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Valuative invariants for polymatroids

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Abstract. Many important invariants for matroids and polymatroids, such as the Tutte polynomial, the Billera-Jia-Reiner quasi-symmetric function, and the invariant $\mathcal{G}$ introduced by the first author, are valuative. In this paper we construct the $\mathbb{Z}$-modules of all $\mathbb{Z}$-valued valuative functions for labeled matroids and polymatroids on a fixed ground set, and their unlabeled counterparts, the $\mathbb{Z}$-modules of valuative invariants. We give explicit bases for these modules and for their dual modules generated by indicator functions of polytopes, and explicit formulas for their ranks. Our results confirm a conjecture of the first author that $\mathcal{G}$ is universal for valuative invariants.

Résumé. Beaucoup des invariants importants des matroïdes et polymatroïdes, tels que le polynôme de Tutte, la fonction quasi-symmetrique de Billera-Jia-Reiner, et l'invariant $\mathcal{G}$ introduit par le premier auteur, sont valuatifs. Dans cet article nous construisons les $\mathbb{Z}$-modules de fonctions valuatives aux valeurs entières des matroïdes et polymatroïdes étiquetés définis sur un ensemble fixe, et leurs équivalents pas étiquetés, les $\mathbb{Z}$-modules des invariants valuatifs. Nous fournissons des bases des ces modules et leurs modules duels, engendrés par fonctions charactéristiques des polytopes, et des formules explicites donnants leurs rangs. Nos résultats confirment une conjecture du premier auteur, que $\mathcal{G}$ soit universel pour les invariants valuatifs.

Keywords: polymatroids, polymatroid polytopes, decompositions, valuations

1 Introduction

Matroids were introduced by Whitney in 1935 (see [22]) as a combinatorial abstraction of linear dependence of vectors in a vector space. Some standard references are [21] and [17]. Polymatroids are multiset analogs of matroids and appeared in the late 1960s (see [9, 13]). There are many distinct but equivalent definitions of matroids and polymatroids, for example in terms of bases, independent sets, flats, polytopes or rank functions. For polymatroids, the equivalence between the various definitions is given in [13]. We will stick to the definition in terms of rank functions:

Definition 1.1 *Suppose that X is a finite set (the* ground set*) and* $\mathrm{rk} : 2^X \to \mathbb{N} = \{0, 1, 2, \dots\}$*, where 2^X is the set of subsets of X. Then (X, rk) is called a* polymatroid *if:*

1. $\mathrm{rk}(\emptyset) = 0$*;*
2. rk *is* weakly increasing*: if $A \subseteq B$ then $\mathrm{rk}(A) \leq \mathrm{rk}(B)$;*

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3. rk *is* submodular*:* $\mathrm{rk}(A \cup B) + \mathrm{rk}(A \cap B) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$ *for all* $A, B \subseteq X$.

If moreover $\mathrm{rk}(\{x\}) \leq 1$ *for every* $x \in X$*, then* (X, rk) *is called a* matroid*.*

An *isomorphism* $\varphi : (X, \mathrm{rk}_X) \to (Y, \mathrm{rk}_Y)$ is a bijection $\varphi : X \to Y$ such that $\mathrm{rk}_Y \circ \varphi = \mathrm{rk}_X$. Every polymatroid is isomorphic to a polymatroid with ground set $\underline{d} = \{1, 2, \ldots, d\}$ for some nonnegative integer d. The rank of a polymatroid (X, rk) is $\mathrm{rk}(X)$.

Our matroid notations will receive the subscript $_{\mathrm{M}}$, and our polymatroid notations the subscript $_{\mathrm{PM}}$. We will write $_{(\mathrm{P})\mathrm{M}}(d, r)$ when we want to refer to both in parallel.

Let $S_{(\mathrm{P})\mathrm{M}}(d, r)$ be the set of all (poly)matroids with ground set $\underline{d}$ of rank r, A function f on $S_{(\mathrm{P})\mathrm{M}}(d, r)$ is a (poly)matroid *invariant* if $f\big((\underline{d}, \mathrm{rk})\big) = f\big((\underline{d}, \mathrm{rk}')\big)$ whenever $(\underline{d}, \mathrm{rk})$ and $(\underline{d}, \mathrm{rk}')$ are isomorphic. Let $S^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d, r)$ be the set of isomorphism classes in $S_{(\mathrm{P})\mathrm{M}}(d, r)$. Invariant functions on $S_{(\mathrm{P})\mathrm{M}}(d, r)$ correspond to functions on $S^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d, r)$. Let $Z_{(\mathrm{P})\mathrm{M}}(d, r)$ and $Z^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d, r)$ be the $\mathbb{Z}$-modules freely generated by the symbols $\langle \mathrm{rk} \rangle$ for rk in $S_{(\mathrm{P})\mathrm{M}}(d, r)$ and $S^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d, r)$ respectively. For an abelian group A, every function $f : S^{(\mathrm{sym})}_{(\mathrm{P})\mathrm{M}}(d, r) \to A$ extends uniquely to a group homomorphism $Z^{(\mathrm{sym})}_{(\mathrm{P})\mathrm{M}}(d, r) \to A$.

To a (poly)matroid $(\underline{d}, \mathrm{rk})$ one can associate its *base polytope* $Q(\mathrm{rk})$ in $\mathbb{R}^d$ (see Definition 2.2). For $d \geq 1$, the dimension of this polytope is $\leq d - 1$. The indicator function of a polytope $\Pi \subseteq \mathbb{R}^d$ is denoted by $[\Pi] : \mathbb{R}^d \to \mathbb{Z}$. Let $P_{(\mathrm{P})\mathrm{M}}(d, r)$ be the $\mathbb{Z}$-module generated by all $[Q(\mathrm{rk})]$ with $(\underline{d}, \mathrm{rk}) \in S_{(\mathrm{P})\mathrm{M}}(d, r)$. We also define an analogue $P^{\mathrm{sym}}_{\mathrm{P(M)}}(d, r)$ by a certain pushout (see Section 6).

Definition 1.2 *Suppose that* A *is an abelian group. A function* $f : S_{(\mathrm{P})\mathrm{M}}(d, r) \to A$ *is* strongly valuative *if there exists a group homomorphism* $\widehat{f} : P_{(\mathrm{P})\mathrm{M}}(d, r) \to A$ *such that for all* $(\underline{d}, \mathrm{rk}) \in S_{(\mathrm{P})\mathrm{M}}(d, r)$*,*

$$f\big((\underline{d}, \mathrm{rk})\big) = \widehat{f}([Q(\mathrm{rk})]).$$

Many interesting functions on matroids are valuative. Among these is the Tutte polynomial, one of the most important matroid invariants [5, 7]. Other valuative functions on matroids include the quasi-symmetric function $\mathcal{F}$ for matroids of Billera, Jia and Reiner introduced in [3], and the first author's quasi-symmetric function $\mathcal{G}$ introduced in [8]. Speyer's invariant defined in [19] using K-theory is strictly speaking not valuative, but its composition with a certain automorphism of $Z^{\mathrm{sym}}_{*\mathrm{M}}(d, r)$ is valuative. Valuative invariants and additive invariants can be useful for deciding whether a given matroid polytope has a decomposition into smaller matroid polytopes (see the discussion in [3, Section 7]). Matroid polytope decompositions appeared in the work of Lafforgue ([14, 15]) on compactifications of a fine Schubert cell in the Grassmannian associated to a matroid.

It follows from Definition 1.2 that the dual $P_{(\mathrm{P})\mathrm{M}}(d, r)^\vee = \mathrm{Hom}_{\mathbb{Z}}(P_{(\mathrm{P})\mathrm{M}}(d, r), \mathbb{Z})$ is the space of all $\mathbb{Z}$-valued valuative functions on $S_{(\mathrm{P})\mathrm{M}}(d, r)$. Likewise $P^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d, r)^\vee$ is the space of all $\mathbb{Z}$-valued valuative invariants. Let $p^{(\mathrm{sym})}_{(\mathrm{P})\mathrm{M}}(d, r)$ be the rank of $P^{(\mathrm{sym})}_{(\mathrm{P})\mathrm{M}}(d, r)$.

We will give explicit bases for each of the spaces $P_{(\mathrm{P})\mathrm{M}}(d, r)$ and $P^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d, r)$ and their duals (see Theorems 5.1,6.1, Corollaries 5.3, 6.2). From these we obtain the following formulas:

Theorem 1.3

a. $p^{\mathrm{sym}}_{\mathrm{M}}(d, r) = \binom{d}{r}$ *and* $\displaystyle\sum_{0 \leq r \leq d} p^{\mathrm{sym}}_{\mathrm{M}}(d, r) x^{d-r} y^r = \frac{1}{1 - x - y}.$

b. $p^{\mathrm{sym}}_{\mathrm{PM}}(d,r) = \begin{cases} \binom{r+d-1}{r} & \text{if } d \geq 1 \text{ or } r \geq 1; \\ 1 & \text{if } d = r = 0 \end{cases}$ *and* $\sum_{r=0}^{\infty}\sum_{d=0}^{\infty} p^{\mathrm{sym}}_{\mathrm{PM}}(d,r)x^d y^r = \frac{1-x}{1-x-y}$.

c. $\sum_{0\leq r\leq d} \frac{p_{\mathrm{M}}(d,r)}{d!}x^{d-r}y^r = \frac{x-y}{xe^{-x}-ye^{-y}}$.

d. $p_{\mathrm{PM}}(d,r) = \begin{cases} (r+1)^d - r^d & \text{if } d \geq 1 \text{ or } r \geq 1; \\ 1 & \text{if } d = r = 0, \end{cases}$ *and* $\sum_{d=0}^{\infty}\sum_{r=0}^{\infty} \frac{p_{\mathrm{PM}}(d,r)x^d y^r}{d!} = \frac{e^x(1-y)}{1-ye^x}$.

In particular, the following theorem, which is a corollary of Corollary 6.3, proves a conjecture of the first author in [8]:

Theorem 1.4 *The invariant $\mathcal{G}$ is universal for all valuative (poly)matroid invariants, i.e., the coefficients of $\mathcal{G}$ span the vector space of all valuative (poly)matroid invariants with values in $\mathbb{Q}$.*

Definition 1.5 *Suppose that $d > 0$. A valuative function $f : S_{(\mathrm{P})\mathrm{M}}(d,r) \to A$ is said to be* additive*, if $f\big((\underline{d}, \mathrm{rk})\big) = 0$ whenever the dimension of $Q(\mathrm{rk})$ is $< d-1$.*

In Sections 8 and 9 we construct bigraded modules $T_{(\mathrm{P})\mathrm{M}}$ and $T^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}$ such that $T_{(\mathrm{P})\mathrm{M}}(d,r)^\vee$ is the space of all additive functions on $S_{(\mathrm{P})\mathrm{M}}(d,r)$ and $T^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)^\vee$ is the space of all additive invariants. Let $t_{(\mathrm{P})\mathrm{M}}(d,r)$ be the rank of $T_{(P)M}(d,r)$ and $t^{\mathrm{sym}}_{(P)M}(d,r)$ be the rank of $T^{\mathrm{sym}}_{(P)M}(d,r)$. Then we have the following formulas:

Theorem 1.6

a. $\prod_{0\leq r\leq d}(1-x^{d-r}y^r)^{t^{\mathrm{sym}}_{\mathrm{M}}(d,r)} = 1-x-y.$

b. $\prod_{r,d}(1-x^d y^r)^{t^{\mathrm{sym}}_{\mathrm{PM}}(d,r)} = \frac{1-x-y}{1-y}$.

c. $\sum_{r,d} \frac{t_{\mathrm{M}}(d,r)}{d!}x^{d-r}y^r = \log\left(\frac{x-y}{xe^{-x}-ye^{-y}}\right)$.

d. $t_{\mathrm{PM}}(d,r) = \begin{cases} r^{d-1} & \text{if } d \geq 1 \\ 0 & \text{if } d = 0, \end{cases}$ *and* $\sum_{r,d} \frac{t_{\mathrm{PM}}(d,r)}{d!}x^d y^r = \log\big(\frac{e^x(1-y)}{1-ye^x}\big)$.

2 Polymatroids and their polytopes

By a *polyhedron* we will mean a finite intersection of closed half-spaces. A *polytope* is a bounded polyhedron. It is convenient to have a polyhedral analogue of polymatroid polytopes, so we make the following definition.

Definition 2.1 *A function $2^X \to \mathbb{Z} \cup \{\infty\}$ is called a* megamatroid *if it has the following properties:*

1. $\mathrm{rk}(\emptyset) = 0$;
2. $\mathrm{rk}(X) \in \mathbb{Z}$;

3. rk *is* submodular*: if* $\mathrm{rk}(A), \mathrm{rk}(B) \in \mathbb{Z}$*, then* $\mathrm{rk}(A \cup B), \mathrm{rk}(A \cap B) \in \mathbb{Z}$ *and* $\mathrm{rk}(A \cup B) + \mathrm{rk}(A \cap B) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$.

Obviously, every matroid is a polymatroid, and every polymatroid is a megamatroid. The *rank* of a megamatroid (X, rk) is the integer $\mathrm{rk}(X)$. We will use notations for megamatroids analogous to our notations for (poly)matroids but with the subscript $_{\mathrm{MM}}$. We will use the subscript $_{*\mathrm{M}}$ and say "$*$matroid" when we want to refer to megamatroids or polymatroids or matroids in parallel.

Definition 2.2 *For a megamatroid* $(\underline{d}, \mathrm{rk})$*, we define its* base polyhedron $Q(\mathrm{rk})$ *as the set of all* $(y_1, \ldots, y_d) \in \mathbb{R}^d$ *such that* $y_1 + y_2 + \cdots + y_d = \mathrm{rk}(X)$ *and* $\sum_{i \in A} y_i \leq \mathrm{rk}(A)$ *for all* $A \subseteq X$.

If rk is a polymatroid then $Q(\mathrm{rk})$ is a polytope, called the *base polytope* of rk. In fact $Q(\mathrm{rk})$ is always nonempty. The base polytope of a polymatroid $(\underline{d}, \mathrm{rk})$ of rank r is contained in the simplex

$$\Delta_{\mathrm{PM}}(d, r) = \{(y_1, \ldots, y_d) \in \mathbb{R}^d \mid y_1, \ldots, y_d \geq 0,\ y_1 + y_2 + \cdots + y_d = r\}$$

and the base polytope of a matroid $(\underline{d}, \mathrm{rk})$ of rank r is contained in the hypersimplex

$$\Delta_{\mathrm{M}}(d, r) = \{(y_1, \ldots, y_d) \in \mathbb{R}^d \mid 0 \leq y_1, \ldots, y_d \leq 1,\ y_1 + y_2 + \cdots + y_d = r\}.$$

The next theorem generalises a theorem of Gelfand-Goresky-MacPherson-Serganova [10] on matroids.

Theorem 2.3 *A convex polyhedron contained in* $y_1 + \cdots + y_d = r$ *equals* $Q(M)$ *for some megamatroid* M *if and only if for every face* F *of* Π*, the linear hull* $\mathrm{lhull}(F)$ *is of the form* $z + W$ *where* $z \in \mathbb{Z}^d$ *and* W *is spanned by vectors of the form* $e_i - e_j$.

Polymatroid polyhedra are, up to translation, the lattice polytopes among the *generalized permutohedra* of [18] or the *submodular rank tests* of [16].

3 The valuative property

There are essentially two definitions of the valuative property in the literature, which we will refer to as the *strong valuative* and the *weak valuative* properties. The equivalence of these definitions is shown in [12] and [20] when valuations are defined on sets of polyhedra closed under intersection. We show their equivalence for megamatroid polytopes, which are not such a set.

Definition 3.1 *A* megamatroid polyhedron decomposition *is a decomposition* $\Pi = \Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_k$ *such that* $\Pi, \Pi_1, \ldots, \Pi_k$ *are megamatroid polyhedra, and* $\Pi_i \cap \Pi_j$ *is empty or contained in a proper face of* Π_i *and of* Π_j *for all* $i \neq j$.

A megamatroid polyhedron decomposition $\Pi = \Pi_1 \cup \cdots \cup \Pi_k$ is a *(poly)matroid polytope decomposition* if $\Pi, \Pi_1, \ldots, \Pi_k$ are (poly)matroid polytopes.

For a megamatroid polyhedron decomposition $\Pi = \Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_k$ we define $\Pi_I = \bigcap_{i \in I} \Pi_i$ if $I \subseteq \{1, 2, \ldots, k\}$, and $\Pi_\emptyset = \Pi$. Define

$$m_{\mathrm{val}}(\Pi; \Pi_1, \ldots, \Pi_k) = \sum_{I \subseteq \{1,2,\ldots,k\}} (-1)^{|I|} m_I \in Z_{\mathrm{MM}}(d, r),$$

where $m_I = \langle \mathrm{rk}^I \rangle$ if rk^I is the megamatroid with $Q(\mathrm{rk}^I) = \Pi_I$, and $m_I = 0$ if $\Pi_I = \emptyset$.

Definition 3.2 *A homomorphism of abelian groups $f : Z_{*\mathrm{M}}(d,r) \to A$ is called* weakly valuative *if for every megamatroid polyhedron decomposition $\Pi = \Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_k$ we have*

$$f(m_{\mathrm{val}}(\Pi; \Pi_1, \ldots, \Pi_k)) = 0.$$

For a polyhedron Π in $\mathbb{R}^d$, let $[\Pi]$ denote its indicator function. Define $P_{\mathrm{MM}}(d,r)$ as the $\mathbb{Z}$-module generated by all $[Q(\mathrm{rk})]$, where rk lies in $S_{\mathrm{MM}}(d,r)$. There is a natural $\mathbb{Z}$-module homomorphism

$$\Psi_{*\mathrm{M}} : Z_{*\mathrm{M}}(d,r) \to P_{*\mathrm{M}}(d,r)$$

such that $\Psi_{*\mathrm{M}}(\langle \mathrm{rk} \rangle) = [Q(\mathrm{rk})]$ for all $\mathrm{rk} \in S_{*\mathrm{M}}(d,r)$.

Definition 3.3 *A homomorphism of groups $f : Z_{*\mathrm{M}}(d,r) \to A$ is* strongly valuative *if there exists a group homomorphism $\widehat{f} : P_{*\mathrm{M}}(d,r) \to A$ such that $f = \widehat{f} \circ \Psi_{*\mathrm{M}}$.*

The map $\Psi_{*\mathrm{M}}$ has the weak valuative property, which shows that the strong valuative property implies the weak valuative property. In fact the two valuative properties are equivalent, and in view of this we may speak of *the* valuative property.

Theorem 3.4 *A homomorphism $f : Z_{*\mathrm{M}}(d,r) \to A$ of abelian groups is weakly valuative if and only if it is strongly valuative.*

4 Decompositions into cones

A chain of length $k =: \mathrm{length}(\underline{X})$ in $\underline{d}$ is $\underline{X} : \emptyset \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k = \underline{d}$ (here $\subset$ denotes proper inclusion). If $d > 0$ then every chain has length ≥ 1, but for $d = 0$ there is exactly one chain, namely $\emptyset = \underline{0}$, and this chain has length 0. For a chain $\underline{X}$ of length k and a k-tuple $\underline{r} = (r_1, r_2, \ldots, r_k) \in (\mathbb{Z} \cup \{\infty\})^k$, we define a megamatroid polyhedron

$$R_{\mathrm{MM}}(\underline{X}, \underline{r}) = \Big\{ (y_1, \ldots, y_d) \in \mathbb{R}^d \;\Big|\; \sum_{i=1}^{d} y_i = r_k,\ \forall j \sum_{i \in X_j} y_i \leq r_j \Big\}.$$

We will always use the conventions $r_0 = 0$, $X_0 = \emptyset$. Note that the polytopes $R_{\mathrm{MM}}(\underline{X}, \underline{r})$ are full-dimensional cones in $\{\sum_{y=1}^{d} y_i = r\}$.

The next theorem is an analogue of the Brianchon-Gram Theorem [4, 11] for megamatroid polytopes.

Theorem 4.1 *For any megamatroid $\mathrm{rk} : 2^{\underline{d}} \to \mathbb{Z} \cup \{\infty\}$ we have*

$$[Q(\mathrm{rk})] = \sum_{\underline{X}} (-1)^{d - \mathrm{length}(\underline{X})} [R_{\mathrm{MM}}(\underline{X}, (\mathrm{rk}(X_1), \ldots, \mathrm{rk}(X_k)))].$$

Example 4.2 *Consider the case where $d = 3$ and $r = 3$, and rk is defined by $\mathrm{rk}(\{1\}) = \mathrm{rk}(\{2\}) = \mathrm{rk}(\{3\}) = 2$, $\mathrm{rk}(\{1,2\}) = \mathrm{rk}(\{2,3\}) = \mathrm{rk}\{(1,3\}) = 3$, $\mathrm{rk}(\{1,2,3\}) = 4$. The right of Figure 1 depicts the decomposition using the Brianchon-Gram theorem of a polytope $Q_\varepsilon(\mathrm{rk})$, which is defined by a certain perturbation of the inequalities defining $Q(\mathrm{rk})$. Note how the summands in the decomposition correspond to the faces of $Q_\varepsilon(\mathrm{rk})$. In the limit where the perturbation approaches 0, $Q_\varepsilon(\mathrm{rk})$ tends to $Q(\mathrm{rk})$ and we obtain the left of Figure 1. This is exactly the decomposition in Theorem 4.1. In this decomposition, the summands do not correspond bijectively to the faces of $Q(\mathrm{rk})$.*

The dashed triangle is the triangle defined by $y_1, y_2, y_3 \geq 0$, $y_1 + y_2 + y_3 = 4$.

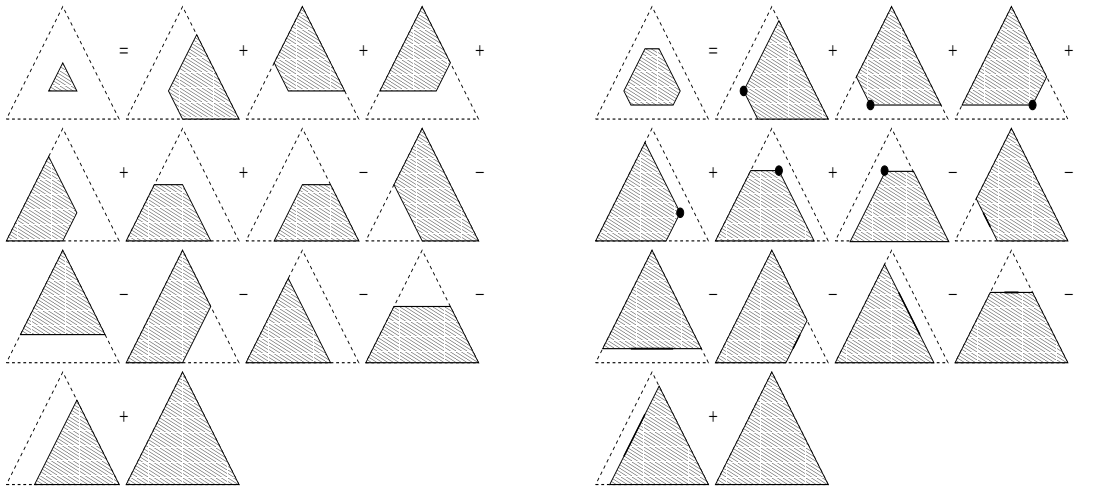


Fig. 1: At left, a decomposition of $Q(\mathrm{rk})$ as in Theorem 4.1. At right, a decomposition of a polytope $Q_\epsilon(\mathrm{rk})$.

5 Valuations

Suppose that $d \geq 1$. Let $\mathfrak{p}_{\mathrm{MM}}(d,r)$ be the set of all pairs $(\underline{X},\underline{r})$ such that $\underline{X}$ is a chain of length k ($1 \leq k \leq d$) and $\underline{r} = (r_1, r_2, \ldots, r_k)$ is an integer vector with $r_k = r$. We define $R_{(\mathrm{P})\mathrm{M}}(\underline{X},\underline{r}) = R_{\mathrm{MM}}(\underline{X},\underline{r}) \cap \Delta_{(\mathrm{P})\mathrm{M}}(d,r)$. If $R_{(\mathrm{P})\mathrm{M}}(\underline{X},\underline{r})$ is nonempty, then it is a (poly)matroid base polytope. Define $\mathfrak{p}_{\mathrm{PM}}(d,r) \subseteq \mathfrak{p}_{\mathrm{MM}}(d,r)$ as the set of all pairs $(\underline{X},\underline{r})$ with $0 \leq r_1 < \cdots < r_k = r$. Let $\mathfrak{p}_{\mathrm{M}}(d,r)$ denote the set of all pairs $(\underline{X},\underline{r}) \in \mathfrak{p}_{\mathrm{MM}}(d,r)$ such that $\underline{r} = (r_1, \ldots, r_k)$ for some k ($1 \leq k \leq d$),

$$0 \leq r_1 < r_2 < \cdots < r_k = r$$

and

$$0 < |X_1| - r_1 < |X_2| - r_2 < \cdots < |X_{k-1}| - r_{k-1} \leq |X_k| - r_k = d - r.$$

For $d = 0$, we define $\mathfrak{p}_{\mathrm{MM}}(0,r) = \mathfrak{p}_{\mathrm{PM}}(0,r) = \mathfrak{p}_{\mathrm{M}}(0,r) = \emptyset$ for $r \neq 0$ and $\mathfrak{p}_{\mathrm{MM}}(0,0) = \mathfrak{p}_{\mathrm{PM}}(0,0) = \mathfrak{p}_{\mathrm{M}}(0,0) = \{(\emptyset \subseteq \underline{0}, ())\}$.

Theorem 5.1 *The group $P_{*\mathrm{M}}(d,r)$ is freely generated by the basis $\big\{[R_{*\mathrm{M}}(\underline{X},\underline{r})] \;\big|\; (\underline{X},\underline{r}) \in \mathfrak{p}_{*\mathrm{M}}(d,r)\big\}$.*

Note that the basis of this theorem is a generating set by Theorem 4.1.

Suppose that $\underline{X}$ is a chain of length k and $\underline{r} = (r_1, \ldots, r_k)$ is an integer vector with $r_k = r$. Define a homomorphism $s_{\underline{X},\underline{r}} : Z_{\mathrm{MM}}(d,r) \to \mathbb{Z}$ by

$$s_{\underline{X},\underline{r}}(\mathrm{rk}) = \begin{cases} 1 & \text{if } \mathrm{rk}(X_j) = r_j \text{ for } j = 1,2,\ldots,k, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.2 *The homomorphism $s_{\underline{X},\underline{r}}$ is valuative.*

Theorem 5.3 *The group $P_{(\mathrm{P})\mathrm{M}}(d,r)^\vee$ is freely generated by the basis $\big\{s_{\underline{X},\underline{r}} : (\underline{X},\underline{r}) \in \mathfrak{p}_{(\mathrm{P})\mathrm{M}}(d,r)\big\}$.*

If $\underline{X}$ is not a maximal chain, then $s_{\underline{X},\underline{r}}$ is a linear combination of functions of the form $s_{\underline{X}',\underline{r}'}$ where $\underline{X}'$ is a maximal chain. The set of such functions $s_{\underline{X}',\underline{r}'}$ appeared as the coordinates of the function H defined in [1, §6], which was introduced there as a labeled analogue of $\mathcal{G}$.

6 Valuative invariants

Let $Y_{*\mathrm{M}}(d,r)$ be the group generated by all $\langle \mathrm{rk}\rangle - \langle \mathrm{rk}\circ\sigma\rangle$ where $\mathrm{rk}: 2^{\underline{d}} \to \mathbb{Z}\cup\{\infty\}$ is a $*$matroid of rank r and σ is a permutation of $\underline{d}$. We define $Z^{\mathrm{sym}}_{*\mathrm{M}}(d,r) = Z_{*\mathrm{M}}(d,r)/Y_{*\mathrm{M}}(d,r)$. Let $\pi_{*\mathrm{M}} : Z_{*\mathrm{M}}(d,r) \to Z^{\mathrm{sym}}_{*\mathrm{M}}(d,r)$ be the quotient homomorphism. If $\mathrm{rk}_X : 2^X \to \mathbb{Z}\cup\{\infty\}$ is any $*$matroid, then we can choose a bijection $\varphi : \underline{d} \to X$, where d is the cardinality of X. Let $r = \mathrm{rk}_X(X)$. The image of $\langle \mathrm{rk}_X \circ\varphi\rangle$ in $Z^{\mathrm{sym}}_{*\mathrm{M}}(d,r)$ does not depend on φ, and will be denoted by $[\mathrm{rk}_X]$. The $*$matroids (X,rk_X) and (Y,rk_Y) are isomorphic if and only if $[\mathrm{rk}_X] = [\mathrm{rk}_Y]$. So we may think of $Z^{\mathrm{sym}}_{*\mathrm{M}}(d,r)$ as the free group generated by all isomorphism classes of rank r $*$matroids on sets with d elements.

Let $P^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$ be the pushout of the diagram

$$\begin{array}{ccc} Z_{(\mathrm{P})\mathrm{M}}(d,r) & \xrightarrow{\pi_{(\mathrm{P})\mathrm{M}}} & Z^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r) \\ \Psi_{(\mathrm{P})\mathrm{M}}\downarrow & & \vdots\,\Psi^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}} \\ P_{(\mathrm{P})\mathrm{M}}(d,r) & \xrightarrow[\rho_{(\mathrm{P})\mathrm{M}}]{} & P^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r) \end{array} . \tag{1}$$

Then the dual space $P^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)^\vee$ is exactly the set of all $\mathbb{Z}$-valued valuative (poly)matroid invariants. Define $\mathfrak{p}^{\mathrm{sym}}_{*\mathrm{M}}(d,r)$ as the set of all pairs $(\underline{X},\underline{r}) \in \mathfrak{p}_{*\mathrm{M}}(d,r)$ such that every X_j is of the form $\{1,2,\ldots,i\}$.

Theorem 6.1 *The $\mathbb{Z}$-module $P^{\mathrm{sym}}_{\star\mathrm{M}}(d,r)$ is freely generated by all $\rho_{\star\mathrm{M}}([R_{\star\mathrm{M}}(\underline{X},\underline{r})])$ with $(\underline{X},\underline{r}) \in \mathfrak{p}^{\mathrm{sym}}_{\star\mathrm{M}}(d,r)$.*

The matroid polytopes $R_{\mathrm{M}}(\underline{X},\underline{r})$ with $(\underline{X},\underline{r}) \in \mathfrak{p}^{\mathrm{sym}}_{\star\mathrm{M}}(d,r)$ are the polytopes of *Schubert matroids*. Schubert matroids were first described by Crapo [6], and have since arisen in several contexts, prominent among these being the stratification of the Grassmannian into Schubert cells [2, §2.4].

For $(\underline{X},\underline{r}) \in \mathfrak{p}^{\mathrm{sym}}_{\mathrm{MM}}(d,r)$, define a homomorphism $s^{\mathrm{sym}}_{\underline{X},\underline{r}} : Z_{\mathrm{MM}}(d,r) \to \mathbb{Z}$ by

$$s^{\mathrm{sym}}_{\underline{X},\underline{r}} = \sum_{\sigma\underline{X}} s_{\sigma\underline{X},\underline{r}}$$

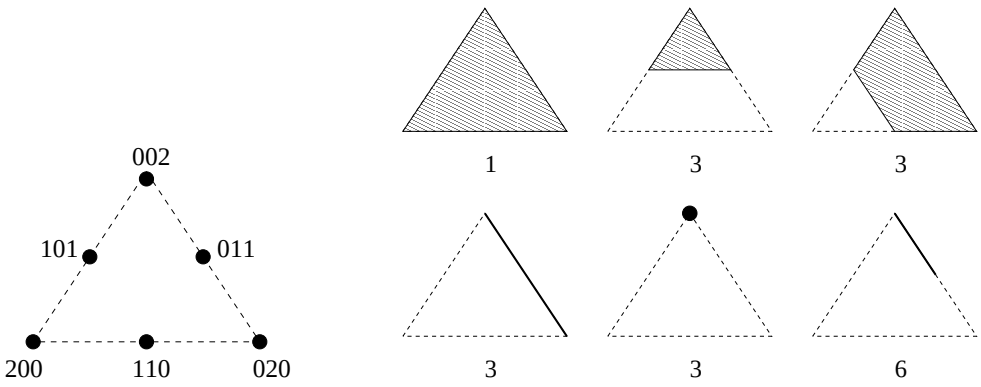
where the sum is over all chains $\sigma\underline{X}$ in the orbit of $\underline{X}$ under the action of the symmetric group.

Theorem 6.2 *The $\mathbb{Q}$-vector space $P^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ of valuations $Z^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r) \to \mathbb{Q}$ has a basis given by the functions $s^{\mathrm{sym}}_{\underline{X},\underline{r}}$ for $(\underline{X},\underline{r}) \in \mathfrak{p}^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$.*

For a sequence $\alpha = (\alpha_1,\ldots,\alpha_d)$ of nonnegative integers with $|\alpha| = \sum_i \alpha_i = r$, we define $u_\alpha = s_{\underline{X},\underline{r}} : Z^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r) \to \mathbb{Z}$, where $\underline{X}_i = \underline{i}$ for $i = 1,2,\ldots,r$ and $\underline{r} = (\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\cdots+\alpha_d)$.

Corollary 6.3 *The $\mathbb{Q}$-vector space $P^{\mathrm{sym}}_{\mathrm{PM}}(d,r)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ of valuations $Z^{\mathrm{sym}}_{\mathrm{PM}}(d,r) \to \mathbb{Q}$ has a $\mathbb{Q}$-basis given by the functions u_α, where α runs over all sequences $(\alpha_1,\ldots,\alpha_d)$ of nonnegative integers with $|\alpha| = r$.*

Corollary 6.4 *The $\mathbb{Q}$-vector space $P^{\mathrm{sym}}_{\mathrm{M}}(d,r)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ of valuations $Z^{\mathrm{sym}}_{\mathrm{M}}(d,r) \to \mathbb{Q}$ has a $\mathbb{Q}$-basis given by all functions u_α where α runs over all sequences $(\alpha_1,\ldots,\alpha_d) \in \{0,1\}^d$ with $|\alpha| = r$.*



$\underline{X}:\ \{1,2,3\}$ $\underline{r}=\ (2)$	$\underline{X}:\ \{1,2\}\subset\{1,2,3\}$ $\underline{r}=\ (1,2)$	$\underline{X}:\ \{1\}\subset\{1,2,3\}$ $\underline{r}=\ (1,2)$
$\underline{X}:\ \{1\}\subset\{1,2,3\}$ $\underline{r}=\ (0,2)$	$\underline{X}:\ \{1,2\}\subset\{1,2,3\}$ $\underline{r}=\ (0,2)$	$\underline{X}:\ \{1\}\subset\{1,2\}\subset\{1,2,3\}$ $\underline{r}=\ (0,1,2)$

Fig. 2: A polymatroid example of Theorems 5.1 and 6.1.

Example 6.5 *Figures 2 and 3 are examples of both Theorems 5.1 and 6.1 for matroids in the cases* $(d,r)=(3,2)$ *and* $(d,r)=(4,2)$*, respectively. At top left are the polyhedra* $\Delta_{(\mathrm{P})\mathrm{M}}(d,r)$*, containing all the (poly)matroid base polytopes. At top right are the polytopes* $R(\underline{X},\underline{r})$ *for* $(\underline{X},\underline{r})\in\mathfrak{p}^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$*, and at bottom the corresponding pairs* $(\underline{X},\underline{r})$*.*

The symmetric group Σ_d *acts on* Δ_d *by permuting the coordinates. If* Σ_d *acts on the generators* $R(\underline{X},\underline{r})$ *with* $(\underline{X},\underline{r})\in\mathfrak{p}_{(\mathrm{P})\mathrm{M}}(d,r)$*, then we get all* $R(\underline{X},\underline{r})$ *with* $(\underline{X},\underline{r})\in\mathfrak{p}_{(\mathrm{P})\mathrm{M}}(d,r)$*. In the figure, we have written under each polytope the cardinality of its* Σ_d*-orbit.*

7 Hopf algebra structures

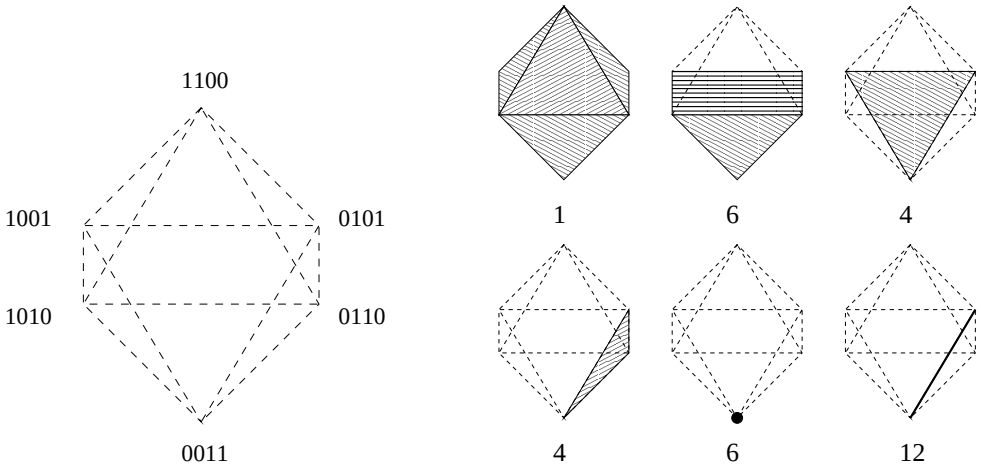
Define $Z_{*\mathrm{M}}=\bigoplus_{d,r}Z_{*\mathrm{M}}(d,r)$, and in a similar way define $Z^{\mathrm{sym}}_{*\mathrm{M}}$, $P_{*\mathrm{M}}$, and $P^{\mathrm{sym}}_{*\mathrm{M}}$. We can view $Z^{\mathrm{sym}}_{*\mathrm{M}}$ as the $\mathbb{Z}$-module freely generated by all isomorphism classes of $*$matroids. In this section we will only speak of the megamatroid objects; in every case, there are analogous matroid and polymatroid objects, which are substructures.

If $\mathrm{rk}_1:2^{\underline{d}}\to\mathbb{Z}\cup\{\infty\}$ and $\mathrm{rk}_2:2^{\underline{e}}\to\mathbb{Z}\cup\{\infty\}$ then we define $\mathrm{rk}_1\boxplus\mathrm{rk}_2:2^{\underline{d+e}}\to\mathbb{Z}\cup\{\infty\}$ by

$$(\mathrm{rk}_1\boxplus\mathrm{rk}_2)(A)=\mathrm{rk}_1(A\cap\underline{d})+\mathrm{rk}_2(\{i\in\underline{e}\mid d+i\in A\})$$

for any set $A\subseteq\underline{d+e}$. Note that $\boxplus$ is not commutative. We have a multiplication $\nabla:Z_{\mathrm{MM}}\otimes_{\mathbb{Z}}Z_{\mathrm{MM}}\to Z_{\mathrm{MM}}$ defined by $\nabla(\langle\mathrm{rk}_1\rangle\otimes\langle\mathrm{rk}_2\rangle)=\langle\mathrm{rk}_1\boxplus\mathrm{rk}_2\rangle$, which makes $Z_{\mathrm{MM}}(d,r)$ into an associative (noncommutative) ring with 1. The multiplication also respects the bigrading of $Z_{\mathrm{MM}}(d,r)$. The unit $\eta:\mathbb{Z}\to Z_{\mathrm{MM}}(d,r)$ is given by $1\mapsto\langle\mathrm{rk}_0\rangle$ where $\mathrm{rk}_0:2^{\underline{0}}\to\mathbb{Z}\cup\{\infty\}$ is the unique megamatroid defined by $\mathrm{rk}(\emptyset)=0$.

Next, we define a comultiplication for Z_{MM}. Suppose that $X=\{i_1,i_2,\ldots,i_d\}$ is a set of integers with $i_1<\cdots<i_d$ and $\mathrm{rk}:2^X\to\mathbb{Z}\cup\{\infty\}$ is a megamatroid. We define a megamatroid $\widehat{\mathrm{rk}}:2^{\underline{d}}\to\mathbb{Z}\cup\{\infty\}$



$\underline{X}: \{1,2,3,4\}$ $\underline{r} = (2)$	$\underline{X}: \{1,2\} \subset \{1,2,3,4\}$ $\underline{r} = (1,2)$	$\underline{X}: \{1,2,3\} \subset \{1,2,3,4\}$ $\underline{r} = (1,2)$
$\underline{X}: \{1\} \subset \{1,2,3,4\}$ $\underline{r} = (0,2)$	$\underline{X}: \{1,2\} \subset \{1,2,3,4\}$ $\underline{r} = (0,2)$	$\underline{X}: \{1\} \subset \{1,2,3\} \subset \{1,2,3,4\}$ $\underline{r} = (0,1,2)$

Fig. 3: A matroid example of Theorems 5.1 and 6.1.

by $\widehat{\mathrm{rk}}(A) = \mathrm{rk}(\{i_j \mid j \in A\})$. If $\mathrm{rk} : 2^X \to \mathbb{Z} \cup \{\infty\}$ is a megamatroid and $B \subseteq A \subseteq X$ then we define $\mathrm{rk}_{A/B} : 2^{A \setminus B} \to \mathbb{Z} \cup \{\infty\}$ by $\mathrm{rk}_{A/B}(C) = \mathrm{rk}(B \cup C) - \mathrm{rk}(B)$ for all $C \subseteq A \setminus B$. We also define $\mathrm{rk}_A := \mathrm{rk}_{A/\emptyset}$ and $\mathrm{rk}_{/B} = \mathrm{rk}_{X/B}$.

We now define $\Delta : Z_{\mathrm{MM}} \to Z_{\mathrm{MM}} \otimes_{\mathbb{Z}} Z_{\mathrm{MM}}$ by

$$\Delta(\langle \mathrm{rk} \rangle) = \sum_{A \subseteq \underline{d};\ \mathrm{rk}(A) < \infty} \langle \widehat{\mathrm{rk}_A} \rangle \otimes \langle \widehat{\mathrm{rk}_{/A}} \rangle.$$

where A runs over all subsets of $\underline{d}$ for which $\mathrm{rk}(A)$ is finite. This comultiplication is coassociative, but not cocommutative. If $\mathrm{rk} : 2^{\underline{d}} \to \mathbb{Z} \cup \{\infty\}$ is a megamatroid, then the counit is defined by $\epsilon(\langle \mathrm{rk} \rangle) = 1$ (if $d = 0$), 0 (otherwise). We omit here the definition of the antipode S.

It is well-known that $Z_{\mathrm{M}}^{\mathrm{sym}}$ has the structure of a Hopf algebra over $\mathbb{Z}$. In fact we have that $Z_{\mathrm{PM}}^{\mathrm{sym}}$ has a Hopf algebra structure, with $Z_{\mathrm{M}}^{\mathrm{sym}}$ as a Hopf subalgebra. This structure is defined analogously to the one on Z_{MM} above, replacing each megamatroid by its isomorphism class: e.g. multiplication is given by the direct sum of megamatroids, and is now commutative. The map π_{MM} of (1) is a Hopf algebra morphism.

The space P_{MM} inherits a Hopf algebra structure from Z_{MM}. Most of this structure can be defined in the expected fashion, but the coproduct requires some care. We define $\Delta : P_{\mathrm{MM}} \to P_{\mathrm{MM}} \otimes P_{\mathrm{MM}}$ by

$$\Delta([R_{\mathrm{MM}}(\underline{X}, \underline{r})]) = \sum_{i=0}^{k} [R_{\mathrm{MM}}(\widehat{\underline{X}_i, \underline{r}_i})] \otimes [R_{\mathrm{MM}}(\widehat{\underline{X}^i, \underline{r}^i})].$$

Since the $R_{\mathrm{MM}}(\underline{X}, \underline{r})$ with $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{MM}} = \bigcup_{d,r} \mathfrak{p}_{\mathrm{MM}}(d, r)$ form a basis of P_{MM}, this is sufficient to linearly extend. From Theorem 4.1 one can check that $(\Psi_{\mathrm{MM}} \otimes \Psi_{\mathrm{MM}}) \otimes \Delta = \Delta \circ \Psi_{\mathrm{MM}}$.

The Hopf algebra structure on P_{MM} naturally induces a Hopf algebra structure on $P^{\mathrm{sym}}_{\mathrm{MM}}$ such that ρ_{MM} and $\Psi^{\mathrm{sym}}_{\mathrm{MM}}$ are Hopf algebra homomorphisms.

8 Additive valuations

For $0 \leq e \leq d$ we define $P_{*\mathrm{M}}(d,r,e) \subseteq P_{*\mathrm{M}}(d,r)$ as the span of all $[\Pi]$ where $\Pi \subseteq \mathbb{R}^d$ is a $*$matroid polytope of dimension $\leq d-e$. We have $P_{*\mathrm{M}}(0,r,0) = P_{*\mathrm{M}}(0,r)$ and $P_{*\mathrm{M}}(d,r,1) = P_{*\mathrm{M}}(d,r)$ for $d \geq 1$. These subgroups form a filtration

$$\cdots \subseteq P_{*\mathrm{M}}(d,r,2) \subseteq P_{*\mathrm{M}}(d,r,1) \subseteq P_{*\mathrm{M}}(d,r,0) = P_{*\mathrm{M}}(d,r).$$

Define $\overline{P}_{*\mathrm{M}}(d,r,e) := P_{*\mathrm{M}}(d,r,e)/P_{*\mathrm{M}}(d,r,e+1)$, and $T_{\star\mathrm{M}}(d,r) = \overline{P}_{\star\mathrm{M}}(d,r,1)$. The image of $[Q(M)]$ in $T_{*\mathrm{M}}(d,r)$ is zero if and only if M is *connected*. The associated graded algebra $\overline{P}_{*\mathrm{M}} = \bigoplus_{d,r,e} \overline{P}_{*\mathrm{M}}(d,r,e)$ has an induced Hopf algebra structure.

Define

$$\overline{P}_{*\mathrm{M}}(\underline{X}) = \bigoplus_{r_1,r_2,\ldots,r_e \in \mathbb{Z}} T_{*\mathrm{M}}(|X_1|,r_1) \otimes \cdots \otimes T_{*\mathrm{M}}(|X_e|,r_e).$$

There is a group homomorphism $\overline{\phi}_{\underline{X}} : \overline{P}_{*\mathrm{M}}(\underline{X}) \to \overline{P}_{*\mathrm{M}}(d,r,e)$. which takes the classes of a list of $*$matroids to the class of their direct sum. The next theorem essentially asserts a unique decomposition of $*$matroids into connected components.

Theorem 8.1 *We have the isomorphism*

$$\left(\sum_{\underline{X}} \overline{\phi}_{\underline{X}}\right) : \bigoplus_{\substack{\underline{X}=(X_1,X_2,\ldots,X_e) \\ \underline{d}=X_1 \sqcup X_2 \sqcup \cdots \sqcup X_e; X_1,\ldots,X_e \neq \emptyset}} \overline{P}_{*\mathrm{M}}(\underline{X}) \to \bigoplus_{r \in \mathbb{Z}} \overline{P}_{*\mathrm{M}}(d,r,e) \tag{2}$$

If $d \geq 1$, let $\mathfrak{t}_{\mathrm{PM}}(d,r)$ be the set of all pairs $(\underline{X},\underline{r}) \in \mathfrak{p}_{\mathrm{PM}}(d,r)$ such that $r_1 > 0$, and $d \notin X_{k-1}$, where k is the length of $\underline{X}$. Similarly, if $d \geq 2$, let $\mathfrak{t}_{\mathrm{M}}(d,r)$ be the set of all pairs $(\underline{X},\underline{r}) \in \mathfrak{t}_{\mathrm{M}}(d,r)$ such that $r_1 > 0$, $|X_{k-1}| - r_{k-1} < d - r$, and $d \notin X_{k-1}$.

Theorem 8.2 *The group $T_{(\mathrm{P})\mathrm{M}}(d,r)$ is freely generated by all $[R_{(\mathrm{P})\mathrm{M}}(\underline{X},\underline{r})]$ with $(\underline{X},\underline{r}) \in \mathfrak{t}_{(\mathrm{P})\mathrm{M}}(d,r)$.*

9 Additive invariants

The algebra $P^{\mathrm{sym}}_{\star M}$ also has a natural filtration:

$$\cdots \subseteq P^{\mathrm{sym}}_{\star\mathrm{M}}(d,r,2) \subseteq P^{\mathrm{sym}}_{\star\mathrm{M}}(d,r,1) \subseteq P^{\mathrm{sym}}_{\star M}(d,r,0) = P^{\mathrm{sym}}_{\star\mathrm{M}}(d,r).$$

Here $P^{\mathrm{sym}}_{\star\mathrm{M}}(d,r,e)$ is spanned by the indicator functions of all $*$matroid base polytopes of rank r and dimension $d-e$. Define $\overline{P}^{\mathrm{sym}}_{\star\mathrm{M}}(d,r,e) = P^{\mathrm{sym}}_{*\mathrm{M}}(d,r,e)/P^{\mathrm{sym}}_{*\mathrm{M}}(d,r,e+1)$. Let $\overline{P}^{\mathrm{sym}}_{\star\mathrm{M}} = \bigoplus_{d,r,e} \overline{P}^{\mathrm{sym}}_{\star\mathrm{M}}(d,r,e)$ be the associated graded algebra.

Define $T^{\mathrm{sym}}_{\star\mathrm{M}} = \bigoplus_{d,r} \overline{P}^{\mathrm{sym}}_{\star\mathrm{M}}(d,r,1)$. The following theorem follows from Theorem 8.1.

Theorem 9.1 *The algebra $\overline{P}^{\mathrm{sym}}_{\star\mathrm{M}}$ is the free symmetric algebra $S(T^{\mathrm{sym}}_{\star\mathrm{M}})$ on $T^{\mathrm{sym}}_{\star\mathrm{M}}$, and there exists an isomorphism*

$$S^e(T^{\mathrm{sym}}_{\star\mathrm{M}}) \cong \bigoplus_{d,r} \overline{P}^{\mathrm{sym}}_{\star\mathrm{M}}(d,r,e). \tag{3}$$

Corollary 9.2 *The algebra $P^{\mathrm{sym}}_{\star\mathrm{M}}$ is a polynomial ring over $\mathbb{Z}$.*

10 Invariants as elements in free algebras

Let

$$(P^{\mathrm{sym}}_{*\mathrm{M}})^{\#} := \bigoplus_{d,r} P^{\mathrm{sym}}_{*\mathrm{M}}(d,r)^{\vee}$$

be the *graded* dual of $P^{\mathrm{sym}}_{*\mathrm{M}}$. Let $\mathfrak{m}_{\star\mathrm{M}} = \bigoplus_{d,r} P^{\mathrm{sym}}_{*\mathrm{M}}(d,r,1)$. Then we have $\mathfrak{m}^2_{\star\mathrm{M}} = \bigoplus_{d,r} P^{\mathrm{sym}}_{*\mathrm{M}}(d,r,2)$ and $T^{\mathrm{sym}}_{*\mathrm{M}} = \mathfrak{m}_{*\mathrm{M}}/\mathfrak{m}^2_{*\mathrm{M}}$. The graded dual $\mathfrak{m}^{\#}_{\star\mathrm{M}}$ can be identified with

$$(P^{\mathrm{sym}}_{\star\mathrm{M}})^{\#}/P^{\mathrm{sym}}_{*\mathrm{M}}(0,0) \cong \bigoplus_{d=1}^{\infty} \bigoplus_{r} P^{\mathrm{sym}}_{\star\mathrm{M}}(d,r)^{\vee}.$$

So $\mathfrak{m}^{\#}_{\mathrm{PM}} \otimes_{\mathbb{Z}} \mathbb{Q}$ will be identified with the ideal $(u_0, u_1, \dots)$ of $\mathbb{Q}\langle u_0, u_1, \dots \rangle$ and $\mathfrak{m}^{\#}_{\mathrm{M}} \otimes_{\mathbb{Z}} \mathbb{Q}$ will be identified with the ideal (u_0, u_1) of $\mathbb{Q}\langle u_0, u_1\rangle$. The graded dual $(T^{\mathrm{sym}}_{\mathrm{PM}})^{\#} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a subalgebra (without 1) of the ideal $(u_0, u_1, \dots)$, and $(T^{\mathrm{sym}}_{\mathrm{PM}})^{\#} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a subalgebra of (u_0, u_1).

Theorem 10.1 *Let $u_0, u_1, u_2, \dots$ be indeterminates, where u_i has bidgree $(1, i)$. We have the following isomorphisms of bigraded associative algebras over $\mathbb{Q}$:*

a. The space $(P^{\mathrm{sym}}_{M})^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathbb{Q}\langle\langle u_0, u_1\rangle\rangle$, the completion (in power series) of the free associative algebra generated by u_0, u_1.

b. The space $(P^{\mathrm{sym}}_{\mathrm{PM}})^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathbb{Q}\langle\langle u_0, u_1, u_2, \dots\rangle\rangle$.

c. The space $(T^{\mathrm{sym}}_{\mathrm{M}})^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathbb{Q}\{\{u_0, u_1\}\}$, the completion of the free Lie algebra generated by u_0, u_1.

d. The space $(T^{\mathrm{sym}}_{\mathrm{PM}})^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathbb{Q}\{\{u_0, u_1, u_2, \dots\}\}$.

Proposition 10.2 *The Hopf algebra $P^{\mathrm{sym}}_{\mathrm{PM}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to the ring $QSym$ of quasi-symmetric functions over $\mathbb{Q}$.*

If we identify $P^{\mathrm{sym}}_{\mathrm{PM}} \otimes_{\mathbb{Z}} \mathbb{Q}$ with $QSym$, then $\mathcal{G}$ is equal to $\Psi^{\mathrm{sym}}_{\mathrm{PM}}$.

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