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On formulas for moments of the Wishart distributions as weighted generating functions of matchings

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Abstract. We consider the real and complex noncentral Wishart distributions. The moments of these distributions are shown to be expressed as weighted generating functions of graphs associated with the Wishart distributions. We give some bijections between sets of graphs related to moments of the real Wishart distribution and the complex noncentral Wishart distribution. By means of the bijections, we see that calculating these moments of a certain class the real Wishart distribution boils down to calculations for the case of complex Wishart distributions.

Résumé. Nous considérons les lois Wishart non-centrale réel et complexe. Les moments sont décrits comme fonctions génératrices de graphes associées avex les lois Wishart. Nous donnons bijections entre ensembles de graphes relatifs aux moments des lois Wishart non-centrale réel et complexe. Au moyen de la bijections, nous voyons que le calcul des moments d'une certaine classe la loi Wishart réel deviennent le calcul de moments de loi Wishart complexes.

Keywords: generating funtion; Hafnian; matching; moments formula; Wishart distribution.

1 Introduction

First we recall the Wishart distributions which originate from the paper by Wishart [18]. Let $X_1 = (x_{i1})_{1\leq i\leq p}$, $X_2 = (x_{i2})_{1\leq i\leq p}, \ldots, X_\nu = (x_{i\nu})_{1\leq i\leq p}$ be p-dimensional random column vectors distributed independently according to the normal (Gauss) distribution $N_p(\mu_1, \Sigma), \ldots, N_p(\mu_\nu, \Sigma)$ with mean vectors $\mu_1 = (\mu_{i1})_{1\leq i\leq p}, \ldots, \mu_\nu = (\mu_{i\nu})_{1\leq i\leq p}$ (respectively) and a common covariance matrix $\Sigma = (\sigma_{ij})$. The distribution of a $p \times p$ symmetric random matrix $W = (w_{ij})_{1\leq i,j\leq p}$ defined by $w_{ij} = \sum_{t=1}^{\nu} x_{it}x_{jt}$ is the real noncentral Wishart distribution $W_p(\nu, \Sigma, \Delta)$, where $\Delta = (\delta_{ij})_{1\leq ij\leq p}$ is the mean square matrix defined by $\delta_{ij} = \sum_{t=1}^{\nu} \mu_{it}\mu_{jt}$. The Wishart distribution for $\Delta = 0$ is said to be central and is denoted by $W_p(\nu, \Sigma)$.

The matrix $\Omega = \Sigma^{-1}\Delta$ is called the noncentrality matrix. It is usually used instead of Δ to parameterize the Wishart distribution. However, in this paper, we use (ν, Σ, Δ) for simplicity in describing formulas. The complex Wishart distribution $CW_p(\nu, \Sigma, \Delta)$ is defined as the distribution of some $p \times p$ Hermitian random matrix constructed from random vectors distributed independently according to the complex normal (complex Gauss) distributions.

The moment generating function of the real Wishart distribution is given by

$$\mathbb{E}[e^{\mathrm{tr}(\Theta W)}] = \det(I - 2\Theta\Sigma)^{-\frac{\nu}{2}} e^{-\frac{1}{2}\mathrm{tr}(I-2\Theta\Sigma)^{-1}\Theta\Delta},$$

where Θ is a $p \times p$ symmetric parameter matrix [13]. Similarly, the moment generating function of the complex Wishart distribution is given as follows:

$$\mathbb{E}[e^{\mathrm{tr}(\Theta W)}] = \det(I - \Theta\Sigma)^{-\nu} e^{-\mathrm{tr}(I-\Theta\Sigma)^{-1}\Theta\Delta},$$

where Θ is a $p \times p$ Hermitian parameter matrix. See also [2] for the central case. Our first objective is to describe the moments $\mathbb{E}[w_{i_1,i_2}w_{i_3,i_4}\cdots w_{i_{2n-1},i_{2n}}]$ of the Wishart distributions of general degrees in explicit forms. Since the Wishart distribution is one of the most important distributions, it has been studied by many researchers not only in the field of mathematical statistics but also in other fields (e.g., [1, 11]). Its moments have been well studied, and methods to calculate the moments in the central cases have been developed by Lu and Richards [10]; Graczyk, Letac, and Massam [3, 4]; Vere-Jones [16]; and many other authors. In particular, Graczyk, Letac and Massam [3, 4] developed a formula for the moments using the representation theory of symmetric group. More recently, Letac and Massam [9] introduced a method to calculate the moments of the noncentral Wishart distributions. In this paper, we introduce another formula for the moments of Wishart distribution; in our formula, the moments are described as special values of the weighted generating function of matchings of graphs. Calculation of the moments boils down to enumeration of graphs via our formulas. As an application of our formulas, we construct some correspondences between some sets of graphs, which implies several identities of moments.

The organization of this paper is as follows. In Section 2, we introduce some notations for the graphs. In Section 3, we define the generating functions of matchings and give the main formulas, which are an extension of Takemura [15] dealing with the central case. In Section 4.1, we give some correspondences between directed and undirected graphs, which implies equations between the moments of the complex Wishart distribution and the moments of the real Wishart distribution for some special parameter. In Sections 4.2 and 4.3, we consider the Wishart distribution with some degenerated parameters. We see that the calculation of its moments is reduced to enumerating graphs satisfying some conditions.

A part of this paper is taken from our previous paper [8]. Please see the paper for the omitted proofs.

2 Notation of graphs

In this paper, we consider both undirected and directed graphs. For $l \in \mathbb{Z}$, we define $\dot{l}$ and $\ddot{l}$ by $\dot{l} = 2l - 1$ and $\ddot{l} = 2l$. Let us fix $n \in \mathbb{Z}_{>0}$. We also fix sets V and V' as follows: $V = [n] = \{ 1, \dots, n \}$, $\dot{V} = [\dot{n}] = \{ \dot{1}, \dots, \dot{n} \}$, $\ddot{V} = [\ddot{n}] = \{ \ddot{1}, \dots, \ddot{n} \}$, and $V' = \dot{V} \amalg \ddot{V} = [\dot{n}] \amalg [\ddot{n}] = [2n]$. We use V and V' as the sets of vertices of directed and undirected graphs, respectively.

First we consider undirected graphs. For $v \neq w$, the undirected edge between v and w is denoted by $\{v, w\} = \{w, v\}$. We do not consider undirected self loops, i.e., $\{v, v\}$. For sets W' and U' of vertices, we define sets $K'_{U'}$ and $K'_{W',U'}$ of undirected edges by $K'_{W',U'} = \{ \{w, u\} \mid w \in W', u \in U', w \neq u \}$, $K'_{U'} = K'_{U',U'} = \{ \{v, u\} \mid v \neq u \in V' \}$. We call a pair $G' = (V', E')$ of a finite set V' and a subset $E' \subset K'_{V'}$ an undirected graph. For an undirected graph $G' = (V', E')$, we define $\mathrm{vertex}(E')$ by $\mathrm{vertex}(E') = \{ v \in V' \mid \{v, u\} \in E' \text{ for some } u \in V' \}$. Let (V', K') be an undirected graph. We call a subset $E' \subset K'$ a matching in (V', K') if no two edges in E' share a common vertex. We define $\mathcal{M}'(V', K')$ to be the set of matchings in (V', K') and $\mathcal{M}'(V')$ to be the set $\mathcal{M}'(V', K'_{V'})$ of matchings in the complete graph $(V', K'_{V'})$. A matching E' in (V', K') is said to be perfect if $\mathrm{vertex}(E') = V'$. We define $\mathcal{P}'(V', K')$ to be the set of perfect matchings in (V', K') and $\mathcal{P}'(V')$ by $\mathcal{P}'(V') = \mathcal{P}'(V', K'_{V'})$.

Next we consider directed graphs. A directed edge from v to w is denoted by (v, u). For $v \neq u$, $(v, u) \neq (u, v)$. In the case of directed graphs, we also consider a directed self loop (v, v). For a directed edge $e = (v, u)$, we respectively call v and u a starting and end points of e. For sets W and U of vertices, we define sets K_U and $K_{W,U}$ of directed edges by $K_{W,U} = \{ (v, u) \mid v \in W, u \in U \}$, $K_U = K_{U,U} = \{ (v, u) \mid v, u \in U \}$. We call a pair $G = (V, E)$ of a finite set V and a subset $E \subset K_V$ a directed graph. For a directed graph $G = (V, E)$, we define $\mathrm{start}(E)$ and $\mathrm{end}(E)$ by

$$\begin{aligned}\mathrm{start}(E) &= \{ v \in V \mid \ (v, u) \in E \text{ for some } u \in V \}, \\ \mathrm{end}(E) &= \{ u \in V \mid \ (v, u) \in E \text{ for some } v \in V \}.\end{aligned}$$

For a directed graph (V, K), we call a subset $E \subset K$ a matching in (V, K) if $v \neq v'$ and $u \neq u'$ for any two distinct directed edges (v, u) and $(v', u') \in E$. We define $\mathcal{M}(V, K)$ to be the set of matchings in (V, K) and $\mathcal{M}(V)$ by $\mathcal{M}(V) = \mathcal{M}(V, K_V)$. A matching E in (V, E) is said to be perfect if $\mathrm{start}(E) = V$ and $\mathrm{end}(E) = V$. We define $\mathcal{P}(V, K)$ to be the set of perfect matchings in (V, K) and $\mathcal{P}(V)$ by $\mathcal{P}(V) = \mathcal{P}(V, K_V)$.

Remark 2.1 *We can identify a directed graph (V, E) with a bipartite graph $(\dot{V}, \ddot{V}, \{ \{\dot{v}, \ddot{u}\} \mid (v, u) \in E \})$. i.e., a graph whose edges connect a vertex in $\dot{V}$ to a vertex in $\ddot{V}$. Via this identification, an element in $\mathcal{M}(V, K)$ is identified with a matching in the bipartite graph. In this sense, we call an element in $\mathcal{M}(V, K)$ a matching in (V, K).*

3 Weighted generating functions and moments

First we consider undirected graphs to describe the moments of real Wishart distributions. For an undirected graph (V', E') and variables $\boldsymbol{x} = (x_{i,j})$, we define the weight monomial $\boldsymbol{x}^{E'}$ by $\boldsymbol{x}^{E'} = \prod_{\{v,u\} \in E'} x_{v,u}$. If $x_{v,u} = x_{u,v}$ for $v, u \in V'$, then the weight monomial $\boldsymbol{x}^{E'}$ is well-defined. We define E'_0 to be $\left\{ \{\dot{1}, \ddot{1}\}, \ldots, \{\dot{n}, \ddot{n}\} \right\} \subset K'_{\dot{V}, \ddot{V}}$. For $E' \in \mathcal{M}'(V')$, we define $\check{E}'$ and $\mathrm{len}(E')$ by

$$\begin{aligned}\check{E}' &= \left\{ \{v, u\} \in K'_{V' \setminus \mathrm{vertex}(E')} \;\middle|\; \text{There exists a chain between } v \text{ and } u \text{ in } E' \cup E'_0. \right\} \subset K'_{V'} \\ \mathrm{len}(E') &= (\text{the number of connected components in } (V', E' \cup E'_0)) - \left|\check{E}'\right|.\end{aligned}$$

Remark 3.1 *For $E' \in \mathcal{M}'(V')$, $\check{E}'$ can be defined as a subset of $K'_{V'}$ satisfying the following conditions:*

- *$\check{E}' \in \mathcal{M}'(V')$,*
- *$\check{E}' \cap E' = \emptyset$,*
- *$\check{E}' \cup E' \in \mathcal{P}'(E')$,*
- *The number of connected components in $(V', E' \cup E'_0)$ equals the number of connected components in $(V', \check{E}' \cup E' \cup E'_0)$.*

Remark 3.2 *For $E' \in \mathcal{M}'(V')$, let us consider the undirected graph $(V', E' \amalg E'_0)$ with multiple edges. The connected components of $(V', E' \amalg E'_0)$ are chains and cycles without chords. The number of cycles in $(V', E' \amalg E'_0)$ equals $\operatorname{len}(E')$. The vertices $V' \setminus \operatorname{vertex}(E')$ which do not appear in E' are terminals of chains in $(V', E' \amalg E'_0)$. The set of pairs of terminals of chains in $(V', E' \amalg E'_0)$ equals $\check{E}'$.*

Definition 3.3 *For a set $K' \subset K'_{V'}$ of undirected edges, we define polynomials $\Phi'_{K'}$ and $\Psi'_{K'}$ by*

$$\Phi'_{K'}(t, \boldsymbol{x}, \boldsymbol{y}) = \sum_{E' \in \mathcal{M}'(V', K')} t^{\operatorname{len}(E')} \boldsymbol{x}^{E'} \boldsymbol{y}^{\check{E}'}, \qquad \Psi'_{K'}(t, \boldsymbol{x}) = \sum_{E' \in \mathcal{P}'(V', K')} t^{\operatorname{len}(E')} \boldsymbol{x}^{E'}.$$

We also respectively define $\Phi'(t, \boldsymbol{x}, \boldsymbol{y})$ and $\Psi'(t, \boldsymbol{x})$ to be $\Phi'_{K'_{V'}}(t, \boldsymbol{x}, \boldsymbol{y})$ and $\Psi'_{K'_{V'}}(t, \boldsymbol{x})$.

Remark 3.4 *By definition, $\Psi'_{K'}(t, \boldsymbol{x}) = \Phi'_{K'}(t, \boldsymbol{x}, 0)$ for each $K' \subset K'_{V'}$.*

We have the following formula that describes the moments of the real noncentral Wishart distribution as the special values of the weighted generating function.

Theorem 3.5 *Let $W = (w_{i,j}) \sim W_p(\nu, \Sigma, \Delta)$, namely, let W be a random matrix distributed according to the real noncentral Wishart distribution $W_p(\nu, \Sigma, \Delta)$. Then*

$$\mathbb{E}[w_{1,2} w_{3,4} \cdots w_{2n-1,2n}] = \mathbb{E}[w_{\dot{1},\ddot{1}} w_{\dot{2},\ddot{2}} \cdots w_{\dot{n},\ddot{n}}] = \Phi'(t, \boldsymbol{x}, \boldsymbol{y})\Big|_{t=\nu,\ x_{u,v}=\sigma_{u,v},\ y_{u,v}=\delta_{u,v}} = \Phi'(\nu, \Sigma, \Delta).$$

Corollary 3.6 *For $W \sim W_p(\nu, \Sigma, \Delta)$*

$$\mathbb{E}[w_{i_1,i_2} w_{i_3,i_4} \cdots w_{i_{2n-1},i_{2n}}] = \mathbb{E}[w_{i_{\dot{1}},i_{\ddot{1}}} w_{i_{\dot{2}},i_{\ddot{2}}} \cdots w_{i_{\dot{n}},i_{\ddot{n}}}] = \Phi'(t, \boldsymbol{x}, \boldsymbol{y})\Big|_{t=\nu,\ x_{u,v}=\sigma_{i_u,i_v},\ y_{u,v}=\delta_{i_u,i_v}}.$$

In the case where $\Delta = 0$, $W_p(\nu, \Sigma, 0)$ is called the real central Wishart distribution and is denoted by $W_p(\nu, \Sigma)$. It follows from Remark 3.4 that the moments of the central real Wishart distribution are written as special values of Ψ'.

Corollary 3.7 *For $W = (w_{i,j}) \sim W_p(\nu, \Sigma)$*

$$\mathbb{E}[w_{1,2} w_{3,4} \cdots w_{2n-1,2n}] = \mathbb{E}[w_{\dot{1},\ddot{1}} w_{\dot{2},\ddot{2}} \cdots w_{\dot{n},\ddot{n}}] = \Psi'(t, \boldsymbol{x})\Big|_{t=\nu,\ x_{u,v}=\sigma_{u,v}} = \Psi'(\nu, \Sigma).$$

Corollary 3.8 *For $W = (w_{i,j}) \sim W_p(\nu, \Sigma)$*

$$\mathbb{E}[w_{i_1,i_2} w_{i_3,i_4} \cdots w_{i_{2n-1},i_{2n}}] = \mathbb{E}[w_{i_{\dot{1}},i_{\ddot{1}}} w_{i_{\dot{2}},i_{\ddot{2}}} \cdots w_{i_{\dot{n}},i_{\ddot{n}}}] = \Psi'(t, \boldsymbol{x})\Big|_{t=\nu,\ x_{u,v}=\sigma_{i_u,i_v}}.$$

Next we consider directed graphs to describe the moments of complex Wishart distributions. For a directed graph (V, E) and variables $\boldsymbol{x} = (x_{i,j})$, we define the weight monomial $\boldsymbol{x}^E$ by $\boldsymbol{x}^E = \prod_{(v,u) \in E} x_{v,u}$. Let $E \in \mathcal{M}(V)$. The pair (V, E) is a directed graph whose connected components are directed chains and directed cycles without chords. We define $\operatorname{len}(E)$ to be the number of cycles (and self loops) in (V, E). The vertices $V \setminus \operatorname{start}(E)$ are the endpoints of the chains in (V, E), while the vertices $V \setminus \operatorname{end}(E)$ are the start points of the chains in (V, E). We define $\check{E}$ by

$$\check{E} = \left\{ (v, u) \in K_{V \setminus \operatorname{start}(E), V \setminus \operatorname{end}(E)} \;\middle|\; \text{There exists a chain from } u \text{ to } v \text{ in } E. \right\} \subset K_V.$$

Remark 3.9 *For $E \in \mathcal{M}(V)$, $\check{E}$ can be defined as a subset of K_V satisfying the following conditions:*

- $\check{E} \in \mathcal{M}(V)$,
- $\check{E} \cap E = \emptyset$,
- $\check{E} \cup E \in \mathcal{P}(E)$,
- *The number of connected components in (V, E) equals the number of connected components in $(V, \check{E} \cup E)$.*

Remark 3.10 *For $E \in \mathcal{M}(V)$, we can also define $\mathrm{len}(E)$ by*

$$\mathrm{len}(E) = (\textit{the number of connected components in } (V, E)) - \left|\check{E}\right|.$$

Remark 3.11 *We can identify $E \in \mathcal{P}(V)$ with the element σ_E of the symmetric group S_n such that $\sigma_E(i) = j$ for each $(i, j) \in E$. For each E, $\mathrm{len}(E)$ is the number of cycles of σ_E.*

Definition 3.12 *For a set $K \subset K_V$ of directed edges, we define polynomials Φ_K and Ψ_K by*

$$\Phi_K(t, \boldsymbol{x}, \boldsymbol{y}) = \sum_{E \in \mathcal{M}(V,K)} t^{\mathrm{len}(E)} \boldsymbol{x}^E \boldsymbol{y}^{\check{E}}, \qquad \Psi_K(t, \boldsymbol{x}) = \sum_{E \in \mathcal{P}(V,K)} t^{\mathrm{len}(E)} \boldsymbol{x}^E.$$

We also respectively define $\Phi(t, \boldsymbol{x}, \boldsymbol{y})$ and $\Psi(t, \boldsymbol{x})$ to be $\Phi_{K_V}(t, \boldsymbol{x}, \boldsymbol{y})$ and $\Psi_{K_V}(t, \boldsymbol{x})$.

Remark 3.13 *By definition, $\Psi_K(t, \boldsymbol{x}) = \Phi_K(t, \boldsymbol{x}, 0)$ for each $K \subset K_V$.*

We describe the moments of complex Wishart distributions as special values of the generating functions.

Theorem 3.14 *Let $W = (w_{i,j})$ be a random matrix distributed according to the complex noncentral Wishart distribution $CW_p(\nu, \Sigma, \Delta)$. Then*

$$\mathbb{E}[w_{1,2} w_{3,4} \cdots w_{2n-1,2n}] = \mathbb{E}[w_{\dot{1},\ddot{1}} w_{\dot{2},\ddot{2}} \cdots w_{\dot{n},\ddot{n}}] = \Phi(t, \boldsymbol{x}, \boldsymbol{y})\Big|_{t=\nu,\ x_{u,v}=\sigma_{\dot{u},\ddot{v}},\ y_{u,v}=\delta_{\dot{u},\ddot{v}}}.$$

Corollary 3.15 *For $W = (w_{i,j}) \sim CW_p(\nu, \Sigma, \Delta)$*

$$\mathbb{E}[w_{i_1,i_2} w_{i_3,i_4} \cdots w_{i_{2n-1},i_{2n}}] = \mathbb{E}[w_{i_{\dot{1}},i_{\ddot{1}}} w_{i_{\dot{2}},i_{\ddot{2}}} \cdots w_{i_{\dot{n}},i_{\ddot{n}}}] = \Phi(t, \boldsymbol{x}, \boldsymbol{y})\Big|_{t=\nu,\ x_{u,v}=\sigma_{i_{\dot{u}},i_{\ddot{v}}},\ y_{u,v}=\delta_{i_{\dot{u}},i_{\ddot{v}}}}.$$

By substituting 0 for Δ in the theorem, we have the following formula for the central complex case.

Corollary 3.16 *For $W = (w_{i,j}) \sim CW_p(\nu, \Sigma)$*

$$\mathbb{E}[w_{1,2} w_{3,4} \cdots w_{2n-1,2n}] = \mathbb{E}[w_{\dot{1},\ddot{1}} w_{\dot{2},\ddot{2}} \cdots w_{\dot{n},\ddot{n}}] = \Psi(t, \boldsymbol{x})\Big|_{t=\nu,\ x_{uv}=\sigma_{\dot{u},\ddot{v}}}.$$

Corollary 3.17 *For $W = (w_{i,j}) \sim CW_p(\nu, \Sigma)$*

$$\mathbb{E}[w_{i_1,i_2} w_{i_3,i_4} \cdots w_{i_{2n-1},i_{2n}}] = \mathbb{E}[w_{i_{\dot{1}},i_{\ddot{1}}} w_{i_{\dot{2}},i_{\ddot{2}}} \cdots w_{i_{\dot{n}},i_{\ddot{n}}}] = \Psi(t, \boldsymbol{x})\Big|_{t=\nu,\ x_{u,v}=\sigma_{i_{\dot{u}},i_{\ddot{v}}}}.$$

Remark 3.18 *For a square matrix $A = (a_{ij})$, the α-determinant (or α-permanent) is defined by*

$$\det_\alpha(A) = \sum_{\sigma \in S_n} \alpha^{n-\mathrm{len}(\sigma)} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

This polynomial is an α-analogue of both the determinant and the permanent. Equivalently, the α-determinant is nothing but the ordinary determinant and permanent for $\alpha = -1$ and 1, respectively. (See also [16, 17].) Through the identification in Remark 3.10, we have $\alpha^n \Psi(\alpha^{-1}, A) = \det_\alpha(A)$. Moreover, by Corollary 3.16, the moments of the complex central Wishart distribution are expressed by α-determinants.

In [12], Matsumoto introduced the α-Pfaffian, which is defined by

$$\mathrm{pf}_\alpha(A) = \sum_{E' \in \mathcal{P}'(V')} (-\alpha)^{n-\mathrm{len}(E')} \operatorname{sgn}(E') A^{E'}$$

for a skew-symmetric matrix A, where $\operatorname{sgn}(E')A^{E'}$ is defined to be $\operatorname{sgn}(x) a_{x_{\dot{1}}, x_{\ddot{1}}} \cdots a_{x_{\dot{n}}, x_{\ddot{n}}}$ for $x \in S_{2n}$ such that $E' = \{ \{x_{\dot{1}}, x_{\ddot{1}}\}, \ldots, \{x_{\dot{n}}, x_{\ddot{n}}\} \}$. Since A is skew symmetric, $\operatorname{sgn}(E')A^{E'}$ is independent from choices of $x \in S_{2n}$. The α-Pfaffian is an analogue of the Pfaffian. Equivalently, in the case when $\alpha = -1$, α-Pfaffian $\mathrm{pf}_{-1}(A)$ is nothing but the ordinary Pfaffian $\mathrm{pf}(A)$, i.e., $\sum \operatorname{sgn}(x) a_{x_{\dot{1}} x_{\ddot{1}}} \cdots a_{x_{\dot{n}} x_{\ddot{n}}}$.

Let us define the polynomial $\mathrm{hf}_\alpha(A)$ by

$$\mathrm{hf}_\alpha(B) = \sum_{E' \in \mathcal{P}'(V')} \alpha^{n-\mathrm{len}(E')} B^{E'}$$

for a symmetric matrix B. The polynomial $\mathrm{hf}_\alpha(B)$ is an α-analogue of the Hafnian. Equivalently, $\mathrm{hf}_\alpha(B)$ is the ordinary Hafnian $\mathrm{hf}(B)$, i.e., $\sum b_{x_{\dot{1}} x_{\ddot{1}}} \cdots b_{x_{\dot{n}} x_{\ddot{n}}}$, for $\alpha = 1$. By definition, $\mathrm{hf}_\alpha(B) = \alpha^n \Psi'(\alpha^{-1}, B)$. In this sense, the moments of the real central Wishart distributions are expressed by α-Hafnians.

4 Application

4.1 Relation between real and complex cases

There exist bijections between directed graphs and undirected graphs which preserve the weight monomials in some special cases. These bijections induce equations between weighted generating functions of matchings. From the equations, we can obtain some formulas for the moments of complex and real Wishart distributions with special parameters.

4.1.1 Prototypical case

As in Remark 2.1, there exists a correspondence between directed graphs and bipartite graphs.

Lemma 4.1 *The map $\mathcal{M}(V, K_V) \ni (u, v) \mapsto \{\dot{u}, \ddot{v}\} \in \mathcal{M}'(V', K'_{\dot{V},\ddot{V}})$ is a bijection. The map induces the bijection $\mathcal{P}(V, K_V) \ni (u, v) \mapsto \{\dot{u}, \ddot{v}\} \in \mathcal{P}'(V', K'_{\dot{V},\ddot{V}})$.*

These bijections imply $\Phi_{K_V}(t,\boldsymbol{x},\boldsymbol{y}) = \Phi'_{K'_{\dot V,\ddot V}}(t,\boldsymbol{x},\boldsymbol{y})$ and $\Psi_{K_V}(t,\boldsymbol{x}) = \Psi'_{K'_{\dot V,\ddot V}}(t,\boldsymbol{x})$. If $\Sigma' = (\sigma'_{uv})$ and $\Delta' = (\delta'_{uv})$ satisfy $\sigma'_{u,v} = 0$ and $\delta'_{u,v} = 0$ for $\{u,v\} \in K'_{\dot V} \cup K'_{\ddot V}$, then

$$\Phi'(t,\boldsymbol{x},\boldsymbol{y})\Big|_{t=\nu,\ x_{u,v}=\sigma'_{u,v},\ y_{u,v}=\delta'_{u,v}} = \Phi'_{K'_{\dot V,\ddot V}}(t,\boldsymbol{x},\boldsymbol{y})\Big|_{t=\nu,\ x_{u,v}=\sigma'_{u,v},\ y_{u,v}=\delta'_{u,v}}.$$

If $\Sigma = (\sigma_{uv})$ and $\Delta = (\delta_{uv})$ satisfy $\sigma_{\dot u,\ddot v} = \sigma'_{u,v}$ and $\delta_{\dot u,\ddot v} = \delta'_{u,v}$ for $u,v \in V$, then the equation implies

$$\Phi(t,\boldsymbol{x},\boldsymbol{y})\Big|_{t=\nu,\ x_{u,v}=\sigma_{\dot u,\ddot v},\ y_{u,v}=\delta_{\dot u,\ddot v}} = \Phi'(t,\boldsymbol{x},\boldsymbol{y})\Big|_{t=\nu,\ x_{u,v}=\sigma_{u,v},\ y_{u,v}=\delta_{u,v}}.$$

Hence we have the following:

Propsition 4.2 *Let $\Sigma' = (\sigma'_{u,v})$, $\Delta' = (\delta'_{u,v})$, $\Sigma = (\sigma_{u,v})$ and $\Delta = (\delta_{u,v})$ satisfy $\sigma'_{u,v} = 0$, $\delta'_{u,v} = 0$ for $\{u,v\} \in K'_{\dot V} \cup K'_{\ddot V}$, and $\sigma_{\dot u,\ddot v} = \sigma'_{\dot u,\ddot v}$, $\delta_{\dot u,\ddot v} = \delta'_{\dot u,\ddot v}$ for $u,v \in V$. For $W = (w_{u,v}) \sim CW_p(\nu,\Sigma,\Delta)$ and $W' = (w'_{u,v}) \sim W_p(\nu,\Sigma',\Delta')$, $\mathbb{E}[w_{\dot 1,\ddot 1}\cdots w_{\dot n,\ddot n}] = \mathbb{E}[w'_{\dot 1,\ddot 1}\cdots w'_{\dot n,\ddot n}]$.*

4.1.2 Central case

Next we consider the central Wishart distribution. In this case, we may consider only perfect matchings. We define $\widetilde{\mathcal{P}}'(V')$ and $\widetilde{\mathcal{P}}(V)$ by

$$\widetilde{\mathcal{P}}'(V') = \left\{ (E',\omega') \;\middle|\; \substack{E'\in\mathcal{P}'(V'),\\ \omega' :\ \{\text{ cycles in } (V',\, E\amalg E_0)\ \}\to\{\ \pm 1\ \}} \right\}, \qquad \widetilde{\mathcal{P}}(V) = \left\{ (E,\omega) \;\middle|\; \substack{E\in\mathcal{P}(V),\\ \omega:\ E\to\{\ \pm 1\ \}} \right\}.$$

Lemma 4.3 *There exists a bijection between $\widetilde{\mathcal{P}}'(V')$ and $\widetilde{\mathcal{P}}(V)$.*

We shall give a bijection ψ between $\widetilde{\mathcal{P}}'(V')$ and $\widetilde{\mathcal{P}}(V)$ in Section 4.1.4. The bijection preserves the weight monomials, equivalently, $t^{\mathrm{len}(E)}\boldsymbol{x}^E = t^{\mathrm{len}(E')}(\boldsymbol{x}')^{E'}$ for elements $E' \in \mathcal{P}'(V)$ corresponding to $E \in \mathcal{P}(V)$, in the case when $\boldsymbol{x} = (x_{u,v})$ and $\boldsymbol{x}' = (x'_{u,v})$ satisfy $x'_{u',v'} = x_{u,v}$ for any $u,v \in V$ and any $\{u',v'\} \in K_{\{\ \dot u\ddot v\ \},\{\ \dot u\ddot v\ \}}$. Hence Proposition 4.4 follows from the following equations:

$$\begin{aligned} 2^n\Psi(t,\boldsymbol{x}) &= 2^n \sum_{E\in\mathcal{P}(V)} t^{\mathrm{len}(E)}\boldsymbol{x}^E = \sum_{E\in\widetilde{\mathcal{P}}(V)} t^{\mathrm{len}(E)}\boldsymbol{x}^E, \\ \Psi'(2t,\boldsymbol{x}) &= 2^n \sum_{E'\in\mathcal{P}'(V')} (2t)^{\mathrm{len}(E')}\boldsymbol{x}^{E'} = \sum_{E'\in\widetilde{\mathcal{P}}(V')} t^{\mathrm{len}(E')}\boldsymbol{x}^{E'}. \end{aligned}$$

Propsition 4.4 *Let $\Sigma = (\sigma_{u,v})$ and $\Sigma' = (\sigma'_{u,v})$ satisfy $\sigma'_{u',v'} = \sigma_{u,v}$ for any $u,v \in V$ and any $\{u',v'\} \in K_{\{\ \dot u\ddot v\ \},\{\ \dot u\ddot v\ \}}$. Then*

$$2^n\Psi(t,\boldsymbol{x})\Big|_{t=\nu,\ x_{u,v}=\sigma_{\dot u,\ddot v}} = \Psi'(2t,\boldsymbol{x})\Big|_{t=\nu,\ x_{u,v}=\sigma'_{u,v}}.$$

Corollary 4.5 *Let $\Sigma = (\sigma_{u,v})$ and $\Sigma' = (\sigma'_{u,v})$ satisfy $\sigma'_{u',v'} = \sigma_{u,v}$ for any $u,v \in V$ and any $\{u',v'\} \in K_{\{\ \dot u\ddot v\ \},\{\ \dot u\ddot v\ \}}$. For $W = (w_{u,v}) \sim CW_p(\nu,\Sigma)$ and $W' = (w'_{u,v}) \sim W_p(\nu,\Sigma')$, $\mathbb{E}[w_{\dot 1,\ddot 1}\cdots w_{\dot n,\ddot n}] = \mathbb{E}[w'_{\dot 1,\ddot 1}\cdots w'_{\dot n,\ddot n}]$.*

4.1.3 Noncentral case

Next we consider the noncentral Wishart distribution. In this case, we consider all matchings in complete graphs. We define $\widetilde{\mathcal{M}}(V)$ and $\widetilde{\mathcal{M}}'(V')$ by

$$\widetilde{\mathcal{M}}'(V') = \left\{ (E', \omega') \;\middle|\; \substack{E' \in \mathcal{M}'(V'), \\ \omega' \colon \{\text{ cycles in } (V', E' \cup E'_0) \} \to \{\pm 1\}} \right\}, \quad \widetilde{\mathcal{M}}(V) = \left\{ (E, \omega) \;\middle|\; \substack{E \in \mathcal{M}(V), \\ \omega \colon E \to \{\pm 1\}} \right\}.$$

Lemma 4.6 *There exists a bijection between $\widetilde{\mathcal{M}}(V)$ and $\widetilde{\mathcal{M}}'(V')$.*

We shall give a bijection between $\widetilde{\mathcal{P}}'(V')$ and $\widetilde{\mathcal{P}}(V)$ in Section 4.1.4. The bijection preserves the weight monomial in a special case. Hence Proposition 4.7 follows from the following equations:

$$\Phi'(2t, \boldsymbol{x}, \boldsymbol{y}) = \sum_{E' \in \mathcal{M}'(V')} (2t)^{\operatorname{len}(E')} \boldsymbol{x}^{E'} \boldsymbol{y}^{\check{E}'} = \sum_{(E', \omega') \in \widetilde{\mathcal{M}}'(V')} t^{\operatorname{len}(E')} \boldsymbol{x}^{E'} \boldsymbol{y}^{\check{E}'},$$

$$\Phi(t, 2\boldsymbol{x}, \boldsymbol{y}) = \sum_{E \in \mathcal{M}(V)} t^{\operatorname{len}(E)} (2\boldsymbol{x})^{E} \boldsymbol{y}^{\check{E}} = \sum_{(E, \omega) \in \widetilde{\mathcal{M}}(V)} t^{\operatorname{len}(E)} \boldsymbol{x}^{E} \boldsymbol{y}^{\check{E}},$$

where $2\boldsymbol{x} = (2x_{u,v})$

Propsition 4.7 *Let $\Sigma = (\sigma_{u,v})$, $\Delta = (\delta_{u,v})$, $\Sigma' = (\sigma'_{u,v})$ and $\Delta' = (\delta'_{u,v})$ satisfy $\sigma'_{u',v'} = \sigma_{u,v}$ $\delta'_{u',v'} = \delta_{u,v}$ for any $u, v \in V$ and any $\{u', v'\} \in K_{\{\dot{u}\ddot{v}\},\{\dot{u}\ddot{v}\}}$. Then*

$$\Psi(t, 2\boldsymbol{x}, \boldsymbol{y})\Big|_{t=\nu,\ x_{u,v}=\sigma_{\dot{u},\ddot{v}},\ y_{u,v}=\delta_{\dot{u},\ddot{v}}} = \Psi'(2t, \boldsymbol{x}, \boldsymbol{y})\Big|_{t=\nu,\ x_{u,v}=\sigma_{u,v},\ y_{u,v}=\delta_{u,v}}.$$

Corollary 4.8 *Let $\Sigma = (\sigma_{u,v})$, $\Delta = (\delta_{u,v})$, $\Sigma' = (\sigma'_{u,v})$ and $\Delta' = (\delta'_{u,v})$ satisfy $\sigma'_{u',v'} = \sigma_{u,v}$ $\delta'_{u',v'} = \delta_{u,v}$ for any $u, v \in V$ and any $\{u', v'\} \in K_{\{\dot{u}\ddot{v}\},\{\dot{u}\ddot{v}\}}$. For $W = (w_{u,v}) \sim CW_p(\nu, 2\Sigma, \Delta)$ and $W' = (w'_{u,v}) \sim W_p(2\nu, \Sigma', \Delta')$, $\mathbb{E}[w_{\dot{1},\ddot{1}} \cdots w_{\dot{n},\ddot{n}}] = \mathbb{E}[w'_{\dot{1},\ddot{1}} \cdots w'_{\dot{n},\ddot{n}}]$.*

4.1.4 Construction of Bijections

Here we construct bijections to prove Lemmas 4.3 and 4.6. First we construct a bijection ψ from $\widetilde{\mathcal{P}}(V)$ to $\widetilde{\mathcal{P}}'(V')$. To define the bijection, we define the following map. For $(E, \omega) \in \widetilde{\mathcal{P}}(V)$, let $h_{E,\omega}$ and $h'_{E,\omega}$ be maps from V to V' defined by

$$h_{E,\omega}(v) = \begin{cases} \dot{v} & \text{if } \omega((u,v)) = 1 \text{ for some } (u,v) \in E, \\ \ddot{v} & \text{otherwise,} \end{cases}$$

$$h'_{E,\omega}(v) = \begin{cases} \ddot{v} & \text{if } \omega((u,v)) = 1 \text{ for some } (u,v) \in E, \\ \dot{v} & \text{otherwise.} \end{cases}$$

Remark 4.9 *For $(E, \omega) \in \widetilde{\mathcal{P}}(V)$ and $v \in V$, $\{h_{E,\omega}(v), h'_{E,\omega}(v)\} \in E'_0$.*

First we construct $E' \in \mathcal{P}'(V')$ for each $(E, \omega) \in \widetilde{\mathcal{P}}(V)$. For each $(E, \omega) \in \widetilde{\mathcal{P}}(V)$, we define a surjection $\psi_{E,\omega} \colon E \to K'_{V'}$, and then we define E' to be the image $\psi_{E,\omega}(E)$. Let (v_1, v_2), (v_2, v_3),..., (v_{k-1}, v_k),

(v_k, v_1) be a cycle in E such that $v_1 = \min\{ v_1, \dots, v_k \}$. For each directed edge in the cycle, we define $\psi_{E,\omega}$ by

$$\begin{aligned} \psi_{E,\omega}((v_1, v_2)) &= \{\dot{v}_1, h_{E,\omega}(v_2)\}, \\ \psi_{E,\omega}((v_i, v_{i+1})) &= \{h'_{E,\omega}(v_{i+1}), h_{E,\omega}(v_{i+1})\} \quad (\text{for } i = 2, \dots, k-1), \\ \psi_{E,\omega}((v_k, v_1)) &= \{h'_{E,\omega}(v_k), \ddot{v}_1\}. \end{aligned}$$

Then the image of the cycle forms a cycle C' in the undirected graph $E' \amalg E'_0$. For the cycle C', we define $\omega'(C')$ to be $\omega((v_k, v_1))$.

Remark 4.10 *It is easy to construct the inverse map of ψ, which implies that ψ is bijective.*

Remark 4.11 *This correspondence ψ is equivalent to the one in [4], which is described in more algebraic terms. Let S_{2m} be the $2m$-th symmetric group, and let $B_m = S_m \wr \mathbb{Z}/2\mathbb{Z}$ the hyperoctehedral group, i.e., the subgroup of the permutations $\pi \in S_{2m}$ such that $|\pi(\dot{n}) - \pi(\ddot{n})| = 1$ for all $n = 1, \dots, m$. For $gB_m \in S_{2m}/B_m$, we can define E_{gB_m} by $E_{gB_m} = \{ \{g(\dot{n}), g(\ddot{n})\} \mid n = 1, \dots m \} \in \mathcal{P}(V)$, and we can identify elements $gB_m \in S_{2m}/B_m$ with perfect matchings in (V, K_V). Through this identification, the correspondence in Section 4 of [4] is equivalent to ours.*

Next we construct a bijection φ from $\widetilde{\mathcal{M}}(V)$ to $\widetilde{\mathcal{M}}'(V')$. For each $(E, \omega) \in \widetilde{\mathcal{M}}(V)$, we shall define $(E', \omega') \in \widetilde{\mathcal{M}}(V)$. For each cycle in E, we construct undirected edges and ω' in the same manner as ψ. To define the undirected edges corresponding to chains, we define $t_{E,\omega}$ and $t'_{E,\omega}$ by

$$t_{E,\omega}(v) = \begin{cases} \dot{v} & \text{if } \omega((u,v)) = 1 \text{ for some } (v,u) \in E, \\ \ddot{v} & \text{otherwise,} \end{cases}$$

$$t'_{E,\omega}(v) = \begin{cases} \ddot{v} & \text{if } \omega((u,v)) = 1 \text{ for some } (v,u) \in E, \\ \dot{v} & \text{otherwise.} \end{cases}$$

Let (v_1, v_2), (v_2, v_3),…,(v_{k-1}, v_k) be a maximal chain in E. If $v_1 < v_k$, then we define $\varphi_{E,\omega}$ by

$$\begin{aligned} \varphi_{E,\omega}((v_1, v_2)) &= \{\dot{v}_1, h_{E,\omega}(v_2)\}, \\ \varphi_{E,\omega}((v_i, v_{i+1})) &= \{h'_{E,\omega}(v_i), h_{E,\omega}(v_{i+1})\} \quad (\text{for } i = 2, \dots, k-1). \end{aligned}$$

If $v_1 > v_k$, then we define $\varphi_{E,\omega}$ by

$$\begin{aligned} \varphi_{E,\omega}((v_{k-1}, v_k)) &= \{t_{E,\omega}(k_{k-1}), \ddot{v}_k\} \\ \varphi_{E,\omega}((v_{i-1}, v_i)) &= \{t_{E,\omega}(v_{i-1}), t'_{E,\omega}(v_i)\} \quad (\text{for } i = k-1, \dots, 2). \end{aligned}$$

Then the image of the maximal chains forms a maximal chain in the undirected graph $E' \amalg E'_0$.

Remark 4.12 *It is easy to construct the inverse map of φ, which implies φ is bijective.*

4.2 Noncentral chi-square distribution

In this section, we consider the Wishart distributions for a special parameter, which is linked with the noncentral chi-square distributions.

Propsition 4.13 *Let $\sigma_{u,v} = \sigma$, $\delta_{u,v} = \delta$. For $W = (w_{u,v}) \sim CW_p(\nu, \Sigma, \Delta)$,*

$$\mathbb{E}[w_{\dot{1},\ddot{1}} w_{\dot{2},\ddot{2}} \cdots w_{\dot{n},\ddot{n}}] = \sum_{m=0}^{n} \sum_{l} g_{lmn} \nu^l \sigma^m \delta^{n-m},$$

where g_{lmn} is the number of $E \subset K_V$ such that $\operatorname{len}(E) = l$, $|E| = m$ and $\left|\check{E}\right| = n - m$.

Corollary 4.14 *For the noncentral complex chi-square distribution $\chi^2_\nu(\delta)$ with ν degrees of freedom and the noncentrality parameter δ, its n-th moment $\mathbb{E}(w^n)$ is given as follows:*

$$\mathbb{E}[w^n] = \sum_{m=0}^{n} \sum_{l} g_{lmn} \nu^l \delta^{n-m}.$$

In the case where we add a new directed edge whose staring point is a fixed vertex, we have just one choice of end-points that increase the number of cycles. Hence we obtain Lemma 4.15.

Lemma 4.15 *Let $0 \leq m \leq n$. Then the generating function $G_{mn}(t)$ of g_{lmn} with respect to the number l of cycles satisfies*

$$G_{mn}(t) = \sum_{l \geq 0} g_{lmn} t^l = \binom{n}{m} \prod_{i=1}^{m} (t + n - i).$$

We also obtain the following corollary, which is well-known expression for the noncentral chi-square distribution (e.g. [5])

Corollary 4.16 *For the n-th moment $\mathbb{E}(w^n)$ of the noncentral chi-square distribution $\chi^2_\nu(\delta)$ with ν degrees of freedom and the noncentrality parameter δ,*

$$\mathbb{E}[w^n] = \sum_{m=0}^{n} \sum_{l} g_{lmn} \nu^l \delta^{n-m} = \sum_{m=0}^{n} G_{mn}(\nu) \delta^{n-m} = \sum_{m=0}^{n} \binom{n}{m} \delta^{n-m} \prod_{i=1}^{m} (\nu + n - i).$$

Remark 4.17 *The numbers $s_n(m, l)$ defined by the following generating function are called the noncentral Stirling numbers of the first kind:*

$$\sum_{l} s_n(m, l) t^l = \prod_{i=1}^{m} (t + n - i).$$

If $m = n$, then $s_n(m, l)$ is the Stirling number of the first kind. Lemma 4.15 implies that $g_{lmn} = \binom{n}{m} s_n(m, l)$. Equivalently, we can explicitly describe the moments of the noncentral chi-square distribution $\chi^2_\nu(\delta)$ with the noncentral Stirling numbers. Koutras pointed out that moments of some noncentral distributions are described with the noncentral Stirling numbers of the first kind [7].

Next consider the real case.

Propsition 4.18 *Let $\sigma_{u,v} = \sigma$, $\delta_{u,v} = \delta$. For $W = (w_{u,v}) \sim W_p(\nu, \Sigma, \Delta)$,*

$$\mathbb{E}[w_{\dot{1},\ddot{1}} w_{\dot{2},\ddot{2}} \cdots w_{\dot{n},\ddot{n}}] = \sum_{m=0}^{n} \sum_{l} g'_{lmn} \nu^l \sigma^m \delta^{n-m},$$

where g'_{lmn} is the number of $E' \subset K'_{V'}$ such that $\operatorname{len}(E') = l$, $|E'| = m$ and $\left|\check{E}'\right| = n - m$.

Corollary 4.19 *For the noncentral chi-square distribution $\chi^2_\nu(\delta)$ with ν degrees of freedom and the noncentrality parameter δ, its n-th moment $\mathbb{E}(w^n)$ is given as:*

$$\mathbb{E}[w^n] = \sum_{m=0}^{n} \sum_{l} g'_{lmn} \nu^l \delta^{n-m},$$

where g'_{lmn} is the number of $E' \subset K'_{V'}$ such that $\operatorname{len}(E') = l$, $|E'| = m$ and $\left|\check{E}'\right| = n - m$.

Proposition 4.7 and Lemma 4.15 imply Lemma 4.20.

Lemma 4.20 *Let $0 \le m \le n$, $n \le 0$. Then the generating function $G'_{mn}(t)$ of g'_{lmn} with respect to the number l of cycles satisfies*

$$G'_{mn}(t) = \sum_{l \ge 0} g'_{lmn} t^l = \binom{n}{m} \prod_{i=1}^{m} (t + 2(n-i)).$$

Corollary 4.21 *For the n-th moment $\mathbb{E}(w^n)$ of the noncentral chi-square distribution $\chi^2_\nu(\delta)$ with ν degrees of freedom and the noncentrality parameter δ,*

$$\mathbb{E}[w^n] = \sum_{m=0}^{n} \sum_{l} g'_{lmn} \nu^l \delta^{n-m} = \sum_{m=0}^{n} G'_{mn}(\nu) \delta^{n-m} = \sum_{m=0}^{n} \binom{n}{m} \delta^{n-m} \prod_{i=1}^{m} (\nu + 2(n-i)).$$

4.3 Bivariate chi-square distribution

We can explicitly describe the moments of Wishart distributions by enumerating the matchings satisfying some conditions. For example, in Proposition 4.23, we obtain the description of the moments of the bivariate real chi-square distribution, which was introduced by Kibble [6]. The formulas imply formulas for the complex distribution by Proposition 4.7. See [8] for details and other applications.

Propsition 4.22 *Let $\Sigma = (\sigma_{uv})$ and $\Delta = (\delta_{uv})$ satisfy*

$$\sigma_{u,v} = \begin{cases} 1 & (u, v \le 2b \text{ or } 2b+1 \le u, v), \\ \rho & (\text{otherwise}), \end{cases} \qquad \delta_{u,v} = 0.$$

For a random matrix $W = (w_{u,v}) \sim W_{b+c}(\nu, \Sigma, \Delta)$,

$$\begin{aligned} &\mathbb{E}[w_{\dot{1},\ddot{1}} \cdots w_{\dot{b},\ddot{b}} \cdot w_{(\dot{b+1}),(\ddot{b+1})} \cdots w_{(\dot{b+c}),(\ddot{b+c})}] \\ &= \sum_{a=0}^{\min(b,c)} \rho^{2a} \frac{2^a b! c!}{(b-a)!(c-a)!a!} \prod_{i=1}^{a} (\nu + a(a-i)) \prod_{i=1}^{b-a} (\nu + a(b-i)) \prod_{i=1}^{c-a} (\nu + a(c-i)). \end{aligned}$$

Propsition 4.23 *Let $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, and $W = (w_{u,v}) \sim W_2(\nu, \Sigma)$. For $b, c \in \mathbb{Z}_{\ge 0}$,*

$$\mathbb{E}[w_{1,1}^b w_{2,2}^c] = \sum_{a=0}^{\min(b,c)} \rho^{2a} \frac{2^a b! c!}{(b-a)!(c-a)!a!} \prod_{i=1}^{a} (\nu + 2(a-i)) \prod_{i=1}^{b-a} (\nu + 2(b-i)) \prod_{i=1}^{c-a} (\nu + 2(c-i)).$$

Remark 4.24 *In [14], Nadarajah and Kotz derived another expression for $\mathbb{E}[w_{1,1}^b w_{2,2}^c]$ with the Jacobi polynomials.*

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