

# On the diagonal ideal of $(\mathbb{C}^2)^n$ and $q, t$ -Catalan numbers

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**Abstract.** Let  $I_n$  be the (big) diagonal ideal of  $(\mathbb{C}^2)^n$ . Haiman proved that the  $q, t$ -Catalan number is the Hilbert series of the graded vector space  $M_n = \bigoplus_{d_1, d_2} (M_n)_{d_1, d_2}$  spanned by a minimal set of generators for  $I_n$ . We give simple upper bounds on  $\dim (M_n)_{d_1, d_2}$  in terms of partition numbers, and find all bi-degrees  $(d_1, d_2)$  such that  $\dim (M_n)_{d_1, d_2}$  achieve the upper bounds. For such bi-degrees, we also find explicit bases for  $(M_n)_{d_1, d_2}$ .

**Résumé.** Soit  $I_n$  l'idéal de la (grande) diagonale de  $(\mathbb{C}^2)^n$ . Haiman a démontré que le  $q, t$ -nombre de Catalan est la série de Hilbert de l'espace vectoriel gradué  $M_n = \bigoplus_{d_1, d_2} (M_n)_{d_1, d_2}$  engendré par un ensemble minimal de générateurs de  $I_n$ . Nous obtenons des bornes supérieures simples pour  $\dim (M_n)_{d_1, d_2}$  en termes de nombres de partitions, ainsi que tous les bi-degrés  $(d_1, d_2)$  pour lesquels ces bornes supérieures sont atteintes. Pour ces bi-degrés, nous trouvons aussi des bases explicites de  $(M_n)_{d_1, d_2}$ .

**Keywords:**  $q, t$ -Catalan number, diagonal ideal

## 1 introduction

### 1.1 Background

The goal of this paper is to study the  $q, t$ -Catalan numbers and the (thick) diagonal ideal in  $(\mathbb{C}^2)^n$ , and discuss some technique that we have developed recently.

Let  $n$  be a positive integer. Consider the set of  $n$ -tuples  $\{(x_i, y_i)\}_{1 \leq i \leq n}$  in the plane  $\mathbb{C}^2$ . They form an affine space  $(\mathbb{C}^2)^n$  with coordinate ring  $\mathbb{C}[\mathbf{x}, \mathbf{y}] = \mathbb{C}[x_1, y_1, \dots, x_n, y_n]$ . There is a natural symmetric group  $S_n$  acting on  $\mathbb{C}[\mathbf{x}, \mathbf{y}]$  by permuting the coordinates in  $\mathbf{x}, \mathbf{y}$  simultaneously. With this group action, a polynomial  $f \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$  is called *alternating* if

$$\sigma(f) = \text{sgn}(\sigma)f \quad \text{for all } \sigma \in S_n.$$

Define  $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$  to be the vector space of alternating polynomials in  $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ .

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There is a more combinatorial way to describe  $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$ . Denote by  $\mathbb{N}$  the set of nonnegative integers. Let  $\mathfrak{D}_n$  be the set of  $n$ -tuples  $D = \{(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\} \subset \mathbb{N} \times \mathbb{N}$ . For  $D \in \mathfrak{D}_n$ , define

$$\Delta(D) := \det \begin{bmatrix} x_1^{\alpha_1} y_1^{\beta_1} & x_1^{\alpha_2} y_1^{\beta_2} & \dots & x_1^{\alpha_n} y_1^{\beta_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\alpha_1} y_n^{\beta_1} & x_n^{\alpha_2} y_n^{\beta_2} & \dots & x_n^{\alpha_n} y_n^{\beta_n} \end{bmatrix}$$

Then  $\{\Delta(D)\}_{D \in \mathfrak{D}_n}$  forms a basis for the  $\mathbb{C}$ -vector space  $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$ .

It is easy to see that any alternating polynomial vanishes on the thick diagonal of  $(\mathbb{C}^2)^n$ . (By thick diagonal we mean the set of  $n$ -tuples of points in  $\mathbb{C}^2$  where at least two points coincide.) A theorem of Haiman asserts that the converse is also true: any polynomial that vanishes on the diagonal of  $(\mathbb{C}^2)^n$  can be generated by alternating polynomials, i.e.

$$\bigcap_{1 \leq i < j \leq n} (x_i - x_j, y_i - y_j) = \text{ideal generated by } \Delta(D)\text{'s.}$$

We call the above ideal the **diagonal ideal** and denote it by  $I_n$ . the number of minimal generators of  $I_n$ , which is the same as the dimension of the vector space  $M_n = I_n/(\mathbf{x}, \mathbf{y})I_n$ , is equal to the  $n$ -th Catalan number. The space  $M_n$  is doubly graded as  $\oplus_{d_1, d_2} (M_n)_{d_1, d_2}$ . The  $q, t$ -Catalan number, originally introduced by A.M.Garsia and M.Haiman in [4], can be defined as

$$C_n(q, t) = \sum_{d_1, d_2} t^{d_1} q^{d_2} \dim(M_n)_{d_1, d_2}.$$

The  $q, t$ -Catalan number  $C_n(q, t)$  also has a combinatorial interpretation using Dyck paths. To be more precise, take the  $n \times n$  square whose southwest corner is  $(0, 0)$  and northeast corner is  $(n, n)$ . Let  $\mathcal{D}_n$  be the collection of Dyck paths, i.e. lattice paths from  $(0, 0)$  to  $(n, n)$  that proceed by NORTH or EAST steps and never go below the diagonal. For any Dyck path  $\Pi$ , let  $a_i(\Pi)$  be the number of squares in the  $i$ -th row that lie in the region bounded by  $\Pi$  and the diagonal. A.M.Garsia and J.Haglund ([2], [3]) among others showed that

$$C_n(q, t) = \sum_{\Pi \in \mathcal{D}_n} q^{\text{area}(\Pi)} t^{\text{dinv}(\Pi)},$$

where

$$\text{dinv}(\Pi) := |\{(i, j) \mid i < j \text{ and } a_i(\Pi) = a_j(\Pi)\}| + |\{(i, j) \mid i < j \text{ and } a_i(\Pi) + 1 = a_j(\Pi)\}|.$$

Haiman posed a question asking for a rule that associate to each Dyck path  $\Pi$  an element  $D(\Pi) \in \mathfrak{D}_n$  such that  $\deg_x \Delta(D(\Pi)) = \text{area}(\Pi)$ ,  $\deg_y \Delta(D(\Pi)) = \text{dinv}(\Pi)$ , and that the set  $\{\Delta(D(\Pi))\}$  generates  $I_n$ . The last condition is equivalent to requiring the images  $\{\overline{\Delta(D(\Pi))}\}$  form a basis of  $M_n$ . It is natural to ask the following more general question:

**Question 1.1** Given a bi-degree  $(d_1, d_2)$ , is there a combinatorially significant construction of the basis for each  $(M_n)_{d_1, d_2}$ ?

### 1.2 Main results

This paper initiates the approach to the study of  $M_n$  by comparing it with  $M_{n'}$  for a large integer  $n'$ . On the one hand, there is a natural map  $M_n \rightarrow M_{n'}$  for any  $n' > n$ . On the other hand, for  $n'$  sufficiently large, the basis of  $(M_{n'})_{d'_1, d_2}$  becomes “stable” if we fix  $d_2$  and fix

$$k = \binom{n}{2} - d_1 - d_2 = \binom{n'}{2} - d'_1 - d_2.$$

Therefore we can take the “limit” of such basis for  $n' \rightarrow \infty$ . This basis is indexed by the partitions of  $k$ . As a consequence,  $(M_{n'})_{d'_1, d_2}$  can be imbedded as a subspace of the polynomial ring with infinite many variables  $\mathbb{C}[\rho_1, \rho_2, \dots]$ . The induced map

$$\bar{\varphi} : (M_n)_{d_1, d_2} \rightarrow \mathbb{C}[\rho_1, \rho_2, \dots],$$

which will be defined explicitly in subsection 1.2.3, provides a powerful tool to study  $M_n$ .

#### 1.2.1 Asymptotic behavior when $k \ll n$

We shall show that if  $k \ll n$ , then  $(M_n)_{d_1, d_2}$  has a basis  $\{\overline{\Delta(D)}\}$  where  $D$  are so-called **minimal staircase forms** that will be defined later.

The essential step is to observe the following three linear relations that turn the questions into combinatorial games. First we introduce some notations.

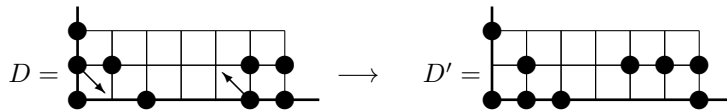
- For  $D = \{P_1, \dots, P_n\} \in \mathfrak{D}_n$  where  $P_i = (\alpha_i, \beta_i)$ , define  $|P_i| = \alpha_i + \beta_i$ .

*Relation 1.* Given positive integers  $1 \leq i \neq j \leq n$  such that  $|P_i| = i - 1, |P_{i+1}| = i, |P_j| = j - 1, |P_{j+1}| = j, \beta_i > 0, \alpha_j > 0$  (we assume  $|P_{n+1}| = n$ ). Let  $D'$  be obtained from  $D$  by moving  $P_i$  to southeast and  $P_j$  to northwest, i.e.

$$D' = \{P_1, \dots, P_{i-1}, P_i + (1, -1), P_{i+1}, \dots, P_{j-1}, P_j + (-1, 1), P_{j+1}, \dots, P_n\}.$$

Then  $\overline{\Delta(D)} = \overline{\Delta(D')}$ .

*Example:*  $n = 9, i = 2, j = 6$ .

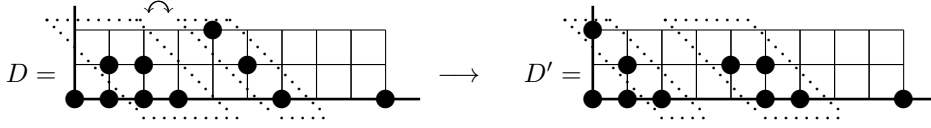


*Relation 2.* Given positive integers  $h, \ell$  and  $m$  such that  $2 \leq h < h + \ell + m \leq n + 1, |P_h| = h - 1, |P_{h+\ell}| = h + \ell - 1, |P_{h+\ell+m}| = h + \ell + m - 1$  (by convention, the last equality holds if  $h + \ell + m = n + 1$ ) and  $\alpha_{h+\ell}, \dots, \alpha_{h+\ell+m-1} \geq \ell$ . Let  $D'$  be obtained from  $D$  by moving the  $m$  points  $P_{h+\ell}, \dots, P_{h+\ell+m-1}$  to the left by  $\ell$  units and moving the  $\ell$  points  $P_h, \dots, P_{h+\ell-1}$  to the right by  $m$  units, i.e.

$$D' = \{P_1, P_2, \dots, P_{h-1}, P_{h+\ell} - (\ell, 0), P_{h+\ell+1} - (\ell, 0), \dots, P_{h+\ell+m-1} - (\ell, 0), P_h + (m, 0), P_{h+1} + (m, 0), \dots, P_{h+\ell-1} + (m, 0), P_{h+\ell+m}, \dots, P_n\}.$$

Then  $\overline{\Delta(D)} = \overline{\Delta(D')}$ .

Example:  $n = 10, h = 3, \ell = 4, m = 3$ .



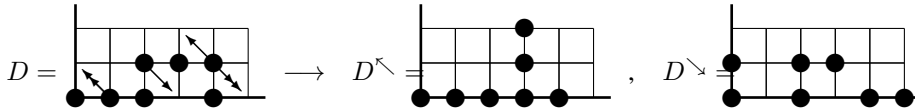
**Relation 3.** Given positive integers  $j$  and  $s$ . Suppose  $P_{s_0}$  is the last point in  $D$  satisfying  $|P_i| = i - 1$ . Define  $j = (s_0 - 1 - |P_{s_0}|) + (s_0 - |P_{s_0+1}|) + \dots + (n - 1 - |P_n|)$ . Suppose  $|P_i| = i - 1$  for  $1 \leq i \leq j + 2$ ,  $P_2 = (1, 0)$ ,  $s_0 \leq s \leq n$ , and  $\alpha_s, \beta_s \geq 1$ . Let

$$D^{\nwarrow} = \{P_1, \dots, P_{j+1}, P_{j+2} + (1, -1), P_{j+3}, \dots, P_{s-1}, P_s + (-1, 1), P_{s+1}, \dots, P_n\},$$

$$D^{\searrow} = \{P_1, (0, 1), P_3, \dots, P_{s-1}, P_s + (1, -1), P_{s+2}, \dots, P_n\}.$$

Then  $2\overline{\Delta(D)} = \overline{\Delta(D^{\nwarrow})} + \overline{\Delta(D^{\searrow})}$ .

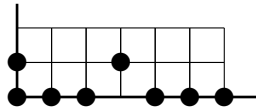
Example:  $n = 9, i = 2, j = 6$ .



We call  $D = \{P_1, \dots, P_n\}$  a *minimal staircase form* if  $|P_i| = i - 1$  or  $i - 2$  for every  $1 \leq i \leq n$ . For a minimal staircase form  $D$ , let  $\{i_1 < i_2 < \dots < i_\ell\}$  be the set of  $i$ 's such that  $|P_i| = i - 1$ , we define the *partition type* of  $D$  to be the partition of  $\binom{n}{2} - \sum |P_i|$  consisting of all the positive integers in the sequence

$$(i_1 - 1, i_2 - i_1 - 1, i_3 - i_2 - 1, \dots, i_\ell - i_{\ell-1} - 1, n - i_\ell).$$

Example: Let  $n = 8$  and  $D = \{P_1, \dots, P_8\}$  satisfying  $(|P_1|, \dots, |P_8|) = (0, 1, 1, 2, 4, 4, 5, 6)$ . Then  $D$  is a minimal staircase form. The set  $\{i \mid |P_i| = i - 1\}$  equals  $\{1, 2, 5\}$ . The positive integers in the sequence  $(1 - 1, 2 - 1 - 1, 5 - 2 - 1, 8 - 5)$  are  $(2, 3)$ , so the partition type of  $D$  is  $(2, 3)$ .



Let  $p(k)$  denote the number of partitions of an integer  $k$  and  $\Pi_k$  denote the set of partitions of  $k$ .

**Theorem 1.2** Let  $k$  be any positive integer. There are positive constants  $c_1 = 8k + 5, c_2 = 2k + 1$  such that the following holds:

For integers  $n, d_1, d_2$  satisfying  $n \geq c_1, d_1 \geq c_2 n, d_2 \geq c_2 n$  and  $d_1 + d_2 = \binom{n}{2} - k$ , the vector space  $(M_n)_{d_1, d_2}$  has dimension  $p(k)$ , and the  $p(k)$  elements

$$\{ \text{a minimal staircase form of bi-degree } (d_1, d_2) \text{ and of partition type } \mu \}_{\mu \in \Pi_k}$$

form a basis of  $(M_n)_{d_1, d_2}$ .

Note that N.Bergeron and Z.Chen have found explicit bases for  $(M_n)_{d_1, d_2}$  for certain bi-degrees using a different method [1].

### 1.2.2 For arbitrary $k$ and $n$

Denote by  $p(k)$  the partition number of  $k$  and by convention  $p(0) = 1$  and  $p(k) = 0$  for  $k < 0$ . Denote by  $p(b, k)$  the partition number of  $k$  into no more than  $b$  parts, and by convention  $p(0, k) = 0$  for  $k > 0$ ,  $p(b, 0) = 1$  for  $b \geq 0$ . One of our main results is as follows.

**Theorem 1.3** *Let  $d_1, d_2$  be non-negative integers  $d_1, d_2$  with  $d_1 + d_2 \leq \binom{n}{2}$ . Define  $k = \binom{n}{2} - d_1 - d_2$  and  $\delta = \min(d_1, d_2)$ . Then the coefficient of  $q^{d_1}t^{d_2}$  in  $C_n(q, t)$  is less than or equal to  $p(\delta, k)$ , and the equality holds if and only if one the following conditions holds:*

- $k \leq n - 3$ , or
- $k = n - 2$  and  $\delta = 1$ , or
- $\delta = 0$ .

This theorem is a consequence of Theorem C. It contains [8, Theorem 6] and a result of N.Bergeron and Z.Chen [1, Corollary 8.3.1] as special cases. In fact it proves [8, Conjecture 8]. We feel that the coefficient of  $q^{d_1}t^{d_2}$  for general  $k$  can also be expressed in terms of partition numbers, only that the expression might be complicated. For example, we give the following conjecture which is verified for  $6 \leq n \leq 10$ .

**Conjecture.** Let  $n, d_1, d_2, \delta, k$  be as in Theorem 1.3. If  $n - 2 \leq k \leq 2n - 8$  and  $\delta \geq k$ , then the coefficient of  $q^{d_1}t^{d_2}$  in  $C_n(q, t)$  equals

$$p(k) - 2[p(0) + p(1) + \dots + p(k - n + 1)] - p(k - n + 2).$$

As a corollary of Theorem 1.3 , we can compute some higher degree terms of the specialization at  $t = q$ .

**Corollary 1.4**

$$C_n(q, q) = \sum_{k=0}^{n-3} \left( p(k) \left( \binom{n}{2} - 3k + 1 \right) + 2 \sum_{i=1}^{k-1} p(i, k) \right) q^{\binom{n}{2}-k} + (\text{lower degree terms}).$$

The following theorem immediately implies Theorem 1.3.

**Theorem 1.5** *Let  $d_1, d_2$  be non-negative integers  $d_1, d_2$  with  $d_1 + d_2 \leq \binom{n}{2}$ . Define  $k = \binom{n}{2} - d_1 - d_2$  and  $\delta = \min(d_1, d_2)$ . Then  $\dim(M_n)_{d_1, d_2} \leq p(\delta, k)$ , and the equality holds if and only if one the following conditions holds:*

- $k \leq n - 3$ , or
- $k = n - 2$  and  $\delta = 1$ , or
- $\delta = 0$ .

*In case the equality holds, there is an explicit construction of a basis of  $(M_n)_{d_1, d_2}$ .*

The idea of the construction of the basis in the above theorem consists of two parts:

(1) Prove that

$$\dim(M_n)_{d_1, d_2} \leq p(\delta, k)$$

using a new characterization of  $q, t$ -Catalan numbers. The characterization is as follows, and is discovered independently by A. Woo [10].

Let  $\mathfrak{D}_n^{\text{catalan}}$  be the set consisting of  $D \subset \mathbb{N} \times \mathbb{N}$ , where  $D$  contains  $n$  points satisfying the following conditions.

- (a) If  $(p, 0) \in D$  then  $(i, 0) \in D, \forall i \in [0, p]$ .
- (b) For any  $p \in \mathbb{N}$ ,

$$\#\{j \mid (p + 1, j) \in D\} + \#\{j \mid (p, j) \in D\} \geq \max\{j \mid (p, j) \in D\} + 1.$$

(If  $\{j \mid (p, j) \in D\} = \emptyset$ , then we require that no point  $(i, j) \in D$  satisfies  $i \geq p$ .) Denote by  $\deg_x D$  (resp.  $\deg_y D$ ) the sum of the first (resp. second) components of the  $n$  points in  $D$ .

**Proposition 1.6** *The coefficient of  $q^{d_1} t^{d_2}$  in the  $q, t$ -Catalan number  $C_n(q, t)$  is equal to*

$$\#\{D \in \mathfrak{D}_n^{\text{catalan}} \mid \deg_x D = d_1, \deg_y D = d_2\}.$$

(2) Construct a set of  $p(\delta, k)$  linearly independent elements in  $(M_n)_{d_1, d_2}$ . It seems difficult (as least to the authors) to test directly whether a given set of elements in  $(M_n)_{d_1, d_2}$  are linearly independent. We define a map  $\varphi$  sending an alternating polynomial  $f \in \mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$  to a polynomial ring

$$\mathbb{C}[\rho] := \mathbb{C}[\rho_1, \rho_2, \rho_3, \dots].$$

The map has two desirable properties: (i) for many  $f$ ,  $\varphi(f)$  can be easily computed, and (ii) for each bi-degree  $(d_1, d_2)$ ,  $\varphi$  induces a morphism  $\bar{\varphi} : (M_n)_{d_1, d_2} \rightarrow \mathbb{C}[\rho]$  of  $\mathbb{C}$ -modules. Then we use the fact the linear dependency is easier to check in  $\mathbb{C}[\rho]$  than in  $(M_n)_{d_1, d_2}$ . The map  $\varphi$  is defined as below.

### 1.2.3 Maps $\varphi$ and $\bar{\varphi}$ .

(a) Define the map  $\varphi : \mathfrak{D}_n \rightarrow \mathbb{Z}[\rho]$  as follows. Let  $D = \{(a_1, b_1), \dots, (a_n, b_n)\} \in \mathfrak{D}_n, k = \binom{n}{2} - \sum_{i=1}^n (a_i + b_i)$ , and define

$$\varphi(D) := (-1)^k \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n \left( \sum \rho_{w_1} \rho_{w_2} \cdots \rho_{w_{b_i}} \right),$$

where  $(w_1, \dots, w_{b_i})$  in the sum  $\sum \rho_{w_1} \rho_{w_2} \cdots \rho_{w_{b_i}}$  runs through the set

$$\{(w_1, \dots, w_{b_i}) \in \mathbb{N}^{b_i} \mid w_1 + \dots + w_{b_i} = \sigma(i) - 1 - a_i - b_i\},$$

with the convention that

$$\sum \rho_{w_1} \cdots \rho_{w_{b_i}} = \begin{cases} 0 & \text{if } \sigma(i) - 1 - a_i - b_i < 0; \\ 0 & \text{if } b_i = 0 \text{ and } \sigma(i) - 1 - a_i - b_i > 0; \\ 1 & \text{if } b_i = 0 \text{ and } \sigma(i) - 1 - a_i - b_i = 0. \end{cases}$$

(b) Here is an equivalent definition of  $\varphi(D)$ . Define the weight of  $\rho_i$  to be  $i$  for  $i \in \mathbb{N}^+$  and define the weight of  $\rho_0 = 1$  to be 0. Naturally the weight of any monomial  $c\rho_{i_1}\dots\rho_{i_n}$  ( $c \in \mathbb{Z}$ ) is defined to be  $i_1 + \dots + i_n$ . For  $w \in \mathbb{N}$  and a power series  $f \in \mathbb{Z}[[\rho_1, \rho_2, \dots]]$ , denote by  $\{f\}_w$  the sum of terms of weight- $w$  in  $f$ , which is a polynomial. Define

$$h(b, w) := \{(1 + \rho_1 + \rho_2 + \dots)^b\}_w, \quad b \in \mathbb{N}, w \in \mathbb{Z}.$$

Naturally  $h(b, w) = 0$  if  $w < 0$ . Also assume  $(1 + \rho_1 + \rho_2 + \dots)^0 = 1$ . Then

$$\varphi(D) = (-1)^k \begin{vmatrix} h(b_1, -|P_1|) & h(b_1, 1 - |P_1|) & h(b_1, 2 - |P_1|) & \dots & h(b_1, n - 1 - |P_1|) \\ h(b_2, -|P_2|) & h(b_2, 1 - |P_2|) & h(b_2, 2 - |P_2|) & \dots & h(b_2, n - 1 - |P_2|) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h(b_n, -|P_n|) & h(b_n, 1 - |P_n|) & h(b_n, 2 - |P_n|) & \dots & h(b_n, n - 1 - |P_n|) \end{vmatrix}.$$

(c) Let  $D_1, \dots, D_\ell \in D'$  be of the same bi-degree and  $\sum_{i=1}^\ell c_i D_i$  be the formal sum for any  $c_i \in \mathbb{C}$  ( $1 \leq i \leq \ell$ ). Define

$$\varphi\left(\sum_{i=1}^\ell c_i D_i\right) := \sum_{i=1}^\ell c_i \varphi(D_i).$$

For any bi-homogeneous alternating polynomials  $f = \sum_{i=1}^\ell c_i \Delta(D_i) \in \mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$ , we define

$$\varphi(f) := \varphi\left(\sum_{i=1}^\ell c_i D_i\right) = \sum_{i=1}^\ell c_i \varphi(D_i)$$

by abuse of notation. □

**Proposition 1.7** Fix any pair of nonnegative integers  $(d_1, d_2)$ , the map  $\varphi$  induces a well-defined linear map

$$\bar{\varphi} : (M_n)_{d_1, d_2} \longrightarrow \mathbb{C}[\rho].$$

Moreover, this map  $\bar{\varphi}$  is conjecturally injective. And our future work is to generalizing it to the case  $I_n^m / (\mathbf{x}, \mathbf{y}) I_n^m$  for any positive integer  $m$ .

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