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Weighted branching formulas for the hook lengths

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Abstract. The famous hook-length formula is a simple consequence of the branching rule for the hook lengths. While the Greene-Nijenhuis-Wilf probabilistic proof is the most famous proof of the rule, it is not completely combinatorial, and a simple bijection was an open problem for a long time. In this extended abstract, we show an elegant bijective argument that proves a stronger, weighted analogue of the branching rule. Variants of the bijection prove seven other interesting formulas. Another important approach to the formulas is via weighted hook walks; we discuss some results in this area. We present another motivation for our work: J-functions of the Hilbert scheme of points.

Résumé. La formule bien connue de la longueur des crochets est une conséquence simple de la règle de branchement des longueurs des crochets. La preuve la plus répandue de cette règle est de nature probabiliste et est due à Greene-Njenhuis-Wilf. Elle n'est toutefois pas complètement combinatoire et une simple bijection a été pendant longtemps un problème ouvert. Dans ce résumé étendu, nous proposons un argument bijectif élégant qui démontre une version à poids plus forte de cette règle. Des variantes de cette bijection permettent d'obtenir sept autres formules intéressantes. Une autre approche importante de ces formules est via les marches des crochets à poids. Nous discutons certains résultats dans cette direction. Enfin, nous présentons aussi une autre motivation à l'origine de ce travail: les J-fonctions du schéma d'Hilbert des points.

Resumen. La famosa fórmula de la longitud de codos es una consecuencia simple de la ley de ramificación de las longitudes de los codos. Mientras que la prueba probabilística de la fórmula de Greene-Nijenhuis-Wilf es la más famosa, ésta no es del todo combinatoria. Por mucho tiempo el problema de encontrar una prueba biyectiva de la formula estuvo abierto. En este resumen extendido, mostramos un argumento biyectivo elegante que prueba una variante ponderada más robusta de la ley de ramificación. Variantes de la biyección prueban otras siete fórmulas interesantes. Otro enfoque importante a las fórmulas es a traves de caminos ponderados de codos: discutimos unos resultados en esta área. Presentamos otra motivación: las J-funciones del esquema de Hilbert de puntos.

Keywords: Hilbert scheme of points, hook-length formula, bijective proofs

1 Introduction and main results

The classical hook-length formula gives an elegant product formula for the number of standard Young tableaux. Since its discovery by Frame, Robinson and Thrall in [9], it has been reproved, generalized and extended in several different ways, and applications have been found in a number of fields of mathematics.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$, be a partition of n, $\lambda \vdash n$, and let $[\lambda] = \{(i,j) \in \mathbb{Z}^2 \colon 1 \leq i \leq \ell, 1 \leq j \leq \lambda_i\}$ be the corresponding *Young diagram*. The *conjugate partition* λ' is defined by $\lambda'_j = \max\{i : \lambda_i \geq j\}$. The *hook* $H_{\mathbf{z}} \subseteq [\lambda]$ is the set of squares weakly to the right and below of $\mathbf{z} = (i,j) \in [\lambda]$, and the *hook length* $h_{\mathbf{z}} = h_{ij} = |H_{\mathbf{z}}| = \lambda_i + \lambda'_j - i - j + 1$ is the size of the hook.

A *standard Young tableau* of shape λ is a bijective map $f : [\lambda] \to \{1, \ldots, n\}$, such that $f(i_1, j_1) < f(i_2, j_2)$ whenever $i_1 \leq i_2$, $j_1 \leq j_2$, and $(i_1, j_1) \neq (i_2, j_2)$. We denote the number of standard Young tableaux of shape λ by f^λ. The hook-length formula states that if λ is a partition of n, then

$$f^\lambda = \frac{n!}{\prod_{\mathbf{z} \in [\lambda]} h_{\mathbf{z}}}.$$

For example, for $\lambda = (3,2,2) \vdash 7$, the hook-length formula gives $f^{322} = \frac{7!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1} = 21$.

One way to prove the hook-length formula is by induction on n. Namely, it is obvious that in a standard Young tableau, n must be in one of the *corners*, squares (i,j) of $[\lambda]$ satisfying $(i+1,j), (i,j+1) \notin [\lambda]$. Therefore $f^\lambda = \sum_{\mathbf{c} \in \mathcal{C}[\lambda]} f^{\lambda - \mathbf{c}}$, where $\mathcal{C}[\lambda]$ is the set of all corners of λ, and $\lambda - \mathbf{c}$ is the partition whose diagram is $[\lambda] \setminus \{\mathbf{c}\}$. That means that in order to prove the hook-length formula, we have to prove that $F^\lambda = n!/\prod h_{\mathbf{z}}$ satisfy the same recursion. It is easy to see that this is equivalent to the following *branching rule for the hook lengths*:

$$\sum_{(r,s) \in \mathcal{C}[\lambda]} \frac{1}{n} \prod_{i=1}^{r-1} \frac{h_{is}}{h_{is} - 1} \prod_{j=1}^{s-1} \frac{h_{rj}}{h_{rj} - 1} = 1. \tag{1}$$

In an important development, Green, Nijenhuis and Wilf introduced the *hook walk* which proves (1) by a combination of a probabilistic and a short but delicate induction argument [13]. Zeilberger converted the hook walk proof into a bijective proof [26], but laments on the "enormous size of the input and output" and "the recursive nature of the algorithm" (ibid, §3). With time, several variations of the hook walk have been discovered, most notably the q-version of Kerov [16], and its further generalization, the (q,t)-version of Garsia and Haiman [10]. In the recent paper [7], a direct bijective proof of (1) is presented. In fact, a bijective proof is presented of the following more general identity, called the *weighted branching formula*.

$$\left[\sum_{(p,q) \in [\lambda]} x_p y_q\right] \cdot \left[\prod_{(i,j) \in [\lambda] \setminus \mathcal{C}[\lambda]} \Big(x_{i+1} + \ldots + x_{\lambda'_j} + y_{j+1} + \ldots + y_{\lambda_i}\Big)\right]$$
$$= \sum_{(r,s) \in \mathcal{C}[\lambda]} \left[\prod_{\substack{(i,j) \in [\lambda] \setminus \mathcal{C}[\lambda] \\ i \neq r, j \neq s}} \Big(x_{i+1} + \ldots + x_{\lambda'_j} + y_{j+1} + \ldots + y_{\lambda_i}\Big)\right]$$
$$\cdot \left[\prod_{i=1}^{r} (x_i + \ldots + x_r + y_{s+1} + \ldots + y_{\lambda_i})\right] \cdot \left[\prod_{j=1}^{s} \Big(x_{r+1} + \ldots + x_{\lambda'_j} + y_j + \ldots + y_s\Big)\right]$$

We refer to this formula as WBR. Here $x_1, \ldots, x_{\ell(\lambda)}, y_1, \ldots, y_{\lambda_1}$ are some commutative variables. If we substitute all x_i and y_j by 1, we get

$$n \cdot \prod_{\mathbf{z} \in [\lambda] \setminus \mathcal{C}[\lambda]} (h_{\mathbf{z}} - 1) \;=\; \sum_{(r,s) \in \mathcal{C}[\lambda]} \left[\prod_{\substack{(i,j) \in [\lambda] \setminus \mathcal{C}[\lambda] \\ i \neq r, j \neq s}} (h_{\mathbf{z}} - 1) \right] \prod_{i=1}^{r} h_{is} \prod_{j=1}^{s} h_{rj},$$

which is equivalent to (1).

In [18], a weighted analogue of the formula

$$\prod_{\mathbf{z} \in [\lambda]} (h_{\mathbf{z}} + 1) \;=\; \sum_{(r,s) \in \mathcal{C}'[\lambda]} \left[\prod_{\substack{(i,j) \in [\lambda] \\ i \neq r, j \neq s}} (h_{\mathbf{z}} + 1) \right] \prod_{i=1}^{r-1} h_{is} \prod_{j=1}^{s-1} h_{rj} \tag{2}$$

is proved. Here $\mathcal{C}'[\lambda]$ is the set of *outer corners* of λ, squares $(i,j) \notin [\lambda]$ satisfying $i = 1$ or $(i-1,j) \in [\lambda]$, and $j = 1$ or $(i, j-1) \in [\lambda]$. The motivation for this formula is as follows, see [22]. Division by $\prod_{\mathbf{z} \in [\lambda]} (h_{\mathbf{z}} + 1)$ and $\prod_{\mathbf{z} \in [\lambda]} h_{\mathbf{z}}$ yields

$$\frac{1}{\prod_{\mathbf{z} \in [\lambda]} h_{\mathbf{z}}} \;=\; \sum_{(r,s) \in \mathcal{C}'[\lambda]} \prod_{i=1}^{r-1} \frac{1}{h_{is} + 1} \prod_{j=1}^{s-1} \frac{1}{h_{rj} + 1} \prod_{\substack{(i,j) \in [\lambda] \\ i \neq r, j \neq s}} \frac{1}{h_{\mathbf{z}}}$$

We multiply by $(n+1)!$ and use the hook-length formula. We get $(n+1) f^{\lambda} = \sum_{\mathbf{c} \in \mathcal{C}'[\lambda]} f^{\lambda + \mathbf{c}}$, where $\lambda + \mathbf{c}$ is the partition whose diagram is $[\lambda] \cup \{\mathbf{c}\}$. Let us introduce the notation $\mu \to \lambda$ or $\lambda \leftarrow \mu$ if $\lambda = \mu - \mathbf{c}$ for a corner $\mathbf{c}$ of μ, or, equivalently, if $\mu = \lambda + \mathbf{c}$ for an outer corner $\mathbf{c}$ of λ. We then have

$$\sum_{\mu \vdash n+1} (f^{\mu})^2 = \sum_{\mu \vdash n+1} f^{\mu} \left(\sum_{\lambda \leftarrow \mu} f^{\lambda} \right) = \sum_{\lambda \vdash n} f^{\lambda} \left(\sum_{\mu \to \lambda} f^{\mu} \right) = (n+1) \sum_{\lambda \vdash n} (f^{\lambda})^2.$$

Induction proves the famous formula $\sum_{\lambda \vdash n} (f^{\lambda})^2 = n!$.

It turns out that the correct weighted analogue is

$$\prod_{(i,j) \in [\lambda]} \left(x_i + \ldots + x_{\lambda'_j} + y_j + \ldots + y_{\lambda_i} \right) \;=\; \sum_{(r,s) \in \mathcal{C}'[\lambda]} \prod_{\substack{(i,j) \in [\lambda] \\ i \neq r, j \neq s}} \left(x_i + \ldots + x_{\lambda'_j} + y_j + \ldots + y_{\lambda_i} \right)$$
$$\cdot \left[\prod_{i=1}^{r-1} \left(x_{i+1} + \ldots + x_{r-1} + y_s + \ldots + y_{\lambda_i} \right) \right] \cdot \left[\prod_{j=1}^{s-1} \left(x_r + \ldots + x_{\lambda'_j} + y_{j+1} + \ldots + y_{s-1} \right) \right].$$

We refer to this result as complementary weighted branching rule, or CWBR.

This extended abstract is organized as follows. In Section 2, we describe the work that led us to WBR. In Section 3, we give bijective proofs of WBR and CWBR. Simple variants of the proofs lead to six other interesting identities. In Section 4, we present new theorems on weighted hook walks, and some recursions for f^{λ} which arise as corollaries. We finish with some final remarks in Section 5.

This extended abstract is based on papers [6], [7] and [18].

2 Motivation: J-functions of the Hilbert scheme of points

In the last fifteen years, deep relations have been uncovered between representation theory and the geometry of the Hilbert scheme of points in the complex affine plane $\mathrm{Hilb}_n(\mathbb{C}^2)$. See, say, Nakajima, [19] and Haiman, [15]. The equivariant quantum cohomology $QH^*_{(\mathbb{C}^*)^2}(\mathrm{Hilb}_n)(\mathbb{C}^2)$ of the Hilbert scheme has been recently determined by Okounkov and Pandharipande, and the authors have also shown that it agrees with the (equivariant) *relative Donaldson-Thomas theory* of $\mathbb{P}^1 \times \mathbb{C}^2$, see [20], [21].

A different perspective on the study of the relationship between $QH^*_{(\mathbb{C}^*)^2}(\mathrm{Hilb}_n(\mathbb{C}^2))$ and DT-theory is undertaken in [3]. The main point there is to exploit the fact that the Hilbert scheme is a Geometric Invariant Theory (GIT) quotient via the celebrated "ADHM construction" of Atiyah-Drinfeld-Hitchin-Manin [1].

On the one hand, this allows one to employ the machinery of the abelian/nonabelian correspondence in Gromov-Witten theory of [5], [2] to analyze the quantum cohomology of $\mathrm{Hilb}_n(\mathbb{C}^2)$. In particular, one can give a formula (a priori conjectural) for the *J-function* of the Hilbert scheme – a certain generating function for Gromov-Witten invariants of a nonsingular algebraic variety, essentially encoding the same information as the quantum cohomology ring. On the other hand, the ADHM construction of $\mathrm{Hilb}_n(\mathbb{C}^2)$ is also highly relevant to the Donaldson-Thomas side of the story, due to work of Diaconescu. Namely, in [8] he used it to obtain a gauge-theoretic partial compactification of the space of maps $\mathbb{P}^1 \to \mathrm{Hilb}_n$, his *moduli space of ADHM sheaves* on $\mathbb{P}^1$, and then provided a direct geometric identification of the DT-theory of $\mathbb{P}^1 \times \mathbb{C}^2$ with the intersection theory of this new moduli space.

The main result of [3], and the jumping board for the paper [6], is a proof of the above-mentioned formula for the J-function of $\mathrm{Hilb}_n(\mathbb{C}^2)$. The following identity involving partitions is obtained as a corollary. Choose a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ and let α, β be indeterminates. For a square $\mathbf{z} = (i, j)$ in $|\lambda|$, we define its *weight* to be $w_{\mathbf{z}} = -(i-1)\alpha - (j-1)\beta$. Then, for each $n \geq 1$ and each partition λ of n, we have

$$\sum_{\mathbf{c} \in \mathcal{C}[\lambda]} (w_{\mathbf{c}} - (\alpha+\beta)) \prod_{\mathbf{z} \in [\lambda] \setminus \{\mathbf{c}\}} \frac{(w_{\mathbf{c}} - w_{\mathbf{z}} - \alpha)(w_{\mathbf{c}} - w_{\mathbf{z}} - \beta)}{(w_{\mathbf{c}} - w_{\mathbf{z}})(w_{\mathbf{c}} - w_{\mathbf{z}} - (\alpha+\beta))} = -n(\alpha+\beta). \tag{3}$$

If λ has ℓ corners, there are ℓ different parts of λ. Let x_ℓ denote the smallest part, $x_{\ell-1} + x_\ell$ the second smallest etc., and $x_1 + x_2 + \ldots + x_\ell$ the largest part. Furthermore, let y_1 be the number of times the largest part appears in λ, y_2 the number of times the second largest part appears, etc. A careful analysis of the cancellations and the substitution of x_i for $x_i\alpha$ and y_i for $y_i\beta$ gives the rational function identity

$$\sum_{k=1}^{\ell} x_k y_k \prod_{p=1}^{k-1} \left(1 + \tfrac{y_p}{x_p + \ldots + x_{k-1} + y_{p+1} + \ldots + y_k}\right) \prod_{q=k+1}^{\ell} \left(1 + \tfrac{x_q}{x_k + \ldots + x_{q-1} + y_{k+1} + \ldots + y_q}\right) = \sum_{1 \leq p \leq q \leq \ell} x_q y_p.$$

This is exactly WBR for the staircase shape $(\ell, \ell-1, \dots, 1)$. See [6] for a more detailed explanation.

3 Bijective proofs of weighted branching formulas

Now we present a bijective proof of WBR, by interpreting both sides as certain sets of arrangements of labels, and then constructing a bijection between two sets of labels.

For the left-hand side of WBR, we are given: special labels x_p, y_q, corresponding to the first summation $\sum_{(p,q) \in [\lambda]} x_p y_q$, and a label x_k for some $i < k \leq \lambda'_j$, or y_l for some $j < l \leq \lambda_i$, in every non-corner square (i, j). Denote by F the resulting arrangement of labels, see Figure 1, left.

We can interpret the special labels x_p, y_q as the starting square (p, q). Furthermore, we can interpret all other labels as arrows: if the label in square (i, j) is x_k, the arrow points to (k, j), and if the label is y_l, the arrow points to (i, l). The arrow from (p, q) points to a square (p', q') in the hook of (p, q), the arrow from (p', q') points to a square (p'', q'') in the hook of (p', q'), etc. Eventually we obtain a *hook walk* which reaches a corner $(r, s) \in \mathcal{C}[\lambda]$ (Figure 1, second drawing). Shade row r and column s. Now we shift the labels in the hook walk and in its projection onto the shaded row and column. If the hook walk has a horizontal step from (i, j) to (i, j'), move the label in (i, j) right and down from (i, j) to (r, j'), and the label from (r, j) up to (i, j). If the hook walk has a vertical step from (i, j) to (i', j), move the label from (i, j) down and right to (i', s), and the label from (i, s) left to (i, j). If the hook walk has a horizontal step from (r, j) to (r, j'), move the label in (r, j) right to (r, j'). If the hook walk has a vertical step from (i, s) to (i', s), move the label in (i, s) down to (i', s). Finally, move the label x_p to (p, s), and the label y_q to (r, q). See Figure 1, third drawing. Denote the resulting arrangement G (Figure 1, right). It turns out that G represents a term on the right-hand side. Furthermore, φ is a bijection.

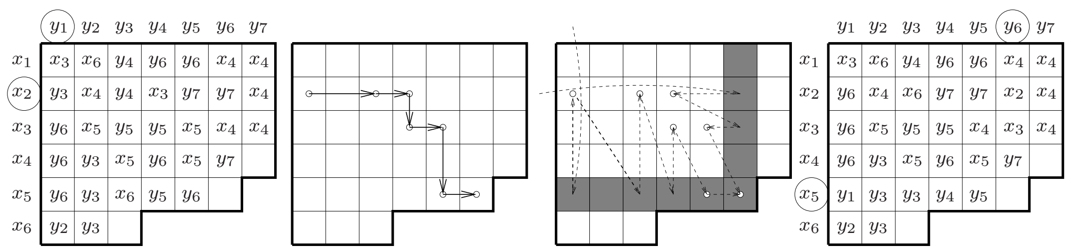

Fig. 1: An arrangement corresponding to the left-hand side of WBR; hook walk; shift of labels; final arrangement.

There are three more identities in the same spirit. To save on space, let us write them down in an abbreviated fashion. If WBR is the identity

$$\left[\sum_{(p,q)\in[\lambda]} x_p y_q\right] \cdot \left[\prod_{(i,j)\in[\lambda]\setminus\mathcal{C}[\lambda]} *\right] = \sum_{(r,s)\in\mathcal{C}[\lambda]} \left[\prod_{\substack{(i,j)\in[\lambda]\setminus\mathcal{C}[\lambda]\\ i\neq r, j\neq s}} *\right] \cdot \left[\prod_{i=1}^{r} *\right] \cdot \left[\prod_{j=1}^{s} *\right],$$

then the following identities are also true:

$$\left[\sum_{p=1}^{\ell(\lambda)} x_p\right] \cdot \left[\prod_{(i,j)\in[\lambda]\setminus\mathcal{C}[\lambda]} *\right] = \sum_{(r,s)\in\mathcal{C}[\lambda]} \left[\prod_{\substack{(i,j)\in[\lambda]\setminus\mathcal{C}[\lambda]\\ i\neq r, j\neq s}} *\right] \cdot \left[\prod_{i=1}^{r} *\right] \cdot \left[\prod_{j=2}^{s} *\right], \tag{4}$$

$$\left[\sum_{q=1}^{\lambda_1} y_q\right] \cdot \left[\prod_{(i,j)\in[\lambda]\setminus\mathcal{C}[\lambda]} *\right] = \sum_{(r,s)\in\mathcal{C}[\lambda]} \left[\prod_{\substack{(i,j)\in[\lambda]\setminus\mathcal{C}[\lambda]\\ i\neq r, j\neq s}} *\right] \cdot \left[\prod_{i=2}^{r} *\right] \cdot \left[\prod_{j=1}^{s} *\right], \tag{5}$$

$$\left[\prod_{(i,j)\in[\lambda]\setminus\mathcal{C}[\lambda]} *\right] = \sum_{(r,s)\in\mathcal{C}[\lambda]} \left[\prod_{\substack{(i,j)\in[\lambda]\setminus\mathcal{C}[\lambda]\\ i\neq r, j\neq s}} *\right] \cdot \left[\prod_{i=2}^{r} *\right] \cdot \left[\prod_{j=2}^{s} *\right]. \tag{6}$$

The proofs are very similar. We start the hook walk in square $(p, 1)$ for (4), $(1, q)$ for (5), and $(1, 1)$ for (6). We proceed as in the proof of WBR, except that in the final arrangement, the square $(r, 1)$ (respectively, $(1, s)$, respectively, both $(r, 1)$ and $(1, s)$) does not get a label.

A direct bijective proof of CWBR shares many characteristics with the bijective proof of WBR. We interpret both left-hand and right-hand sides as labelings of the diagram; we start the bijection with a (variant of the) hook walk; and the hook walk determines a relabeling of the diagram. There are, however, some important differences. First, the walk always starts in the square $(1, 1)$. Second, the hook walk can never pass through a square that is not in the same row as an outer corner and the same column as an outer corner. Third, the rule for one step of the hook walk is different from the one in [7]. And finally, there is an extra shift in the relabeling process.

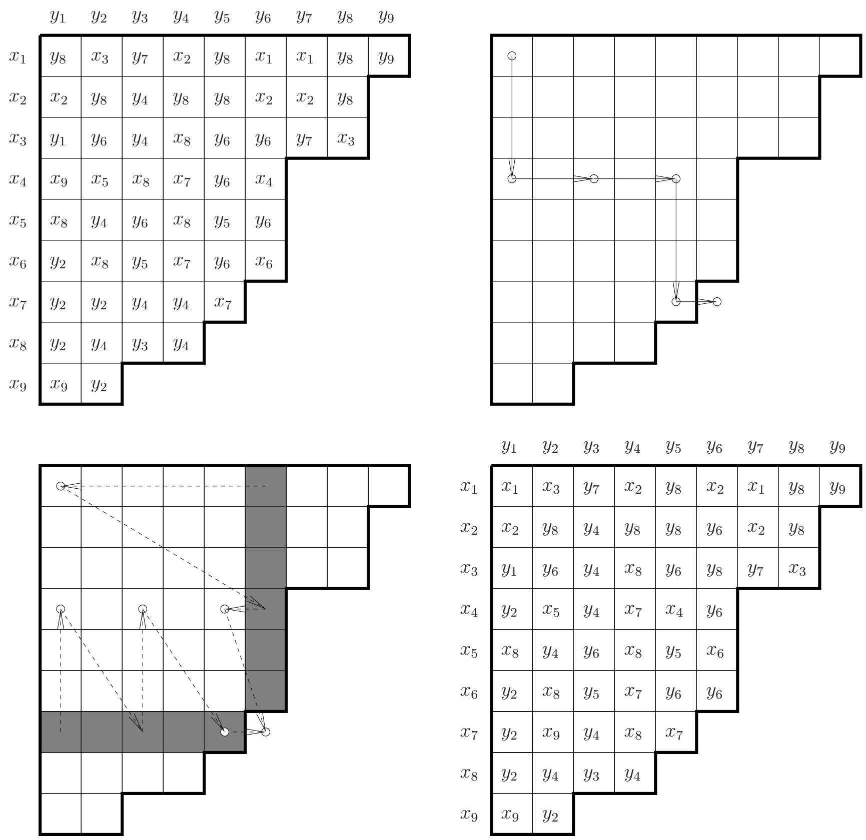

Fig. 2: An example of an arrangement corresponding to the left-hand side of CWBR for $\lambda = 988666542$; hook walk; shift of labels; final arrangement.

For the left-hand side of CWBR, we are given a label x_k for some $i \leq k \leq \lambda'_j$, or y_l for some $j \leq l \leq \lambda_i$, for every square $(i, j) \in [\lambda]$. Denote by F the resulting arrangement of n labels (see Figure 2, top left).

Again, we first construct a hook walk. Start in $(1,1)$, and move only through squares which are in the same row as an outer corner and in the same column as an outer corner. The rule is as follows. If the current square is (i,j) and the label of (i,j) in F is x_k for $i \le k \le \lambda'_j$, move to $(i, \lambda_k + 1)$. If the label of (i,j) in F is y_l for $j \le l \le \lambda'_j$, move to $(\lambda'_l + 1, j)$. Note that $i \le k$ implies $\lambda_k \le \lambda_i$ and $j \le l$ implies $\lambda'_l \le \lambda'_j$, so the square we move to is either in $[\lambda]$ or is the outer corner to the right or below (i,j). The process continues until we arrive in an outer corner (r,s), see the top right drawing in Figure 2.

Shade row r and column s. Now we shift the labels in the hook walk and in its projection onto the shaded row and column. If the hook walk has a horizontal step from (i,j) to (i,j'), $i \ne r$, move the label in (i,j) right and down to (r,j'), and the label from (r,j) up to (i,j). If the hook walk has a vertical step from (i,j) to (i',j), $j \ne s$, move the label from (i,j) down and right to (i',s), and the label from (i,s) left to (i,j). If the hook walk has a horizontal step from (r,j) to (r,j'), move the label in (r,j) right to (r,j'). If the hook walk has a vertical step from (i,s) to (i',s), move the label in (i,s) down to (i',s). See Figure 2, bottom left.

After these changes, we have the following situation. If $r = 1$, there is no label in $(1,1)$, and in $(1,s)$ the label is x_k, $1 \le k \le \lambda'_{\lambda_1}$. Move all the labels in row 1 one square to the left. If $s = 1$, there is no label in $(1,1)$, and in $(r,1)$ the label is y_l, $1 \le l \le \lambda_{\ell(\lambda)}$. Move all the labels in column 1 one square up. If $r > 1$ and $s > 1$, there are no labels in $(r,1)$ and $(1,s)$. In (r,s), there are two labels: one of the form x_k for $r \le k \le \lambda'_{s-1}$, and one of the form y_l for $s \le l \le \lambda_{r-1}$. Push all the labels in row r, including x_k in (r,s), one square to the left; and push all labels in column s, including y_l in (r,s), one square up. See Figure 2, bottom right, for the final arrangement, which we denote G. It turns out that the final arrangement represents a term on the right-hand side of CWBR, and the map $F \mapsto G$ is a bijection.

Again, there are variants of the formula with similar bijective proofs. Namely, if CWBR is

$$\prod_{(i,j)\in[\lambda]} * = \sum_{(r,s)\in\mathcal{C}'[\lambda]} \left[\prod_{\substack{(i,j)\in[\lambda]\\ i\ne r, j\ne s}} *\right] \cdot \left[\prod_{i=1}^{r-1} *\right] \cdot \left[\prod_{j=1}^{s-1} *\right],$$

we also have

$$\left[\sum_{p=1}^{\ell(\lambda)} x_p\right] \cdot \left[\prod_{(i,j)\in[\lambda], j\ne 1} *\right] = \sum_{(r,s)\in\mathcal{C}'[\lambda], s\ne 1} \left[\prod_{\substack{(i,j)\in[\lambda]\\ i\ne r, j\ne 1,s}} *\right] \cdot \left[\prod_{i=1}^{r-1} *\right] \cdot \left[\prod_{j=1}^{s-1} *\right], \tag{7}$$

$$\left[\sum_{q=1}^{\lambda_1} y_q\right] \cdot \left[\prod_{(i,j)\in[\lambda], i\ne 1} *\right] = \sum_{(r,s)\in\mathcal{C}'[\lambda], r\ne 1} \left[\prod_{\substack{(i,j)\in[\lambda]\\ i\ne 1,r, j\ne s}} *\right] \cdot \left[\prod_{i=1}^{r-1} *\right] \cdot \left[\prod_{j=1}^{s-1} *\right], \tag{8}$$

$$\left[\sum_{(p,q)\notin[\lambda]} x_p y_q\right] \cdot \left[\prod_{(i,j)\in[\lambda], i,j\ne 1} *\right] = \sum_{(r,s)\in\mathcal{C}'[\lambda], r,s\ne 1} \left[\prod_{\substack{(i,j)\in[\lambda]\\ i\ne 1,r, j\ne 1,s}} *\right] \cdot \left[\prod_{i=1}^{r-1} *\right] \cdot \left[\prod_{j=1}^{s-1} *\right]. \tag{9}$$

The sum on the left-hand side of (9) is over all (i,j) such that $1 \le i \le \ell(\lambda)$, $1 \le j \le \lambda_1$, $(i,j) \notin [\lambda]$. The proofs of these identities are almost identical to the one for CWBR. We start the hook walk in square $(1, \lambda_p + 1)$ (respectively, in $(\lambda'_q + 1, 1)$, respectively, in $(\lambda'_q + 1, \lambda_p + 1)$); we construct the hook walk in exactly the same fashion as before; we perform the relabeling as before; but before the final shift to the left and up by one, we label (r, λ_p+1) (respectively, (λ'_q+1, s), respectively, both) with x_p (respectively, y_q, respectively, both).

4 Weighted hook walks

Choose a partition λ and draw the borders of its diagram in the plane. Now add lines $x = 0$, $x = \ell(\lambda)$, $y = 0$, $y = \lambda_1$; this divides the plane into ten regions $R_1, \ldots, R_{10}$ (R_5 is empty if $\lambda = a^b$ for some a and b). See Figure 3 for an example and the labelings of these regions. Draw the following lines in bold: the half-line $x = 0$, $y \geq \lambda_1$, the half-line $x = \ell(\lambda)$, $y \leq 0$, the half-line $y = 0$, $x \geq \ell(\lambda)$, the half-line $y = \lambda_1$, $x \leq 0$, and the zigzag line separating regions R_1 and R_5.

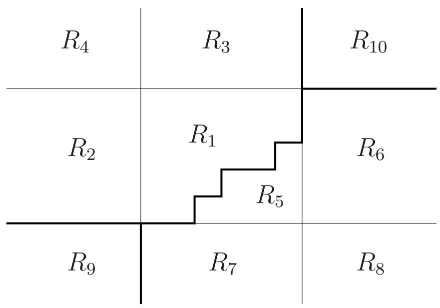

Fig. 3: Division of the plane into regions $R_1, \ldots, R_{10}$ for $\lambda = 66532$, with some lines in bold.

Define a *weighted hook walk* as follows. Choose positive weights $(x_i)_{i=-\infty}^{\infty}$, $(y_j)_{j=-\infty}^{\infty}$ satisfying $\sum_i x_i < \infty$, $\sum_j y_j < \infty$. Select the starting square for the hook walk so that the probability of selecting the square (i, j) is proportional to $x_i y_j$. In each step, move in a vertical or horizontal direction toward the bolded line; in regions R_1, R_2, R_3 and R_4, right or down; in regions R_5, R_6, R_7 and R_8, left or up; in region R_9, right or up; and in region R_{10}, left or down. More specifically, if the current position is (i, j), move to the square (i', j) between (i, j) and the bolded line with probability proportional to $x_{i'}$, and to the square (i, j') between (i, j) and the bolded line with probability proportional to $y_{j'}$. The process stops if we are either in one of the corners of λ (if the initial square was in regions R_1, R_2, R_3 or R_4), one of the outer corners of λ (if the initial square was in regions R_5, R_6, R_7 or R_8), the square $(\ell(\lambda) + 1, 0)$ (if the initial square was in region R_9) or $(0, \lambda_1 + 1)$ (if the initial square was in region R_{10}). These last two possibilities are not particularly interesting.

Below, we give the probabilities of terminating in a particular corner conditional on starting in R_1, R_2, R_3 and R_4, as well as probabilities of ending in a particular outer corner, conditional on starting in R_5, R_6, R_7 and R_8. The most interesting observation is that these probabilities turn out to depend only on $x_1, \ldots, x_{\ell(\lambda)}, y_1, \ldots, y_{\lambda_1}$. As a corollary, we obtain the conditional probabilities in the case where all these values are equal. They represent generalizations of classical results due to Greene, Nijenhuis and Wilf from [13], [14].

We extend the definition of λ_i, λ'_j to all $i, j \in \mathbb{Z}$ in a natural way as follows: for $i \leq 0$, $\lambda_i = \lambda_1$; for $i \geq \ell(\lambda) + 1$, $\lambda_i = 0$; for $j \leq 0$, $\lambda'_j = \ell(\lambda)$; for $j \geq \lambda_1 + 1$, $\lambda'_j = 0$. The following two theorems tell us how to compute probabilities of ending in corners and outer corners.

Theorem 1 *For a corner $\mathbf{c} = (r,s)$ of λ, denote by $P(\mathbf{c}|R)$ the probability that the weighted hook walk terminates in $\mathbf{c}$, conditional on the starting point being in R. Write*

$$\prod\nolimits_{rs} = x_r y_s \prod_{i=1}^{r-1} \left(1 + \tfrac{x_i}{x_{i+1}+\ldots+x_r+y_{s+1}+\ldots+y_{\lambda_i}}\right) \cdot \prod_{j=1}^{s-1} \left(1 + \tfrac{y_j}{x_{r+1}+\ldots+x_{\lambda'_j}+y_{j+1}+\ldots+y_s}\right).$$

Then:

(a) $P(\mathbf{c}|R_1) = \frac{1}{\sum_{(p,q)\in[\lambda]} x_p y_q} \cdot \prod_{rs}$

(b) $P(\mathbf{c}|R_2) = \frac{1}{\left(\sum_{p=1}^{\ell(\lambda)} x_p\right)(x_{r+1}+\ldots+x_{\ell(\lambda)}+y_1+\ldots+y_s)} \cdot \prod_{rs}$

(c) $P(\mathbf{c}|R_3) = \frac{1}{\left(\sum_{q=1}^{\lambda_1} y_q\right)(x_1+\ldots+x_r+y_{s+1}+\ldots+y_{\lambda_1})} \cdot \prod_{rs}$

(d) $P(\mathbf{c}|R_4) = \frac{1}{(x_{r+1}+\ldots+x_{\ell(\lambda)}+y_1+\ldots+y_s)(x_1+\ldots+x_r+y_{s+1}+\ldots+y_{\lambda_1})} \cdot \prod_{rs}$

In particular, the sum of each of the above terms over all corners of λ equals 1; note that this proves WBR, (4), (5) *and* (6)*. Also,*

(e) $P(\mathbf{c}) = \frac{1}{\left(\sum_p x_p\right)\cdot\left(\sum_q y_q\right)} \cdot \left(1 + \frac{\sum_{p\leq 0} x_p}{x_1+\ldots+x_r+y_{s+1}+\ldots+y_{\lambda_1}}\right) \cdot \left(1 + \frac{\sum_{q\leq 0} y_q}{x_{r+1}+\ldots+x_{\ell(\lambda)}+y_1+\ldots+y_s}\right) \cdot \prod_{rs}.$

Theorem 2 *For an outer corner, $\mathbf{c} = (r,s)$ of λ, denote by $P(\mathbf{c}|R)$ the probability that the weighted hook walk terminates in $\mathbf{c}$, conditional on the starting point being in R. Write*

$$\prod\nolimits'_{rs} = \prod_{i=1}^{r-1} \left(1 - \tfrac{x_i}{x_i+\ldots+x_{r-1}+y_s+\ldots+y_{\lambda_i}}\right) \cdot \prod_{j=1}^{s-1} \left(1 - \tfrac{y_j}{x_r+\ldots+x_{\lambda'_j}+y_j+\ldots+y_{s-1}}\right).$$

Then:

(a) $P(\mathbf{c}|R_5) = \frac{(x_r+\ldots+x_{\ell(\lambda)}+y_1+\ldots+y_{s-1})(x_1+\ldots+x_{r-1}+y_s+\ldots+y_{\lambda_1})}{\sum_{(p,q)\notin[\lambda]} x_p y_q} \cdot \prod'_{rs}$

(b) $P(\mathbf{c}|R_6) = \frac{x_r+\ldots+x_{\ell(\lambda)}+y_1+\ldots+y_{s-1}}{\sum_{i=1}^{\ell(\lambda)} x_p} \cdot \prod'_{rs}$

(c) $P(\mathbf{c}|R_7) = \frac{x_1+\ldots+x_{r-1}+y_s+\ldots+y_{\lambda_1}}{\sum_{q=1}^{\lambda_1} y_q} \cdot \prod'_{rs}$

(d) $P(\mathbf{c}|R_8) = \prod'_{rs}$

In particular, the sum of each of the above terms over all outer corners of λ equals 1; note that this proves CWBR, (7), (8) *and* (9)*. Also,*

(e) $P(\mathbf{c}) = \frac{(x_1+\ldots+x_{r-1}+\sum_{q=s}^{\infty} y_q)\cdot(\sum_{p=r}^{\infty} x_p+y_1+\ldots+y_{s-1})}{\left(\sum_p x_p\right)\cdot\left(\sum_q y_q\right)} \cdot \prod'_{rs}.$

Corollary 3 *If $x_1 = \ldots = x_{\ell(\lambda)} = y_1 = \ldots = y_{\lambda_1}$, then we have the following. For a corner $\mathbf{c} = (r, s)$ of λ,*

$$P(\mathbf{c}|R_1) = \frac{f^{\lambda-\mathbf{c}}}{f^\lambda}, \qquad P(\mathbf{c}|R_2) = \frac{nf^{\lambda-\mathbf{c}}}{\ell(\lambda)(\ell(\lambda)-r+s)f^\lambda}$$
$$P(\mathbf{c}|R_3) = \frac{nf^{\lambda-\mathbf{c}}}{\lambda_1(\lambda_1+r-s)f^\lambda}, \qquad P(\mathbf{c}|R_4) = \frac{nf^{\lambda-\mathbf{c}}}{(\ell(\lambda)-r+s)(\lambda_1+r-s)f^\lambda}$$

In particular, the sum of each of the above terms over all corners of λ equals 1.
For an outer corner, $\mathbf{c} = (r, s)$ of λ,

$$P(\mathbf{c}|R_5) = \frac{(\ell(\lambda)-r+s)(\lambda_1+r-s)f^{\lambda+\mathbf{c}}}{(n+1)(\ell(\lambda)\lambda_1-n)f^\lambda}, \qquad P(\mathbf{c}|R_6) = \frac{(\ell(\lambda)-r+s)f^{\lambda+\mathbf{c}}}{(n+1)\ell(\lambda)f^\lambda}$$
$$P(\mathbf{c}|R_7) = \frac{(\lambda_1+r-s)f^{\lambda+\mathbf{c}}}{(n+1)\lambda_1 f^\lambda}, \qquad P(\mathbf{c}|R_8) = \frac{f^{\lambda+\mathbf{c}}}{(n+1)f^\lambda}$$

In particular, the sum of each of the above terms over all outer corners of λ equals 1.

The corollary (for probabilities conditional on starting in $R_2, R_3, \ldots, R_7$) gives six new recursive formulas for numbers of standard Young tableaux. The sums over outer corners have the following interesting interpretation. Recall that the content of a square (i, j) of a diagram $[\lambda]$ is defined as $i - j$.

Corollary 4 *Fix a partition $\lambda \vdash n$. Choose a standard Young tableau of shape λ uniformly at random, and an integer i, $1 \le i \le n+1$ uniformly at random. In the standard Young tableau, increase all integers $\ge i$ by 1, and use the bumping process of the Robinson-Schensted algorithm to insert i in the tableau. Define the random variable X as the content of the square that is added to λ. Then*

$$E(X) = 0, \qquad \operatorname{var}(X) = n.$$

Proof: The bumping process is a bijection $\mathrm{SYT}(\lambda) \times \{1, 2, \ldots, n+1\} \longrightarrow \bigcup_{\mathbf{c} \in \mathcal{C}'[\lambda]} \mathrm{SYT}(\lambda + \mathbf{c})$. This means that the probability that $\mathbf{c}$ is the square added to λ is equal to $\frac{f^{\lambda+\mathbf{c}}}{(n+1)f^\lambda}$. We have

$$(n+1)\lambda_1 f^\lambda = \sum(\lambda_1+r-s)f^{\lambda+\mathbf{c}} = \lambda_1 \sum f^{\lambda+\mathbf{c}} + \sum(r-s)f^{\lambda+\mathbf{c}} = (n+1)\lambda_1 f^\lambda + \sum(r-s)f^{\lambda+\mathbf{c}}$$

and therefore $\sum(r-s)f^{\lambda+\mathbf{c}} = 0$, which is equivalent to $E(X) = 0$. On the other hand, we know that

$$(n+1)(\ell(\lambda)\lambda_1 - n)f^\lambda = \sum(\ell(\lambda)-r+s)(\lambda_1+r-s)f^{\lambda+\mathbf{c}} =$$

$$= \ell(\lambda)\lambda_1 \sum f^{\lambda+\mathbf{c}} + (\ell(\lambda)-\lambda_1)\sum(r-s)f^{\lambda+\mathbf{c}} - \sum(r-s)^2 f^{\lambda+\mathbf{c}}$$

and so $\sum(r-s)^2 f^{\lambda+\mathbf{c}} = (n+1)nf^\lambda$. Division by $(n+1)f^\lambda$ shows that $\operatorname{var}(X) = n$. □

5 Final remarks

As Knuth wrote in 1973, "Since the hook-lengths formula is such a simple result, it deserves a simple proof ..." (see p. 63 of the first edition of [17], cited also in [26]). Unfortunately, the desired simple proofs have been sorely lacking. It is our hope that Section 3 can be viewed as one such proof.

Surveying the history of the hook length formula is a difficult task, even if one is restricted to purely combinatorial proofs. This extended abstract is too short to even attempt such an endeavor. See [7, §6] for a brief outline, and the references therein.

There are several directions in which our results can be potentially extended. First, it would be interesting to obtain the analogues of our results for *shifted* Young diagrams and Young tableaux, for which there is a analogue of the hook length formula due to Thrall [25] (see also [23]). Similarly, most hook formula results easily extend to trees, and one can try to obtain a weighted analogue in this case as well. However, we are less confident this approach will give new and interesting (or at least non-trivial) formulas. Extending to semi-standard and skew tableaux is another possibility, in which case one would be looking for a weighted analogue of Stanley's hook-content formula [24]. Finally, let us mention several new extensions of the hook length formula recently introduced by Guo-Niu Han in [11, 12].

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