy_m}\right).$$

An application of Lemma 4 with $A(x,y)=e^{xy}$ *immediately gives us the factorisation*

$$\det_{1\le j,m\le k}\left(e^{x_jy_m}\right)=\left(\prod_{1\le j<m\le k}(x_m-x_j)\right)\det_{1\le j,m\le k}\left(\frac{1}{2\pi i}\int_{|\xi|=R}\frac{e^{\xi y_m}d\xi}{\prod_{\ell=1}^{j}(\xi-x_\ell)}\right),$$

where $R>\max_j|x_j|$*. Note that the second contour integral occurring in the factorisation given in Lemma 4 is equal to zero because the function* $A(x,y)=e^{xy}$ *is an entire function.*

Now we want to demonstrate how one can derive asymptotics for $\det_{1\le j,m\le k}\left(e^{x_j y_m}\right)$ *as* $x_1,\dots,x_k\to 0$ *from this factorisation. The geometric series expansion gives us*

$$\frac{1}{2\pi i}\int_{|\xi|=R}\frac{e^{\xi y}d\xi}{\prod_{\ell=1}^{j}(\xi-x_\ell)}=\frac{1}{2\pi i}\int_{|\xi|=R}e^{\xi y}\frac{d\xi}{\xi^j}+O\left(\sum_{j=1}^{k}|x_k|\right)=\frac{y^{j-1}}{(j-1)!}+O\left(\sum_{j=1}^{k}|x_k|\right)$$

as $x_1,\dots,x_k\to 0$*. Consequently, we have by virtue of the Vandermonde formula*

$$\det_{1\le j,m\le k}\left(e^{x_j y_m}\right)=\left(\prod_{1\le j<m\le k}(x_m-x_j)\right)\left(\left(\prod_{1\le j<m\le k}\frac{y_m-y_j}{m-j}\right)+O\left(\sum_{j=1}^{k}|x_j|\right)\right)$$

as $x_1,\dots,x_k\to\infty$*.*

If we would have considered the function $\det\limits_{1\le j,m\le k}\left(e^{\xi^2 y}\right)$*,* $k>1$*, instead of* $\det\limits_{1\le j,m\le k}(e^{x_j y_m})$ *as in the example above, we would have got only the upper bound*

$$\det_{1\le j,m\le k}\left(e^{x_j^2 y_m}\right)=O\left(\left(\prod_{1\le j<m\le k}(x_m-x_j)\right)\sum_{j=1}^{k}|x_j|\right)$$

as $x_1,\dots,x_k\to 0$*.The reason for this is that the function* $A(x,y)=e^{x^2y}$ *satisfies the symmetry* $A(-x,y)=A(x,y)$ *which induces additional cancellations of asymptotically leading terms.*

In order to obtain precise asymptotic formulas in cases where the functions $A_m(x,y)$ *exhibit certain symmetries, we have to take into account these symmetries. This can easily be accomplished by a small modification to our factorisation technique presented in Lemma 4. In fact, the only thing we have to do is to modify the representation* (5)*, the rest of our technique remains - mutatis mutandis - unchanged.*

Lemma 5 *For all* $u_1,\dots,u_k\in\mathbb{C}$ *we have as* $(\varphi_1,\dots,\varphi_k)\to(0,\dots,0)$ *the asymptotics*

$$\det_{1\le j,m\le k}(\sin(u_m\varphi_j))=\left(\prod_{j=1}^{k}\varphi_j\right)\left(\prod_{1\le j<m\le k}(\varphi_m^2-\varphi_j^2)\right)\left(\prod_{j=1}^{k}\frac{(-1)^{j-1}}{(2j-1)!}\right)\times\left(\left(\prod_{j=1}^{k}u_j\right)\left(\prod_{1\le j<m\le k}(u_m^2-u_j^2)\right)+O\left(\max_j|\varphi_j|^2\right)\right).$$

Proof: The technique from Lemma 4 gives us

$$\det_{1\le j,m\le k}(\sin(u_m\varphi_j))=\left(\prod_{j=1}^{k}\varphi_j\right)\left(\prod_{1\le j<m\le k}(\varphi_m^2-\varphi_j^2)\right)\det_{1\le j,m\le k}\left(\frac{1}{2\pi i}\int\limits_{|\eta|=1}\frac{\sin(u_m\eta)d\eta}{\prod\limits_{\ell=1}^{j}(\eta^2-\varphi_\ell^2)}\right).$$

Since we my assume that $\max_j |\varphi_j| < 1$, we deduce by the geometric series expansion that

$$\frac{1}{2\pi i}\int\limits_{|\eta|=1}\frac{\sin(u_m\eta)d\eta}{\left(\prod\limits_{\ell=1}^{j}(\eta^2-\varphi_\ell^2)\right)}=\frac{(-1)^{j-1}u_m^{2j-1}}{(2j-1)!}+O\left(\max_j|\varphi_j|^2\right)$$

as $(\varphi_1,\ldots,\varphi_k)\to(0,\ldots,0)$. Since the last determinant is of Vandermonde type it can be evaluated to a closed form expression. This completes the proof. □

4 The main result

In this section, we are going to derive asymptotics for $P_n^+(\mathbf{u}\to\mathbf{v})$ as n tends to infinity (see Theorem 1 below). The asymptotics are derived by applying saddle point techniques to the integral representation (3) together with the techniques developed in Section 3.

Theorem 1 *Let $\mathcal{S}$ be a composite step set over the atomic step set $\mathcal{A}$, and let $w:\mathcal{S}\to\mathbb{R}_+$ be a weight function. By $\mathcal{L}$ we denote the $\mathbb{Z}$-lattice spanned by $\mathcal{A}$. The composite step generating function associated with $\mathcal{S}$ is denoted by $S(z_1,\ldots,z_k)$. Finally, let $\mathcal{M}\subseteq\{0,\pi\}^k$ denote the set of points such that the function $(\varphi_1,\ldots,\varphi_k)\mapsto|S(e^{i\varphi_1},\ldots,e^{i\varphi_k})|$ attains a maximum value, and let $|\mathcal{M}|$ denote the cardinality of the set $\mathcal{M}$.*

If $\mathcal{A}$, $\mathcal{S}$ and w satisfy Assumption 1 and $S(1,\ldots,1)>0$, then for any two points $\mathbf{u},\mathbf{v}\in\mathcal{W}^0\cap\mathcal{L}$ we have the asymptotic formula

$$P_n^+(\mathbf{u}\to\mathbf{v})=|\mathcal{M}|S(1,\ldots,1)^n\left(\frac{2}{\pi}\right)^{k/2}\left(\frac{S(1,\ldots,1)}{nS''(1,\ldots,1)}\right)^{k^2+k/2}$$
$$\times\frac{\left(\prod\limits_{1\le j<m\le k}(u_m^2-u_j^2)(v_m^2-v_j^2)\right)\left(\prod\limits_{j=1}^{k}u_jv_j\right)}{\left(\prod_{j=1}^{k}(2j-1)!\right)}\left(1+O(n^{-2/3})\right)\quad(6)$$

as $n\to\infty$ in the set $\{n:P_n^+(\mathbf{u}\to\mathbf{v})>0\}$. Here, $S''(z_1,\ldots,z_k)$ denotes the second derivative of $S(z_1,\ldots,z_k)$ with respect to any of the z_j.

Before actually proving Theorem 1, we collect some auxiliary results that will be needed for proving the theorem. We begin with a result on the structure of atomic step generating functions which is a direct consequence of Lemma 2.

Lemma 6 *Let $\mathcal{A}$ be an atomic step set satisfying Assumption 1. Then the associated atomic step generating function $A(z_1,\ldots,z_k)$ is equal either to*

$$\sum_{j=1}^{k}\left(z_j+\frac{1}{z_j}\right)\qquad\text{or to}\qquad\prod_{j=1}^{k}\left(z_j+\frac{1}{z_j}\right).\quad(7)$$

As a direct consequence of this last lemma, we obtain the following result.

Lemma 7 *Let $\mathcal{S}$ be composite step set over the atomic step set $\mathcal{A}$, and let $w : \mathcal{S} \to \mathbb{R}_+$ be a weight function. If $\mathcal{S}$, $\mathcal{A}$ and w satisfy Assumption 1, then there exists a polynomial $P(x)$ with non-negative coefficients such that either*

$$S(z_1,\dots,z_k) = P\left(\sum_{j=1}^{k}\left(z_j + \frac{1}{z_j}\right)\right) \qquad \text{or} \qquad S(z_1,\dots,z_k) = P\left(\prod_{j=1}^{k}\left(z_j + \frac{1}{z_j}\right)\right).$$

Proof (Sketch): The reflection principle 1 requires that all steps having the same length (viewed as sequences over $\mathcal{A}$) have to have the same positive weight, and also that if one step of a certain length (again viewed as a sequence over $\mathcal{A}$) is allowed then all steps of this length have to be allowed. $\square$

Lemma 8 *Let $\mathcal{S}$ be a composite step set, and let $S(z_1,\dots,z_k)$ denote the associated composite step generating function. Further, let w be a weight function.*

If $\mathcal{S}$ and w satisfy Assumptions 1, then all maxima of the function $(\varphi_1,\dots,\varphi_k) \mapsto |S(e^{i\varphi_1},\dots,e^{i\varphi_k})|$ lie within the set $\{0,\pi\}^k$. The point $(\varphi_1,\dots,\varphi_k) = (0,\dots,0)$ is always a maximum.

Lemma 9 *We have the expansion*

$$S\left(e^{i\varphi_1},\dots,e^{i\varphi_k}\right) = S(1,\dots,1)\exp\left(-\Lambda\sum_{j=1}^{k}\frac{\varphi_j^2}{2}\right)\left(1 + O\left(\max_j |\varphi_j|^4\right)\right)$$

as $\max_j |\varphi_j| \to 0$, where $\Lambda = \frac{S''(1,\dots,1)}{S(1,\dots,1)} > 0$.

Proof: The expansion follows readily from Lemma 7, and short computations show that either

$$\Lambda = 2\frac{P'(2k)}{P(2k)} > 0 \qquad \text{or} \qquad \Lambda = 2^k\frac{P'(2^k)}{P(2^k)} > 0$$

corresponding to the two cases in Lemma 7. $\square$

Proof of Theorem 1: Choosing $\rho = 1$ in Lemma 3 and setting $z_j = e^{i\varphi_j}$ yields

$$P_n^+(\mathbf{u} \to \mathbf{v}) = \left(\frac{i}{\pi}\right)^k \int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi} \det_{1\le j,m\le k}\Big(\sin(u_m\varphi_j)\Big) S\left(e^{i\varphi_1},\dots,e^{i\varphi_k}\right)^n \left(\prod_{j=1}^{k} e^{-iv_j\varphi_j}\,d\varphi_j\right). \tag{8}$$

For large n, the absolute value of the integral is governed by the factor $|S(e^{i\varphi_1},\dots,e^{i\varphi_k})|^n$. By Lemma 8, the set $\mathcal{M}$ of maximal points of $(\varphi_1,\dots,\varphi_k) \mapsto |S(e^{i\varphi_i},\dots,e^{i\varphi_k})|$ is a subset of $\{0,\pi\}^k$. We are now going to prove that, for large n, the asymptotically dominant part of the integral is captured by small neighbourhoods around these maxima. Asymptotics for the integral can then be determined by saddle point techniques.

For notational convenience, we define the sets

$$\mathcal{U}_\varepsilon(\hat\varphi) = \left\{\varphi \in \mathbb{R}^k \;:\; |\hat\varphi - \varphi|_\infty < \varepsilon\right\}, \qquad \hat\varphi = (\hat\varphi_1,\dots,\hat\varphi_k) \in \mathcal{M},$$

where $\varepsilon > 0$ and $|\cdot|_\infty$ denotes the maximum norm on $\mathbb{R}^k$. We claim that the dominant asymptotic term of $P_n^+(\mathbf{u} \to \mathbf{v})$ is captured by

$$\left(\frac{i}{\pi}\right)^k \sum_{\hat{\varphi} \in \mathcal{M}} \int_{\mathcal{U}_\varepsilon(\hat{\varphi})} \cdots \int \det_{1 \le j,m \le k} \Big(\sin(u_m \varphi_j)\Big) S\left(e^{i\varphi_1}, \dots, e^{i\varphi_k}\right)^n \left(\prod_{j=1}^k e^{-iv_j\varphi_j} d\varphi_j\right), \tag{9}$$

where we choose $\varepsilon = \varepsilon(n) = n^{-5/12}$. This claim can be proved by means of the saddle point method: (1) Determine an asymptotically equivalent expression for (9) that is more convenient to work with; (2) Find a bound for the remaining part of the integral (8).

Let us start with task (1). Fix a point $\hat{\varphi} \in \mathcal{M}$ and consider the corresponding summand in the sum (9). By simple transformations, we bring this addend into the more convenient form

$$\frac{1}{\pi^k k!} \int_{\mathcal{U}_\varepsilon(\hat{\varphi})} \cdots \int \det_{1 \le j,m \le k} \Big(\sin(u_m \varphi_j)\Big) \det_{1 \le j,m \le k} \Big(\sin(v_m \varphi_j)\Big) S\left(e^{i\varphi_1}, \dots, e^{i\varphi_k}\right)^n \left(\prod_{j=1}^k d\varphi_j\right).$$

Now, a case-by-case analysis for the possible atomic step sets shows that the expression above is equal to

$$\frac{1}{\pi^k k!} \int_{\mathcal{U}_\varepsilon(\mathbf{0})} \cdots \int \det_{1 \le j,m \le k} \Big(\sin(u_m \varphi_j)\Big) \det_{1 \le j,m \le k} \Big(\sin(v_m \varphi_j)\Big) S\left(e^{i\varphi_1}, \dots, e^{i\varphi_k}\right)^n \left(\prod_{j=1}^k d\varphi_j\right).$$

Hence, we have shown that the sum (9) is equal to $|\mathcal{M}|$ times the expression above. Asymptotics for the integral above can now be determined by replacing the integrand with a sufficiently accurate Taylor series expansion. An application of Lemma 5 and Lemma 9 gives us

$$\left(\prod_{j=1}^k \frac{u_j v_j}{(2j-1)!^2}\right) \left(\prod_{1 \le j < m \le k} (u_m^2 - u_j^2)(v_m^2 - v_j^2)\right) S(1, \dots, 1)^n$$
$$\times \int_{\mathcal{U}_\varepsilon(\mathbf{0})} \cdots \int \left(\prod_{1 \le j < m \le k} (\varphi_m^2 - \varphi_j^2)\right)^2 \Big(1 + O(n^{-2/3})\Big) \left(\prod_{j=1}^k \varphi_j^2 e^{-n\Lambda\varphi_j^2/2}\right)$$

It remains to evaluate the integral

$$\int_{\mathcal{U}_\varepsilon(\mathbf{0})} \cdots \int \left(\prod_{1 \le j < m \le k} (\varphi_m^2 - \varphi_j^2)\right)^2 \left(\prod_{j=1}^k \varphi_j^2 e^{-n\Lambda\varphi_j^2/2}\right).$$

But this task is readily accomplished by noting that the integrand is even with respect to any of the variables φ_j (hence, it suffices to integrate over $[0, \varepsilon]^k$) and making the change of variables $\vartheta_j = n\Lambda\varphi_j^2/2$, which transforms the integral above into

$$\left(\frac{2}{n\Lambda}\right)^{k^2+k/2} \int_0^\infty \cdots \int_0^\infty \left(\prod_{1 \le j < m \le k} (\vartheta_m - \vartheta_j)\right)^2 \left(\prod_{j=1}^k \sqrt{\vartheta_j} e^{-\vartheta_j} d\vartheta_j\right) + O\left(e^{-(k\Lambda/2 - \delta)n^{1/6}}\right)$$

for any $\delta > 0$ as $n \to \infty$. The integral above is readily identified as a Selberg integral and it is well known (see e.g., Metha [10]) that

$$\int_0^\infty \cdots \int_0^\infty \left(\prod_{1 \le j < m \le k} (\vartheta_m - \vartheta_j) \right)^2 \left(\prod_{j=1}^{k} \sqrt{\vartheta_j} e^{-\vartheta_j} d\vartheta_j \right) = \frac{\pi^{k/2}}{2^{k^2}} k! \prod_{j=1}^{k} (2j-1)!.$$

This completes task (1). The proof is now completed by noting that for $(\varphi_1, \dots, \varphi_k) \in [-\pi, \pi] \setminus \bigcup_{\hat{\varphi} \in \mathcal{M}} \mathcal{U}_\varepsilon(\hat{\varphi})$ we have by our saddle point approximation (see Lemma 9)

$$|S(e^{i\varphi_1}, \dots, e^{i\varphi_k})| \le S(1, \dots, 1) e^{-k\Lambda n^{-5/6}/2},$$

which shows that (9) indeed captures the asymptotically dominant part of (8). □

5 Applications

In this section, we present some applications of Theorem 1. For all results presented in this section, we give comments on known results as well as links to the literature.

5.1 Lock step model of vicious walkers with wall restriction

In general, the vicious walkers model is concerned with k random walkers on a d-dimensional lattice. In the lock step model, at each time step all of the walkers move one step in any of the allowed directions, such that at no time any two random walkers share the same lattice point. This model was defined by Fisher [4] as a model for wetting and melting processes.

In this subsection, we consider a *two dimensional lock step model of vicious walkers with wall restriction*, which we briefly describe now. The only allowed steps are $(1, 1)$ and $(1, -1)$, and the lattice is the $\mathbb{Z}$-lattice spanned by these two vectors. Fix two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^k$ such that $0 < u_1 < u_2 < \cdots < u_k$ and $u_i \equiv u_j \mod 2$ for $1 \le i < j \le k$, and analogously for $\mathbf{v}$. For $1 \le j \le k$, the j-th walker starts at $(0, u_j - 1)$ and, after n steps, ends at the point $(n, v_j - 1)$ in a way such that at no time the walker moves below the horizontal axis ("the wall") or shares a lattice point with another walker. The two dimensional lock step model of vicious walkers as described above can easily be reformulated as a model of lattice paths in a Weyl chamber of type B as follows: at each time, the positions of the walkers are encoded by a k-dimensional vector, where the j-th coordinate records the current second coordinate (the *height*) of the j-th walker. Clearly, if $(c_1, \dots, c_k) \in \mathbb{Z}^k$ is such a vector encoding the heights of our walkers at a certain point in time, then we necessarily have $0 \le c_1 < c_2 < \cdots < c_k$ and $c_i \equiv c_j \mod 2$ for $1 \le i < j \le k$. Hence, each realisation of the lock step model with k vicious walkers, where the j-th walker starts at $(0, u_j - 1)$ and ends at $(n, v_j - 1)$, naturally corresponds to a lattice path in

$$\left\{ (x_1, x_2, \dots, x_k) \in \mathbb{Z}^k \ : \ 0 < x_1 < \cdots < x_k \text{ and } x_i \equiv x_j \mod 2 \text{ for } 1 \le i < j \le k \right\}$$

that starts at $\mathbf{u} = (u_1, \dots, u_k)$ and ends at $\mathbf{v} = (v_1, \dots, v_k)$. (Note the shift by $+1$.) The atomic step set is given by

$$\mathcal{A} = \left\{ \sum_{j=1}^{k} \varepsilon_j \mathbf{e}^{(j)} \ : \ \varepsilon_1, \dots, \varepsilon_k \in \{-1, +1\} \right\},$$

and the composite step set $\mathcal{S}$ is set of all sequences of length one of elements in $\mathcal{A}$. Hence, in this case we have

$$S(z_1,\dots,z_k)=\prod_{j=1}^{k}\left(z_j+\frac{1}{z_j}\right),$$

and the set $\mathcal{M}\subseteq\{0,\pi\}^k$ of points maximising the function $(\varphi_1,\dots,\varphi_k)\mapsto|S(e^{i\varphi_1},\dots,e^{i\varphi_k})|$ is given by $\mathcal{M}=\{0,\pi\}^k$. Hence, we have $|\mathcal{M}|=2^k$, and after short calculations we find $S(1,\dots,1)=S''(1,\dots,1)=2^k$. As a consequence of Theorem 1, we obtain the following result.

Corollary 1 *The number of vicious walkers of length n with k walkers that start at $(0,u_1-1),\dots,(0,u_k-1)$ and end at $(n,v_1-1),\dots,(n,v_k-1)$ (we assume that $u_1+v_1\equiv n \mod 2$) is asymptotically equal to*

$$2^{nk+3k/2}\pi^{-k/2}n^{-k^2-k/2}\frac{\left(\prod_{1\le j<m\le k}(v_m^2-v_j^2)(u_m^2-u_j^2)\right)\left(\prod_{j=1}^{k}v_ju_j\right)}{\left(\prod_{j=1}^{k}(2j-1)!\right)},\qquad \textit{as } n\to\infty.$$

The special case $u_j=2a_j+1$, $j=1,\dots,k$, of the corollary above implicitly appears in Rubey [11, Proof of Theorem 4.1, Chapter 2]. Other special instances of Corollary 1 such as for the watermelon or the star configuration can be found in [8].

5.2 Random turns model of vicious walkers with wall restriction

This model is quite similar to the lock step model of vicious walkers. The difference here is, that at each time step exactly one walker is allowed to move (all the other walkers have to stay in place). The translation of this vicious walkers model into a walk model in the Weyl chamber of type B reads exactly as in the previous case, but this time, the atomic step set is given by

$$\mathcal{A}=\left\{\pm\mathbf{e}^{(1)},\pm\mathbf{e}^{(2)},\dots,\pm\mathbf{e}^{(k)}\right\}.$$

Again, the composite step set is the set of all sequences of length one of elements in $\mathcal{A}$. Hence,

$$S(z_1,\dots,z_k)=A(z_1,\dots,z_k)=\sum_{j=1}^{k}\left(z_j+\frac{1}{z_j}\right).$$

Short calculations give us $S(1,\dots,1)=2k$ and $S''(1,\dots,1)=2$. Furthermore, it is easily checked that the set of maximal points is given by $\mathcal{M}=\{(0,\dots,0),(\pi,\dots,\pi)\}$, which implies $|\mathcal{M}|=2$. Consequently, according to Theorem 1, we have the following result.

Corollary 2 *The number of k vicious walkers in the random turns model, where the j-th walker starts at $(0,u_j-1)$ and, after n steps ends at (n,v_j-1), is asymptotically equal to*

$$2(2k)^n\left(\frac{2}{\pi}\right)^{k/2}\left(\frac{k}{n}\right)^{k^2+k/2}\frac{\left(\prod_{1\le j<m\le k}(v_m^2-v_j^2)(u_m^2-u_j^2)\right)\left(\prod_{j=1}^{k}v_ju_j\right)}{\left(\prod_{j=1}^{k}(2j-1)!\right)},\qquad n\to\infty.$$

5.3 k-non-crossing tangled diagrams with isolated points

Tangled diagrams are certain special embeddings of graphs over the vertex set $\{1, 2, \ldots, n\}$ and vertex degrees of at most two. For details we refer the reader to [2] and references therein, and instead quote the following (slightly modified) crucial observation by Chen et al. [2, Observation 2, page 3]:

> "The number of k-non-crossing tangled diagrams over $\{1, 2, \ldots, n\}$ (allowing isolated points), equals the number of simple lattice walks in $0 < x_1 < \cdots < x_{k-1}$, from the $(1, 2, \ldots, k)$ back to the this point, taking n days, where at each day the walker can either feel lazy and stay in place, or make one unit step in any (legal) direction, or else feel energetic and make any two consecutive steps (chosen randomly)."

In order to simplify the presentation, we replace k with $k + 1$, and determine asymptotics for the number of $(k+1)$-non-crossing tangled diagrams. The observation above shows that $(k+1)$-non-crossing tangled diagrams correspond to walks confined to $\mathcal{W}^0$ starting and ending in $\mathbf{u} = (1, 2, \ldots, k)$ with composite step generating function

$$S(z_1, \ldots, z_k) = 1 + \left(\sum_{j=1}^{k} z_j + \frac{1}{z_j} \right) + \left(\sum_{j=1}^{k} z_j + \frac{1}{z_j} \right)^2 .$$

Short calculations show that $S(1, \ldots, 1) = 1 + 2k + 4k^2$ and $S''(1, \ldots, 1) = 2 + 8k$, and it is easily seen that $(1, \ldots, 1)$ is the only point of maximal modulus of $S(z_1, \ldots, z_k)$ on the torus $|z_1| = \cdots = |z_k| = 1$. Consequently, Theorem 1 gives us asymptotics for the number of $(k + 1)$-non-crossing tangled diagrams.

Corollary 3 *The total number of $(k + 1)$-non-crossing tangled diagrams is asymptotically equal to*

$$P_n^+(\mathbf{u} \to \mathbf{u}) \sim (1 + 2k + 4k^2)^n \left(\frac{2}{\pi} \right)^{k/2} \left(\frac{1 + 2k + 4k^2}{n(2 + 8k)} \right)^{k^2+k/2} \left(\prod_{j=1}^{k} (2j - 1)! \right), \qquad n \to \infty.$$

5.4 k-non-crossing tangled diagrams without isolated points

A vertex of a tangled diagram is called *isolated*, if and only if its vertex degree is zero, that is, the vertex is isolated in the graph theoretical sense.

Again, for the sake of convenience, we shift k by one, and consider $(k + 1)$-non-crossing tangled diagrams without isolated points. In an analogous manner as in the previous section, these diagrams can be bijectively mapped onto a set of lattice paths (see [2, Observation 1, p.3]) in the region $0 < x_1 < \cdots < x_k$ that start and end in $\mathbf{u} = (1, 2, \ldots, k)$. The only difference to the situation described in the last example is the fact, that now the walker is not allowed to stay in place. Hence,the composite step generating function is now given by

$$S(z_1, \ldots, z_k) = \left(\sum_{j=1}^{k} z_j + \frac{1}{z_j} \right) + \left(\sum_{j=1}^{k} z_j + \frac{1}{z_j} \right)^2 ,$$

so that $S(1, \ldots, 1) = 2k+4k^2$ and $S''(1, \ldots, 1) = 2+8k$, as well as $\mathcal{M} = \{(0, \ldots, 0)\}$. Asymptotics for the number of $(k+1)$-non-crossing tangled diagrams without isolated points can now easily be determined with the help of Theorem 1.

Corollary 4 *The total number of $(k+1)$-non-crossing tangled diagrams without isolated points is asymptotically equal to*

$$P_n^+(\mathbf{u} \to \mathbf{u}) \sim (2k+4k^2)^n \left(\frac{2}{\pi}\right)^{k/2} \left(\frac{2k+4k^2}{n(2+8k)}\right)^{k^2+k/2} \left(\prod_{j=1}^{k}(2j-1)!\right), \qquad n \to \infty.$$

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