FPSAC 2010, San Francisco, USA DMTCS proc. **AN**, 2010, 343–354

Generalized Energy Statistics and Kostka–Macdonald Polynomials

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Abstract. We give an interpretation of the $t = 1$ specialization of the modified Macdonald polynomial as a generating function of the energy statistics defined on the set of paths arising in the context of Box-Ball Systems (BBS-paths for short). We also introduce one parameter generalizations of the energy statistics on the set of BBS-paths which all, conjecturally, have the same distribution.

Résumé. Nous donnons une intérprétation de la spécialisation à $t = 1$ du polynôme de Macdonald modifié comme fonction génératrice des statistiques d'énergie définies sur l'ensemble des chemins qui apparaissent dans la théorie des Systèmes BBS (BBS-chemins). Nous présentons également des généralisations à un paramètre de la statistique d'énergie sur les chemins BBS qui toutes, conjecturalement, ont la même distribution.

Keywords: modified Macdonald polynomials, box-ball systems

1 Introduction

The purpose of the present paper is two-fold. First of all we would like to draw attention to a rich combinatorics hidden behind the dynamics of Box-Ball Systems, and secondly, to connect the former with the theory of modified Macdonald polynomials. More specifically, our final goal is to give an interpretation of the Kostka–Macdonald polynomials $K_{\lambda,\mu}(q,t)$ as a *refined partition function* of a certain box-ball systems depending on initial data λ and μ.

Box-Ball Systems (BBS for short) were invented by Takahashi–Satsuma [29, 28] as a wide class of discrete integrable soliton systems. In the simplest case, BBS are described by simple combinatorial procedures using boxes and balls. One can see the simplest but still very interesting examples of the BBS by the free software available at [26]. Despite its simple outlook, it is known that the BBS have various remarkably deep properties:

- Local time evolution rule of the BBS coincides with the isomorphism of the crystal bases [7, 2]. Thus the BBS possesses quantum integrability.

†Supported by Grant-in-Aid for Scientific Research (No.21740114), JSPS.

- BBS are ultradiscrete (or tropical) limit of the usual soliton systems [30, 20]. Thus the BBS possesses classical integrability at the same time.
- Inverse scattering formalism of the BBS [19] coincides with the rigged configuration bijection originating in completeness problem of the Bethe states [14, 16], see also [25].

Let us say a few words about the main results of this note.

- We will identify the space of states of a BBS with the corresponding weight subspace in the tensor product of fundamental (or rectangular) representations of the Lie algebra $\mathfrak{gl}(n)$.
- In the case of statistics *tau*, our main result can be formulated as a computation of the corresponding partition function for the BBS in terms of the values of the Kostka–Macdonald polynomials at $t = 1$.
- In the case of the statistics *energy*, our result can be formulated as an interpretation of the corresponding partition function for the BBS as the q-weight multiplicity of a certain irreducible representation of the Lie algebra $\mathfrak{gl}(n)$ in the tensor product of the fundamental representations. We *expect* that the same statement is valid for the BBS corresponding to the tensor product of rectangular representations.

  Let us remind that a *q-analogue of the multiplicity* of a highest weight λ in the tensor product $\bigotimes_{a=1}^{L} V_{s_a\omega_{r_a}}$ of the highest weight $s_a\,\omega_{r_a}$, $a = 1, \ldots, L$, irreducible representations $V_{s_a\omega_{r_a}}$ of the Lie algebra $\mathfrak{gl}(n)$ is defined as

$$q\text{-Mult}\,[V_\lambda\ :\ \bigotimes_{a=1}^{L} V_{s_a\omega_{r_a}}] = \sum_{\eta} K_{\eta,R}\,K_{\eta,\lambda}(q),$$

  where $K_{\eta,R}$ stands for the parabolic Kostka number corresponding to the sequence of rectangles $R := \{(s_a^{r_a})\}_{a=1,\ldots,L}$, see e.g. [15], [18].

A combinatorial description of the modified Macdonald polynomials has been obtained by Haglund–Haiman–Loehr [5]. In Section 5 we give an interpretation of two Haglund's statistics in the context of the box-ball systems, i.e., in terms of the BBS-paths. Namely, we identify the set of BBS paths of weight α with the set $\mathcal{P}(\alpha)$ which is the weight α component in the tensor product of crystals corresponding to vector representations. We have observed that from the proof given in [5] one can prove the following identity

$$\sum_{p\in\mathcal{P}(\alpha)} q^{\mathrm{inv}_\mu(p)} t^{\mathrm{maj}_\mu(p)} = \sum_{\eta\vdash|\mu|} K_{\eta,\alpha}\tilde{K}_{\eta,\mu}(q,t), \tag{1}$$

see Proposition 6.2 and Corollary 6.3. One of the main problems we are interested in is to generalize the identity Eq.(1) on more wider set of the BBS-paths.

Our result about connections of the energy partition functions for BBS and q-weight multiplicities suggests a deep hidden connections between partition functions for the BBS and characters of the Demazure modules, solutions to the q-difference Toda equations, cf.[3], ...

As an interesting open problem we want to give raise a question about an interpretation of the sums $\sum_\eta K_{\eta,R}\,K_{\eta,\lambda}(q,t)$, where $K_{\eta,\lambda}(q,t)$ denotes the Kostka–Macdonald polynomials [21], as *refined partition functions* for the BBS corresponding to the tensor product of rectangular representations $R =$

$\{(s_a^{r_a})\}_{1\leq a\leq n}$. In other words, one can ask: what is a meaning of the second statistics (see [5]) in the Kashiwara theory [11] of crystal bases (of type A) ?

This paper is abbreviated and updated version of our paper [17]. The main novelty of the present paper is the definition of a one parameter family of statistics on the set of BBS-paths which generalizes those introduced in [17], see Conjecture 7.2. It conjecturally gives a new family of MacMahonian statistics on the set of transportation matrices, see [15].

Organization of the present paper is as follows. In Section 2 we remind algorithms of the combinatorial R-matrix and the energy functions. In Section 3, we introduce the energy statistics and the set of the BBS. In Section 4 we remind definition of box-ball systems and state some of their simplest properties. In Section 5 we remind definition of the Haglund's statistics and give their interpretation in terms of the BBS-paths. Sections 6 and 7 contain our main results and conjectures. In particular it is not difficult to see that Haglund's statistics maj_μ and inv_μ do not compatible with the Kostka–Macdonald polynomials for general partitions λ and μ. In Section 6 we state a conjecture which describes the all pairs of partitions (λ,μ) for those the restriction of the Haglund–Haiman–Loehr formula on the set of highest weight paths of shape μ coincide with the Kostka–Macdonald polynomial $\tilde{K}_{\lambda,\mu}(q,t)$.

2 Combinatorial R and energy function

Let $B^{r,s}$ be the Kirillov–Reshetikhin crystals of type $A_n^{(1)}$ (see [11, 12, 10], see also section 2 of [17]). Here $r\in\{1,2,\cdots,n\}$ and $s\in\mathbb{Z}_{>0}$. As the set, $B^{r,s}$ is consisting of all semistandard tableaux of height r and width s. In this section, we recall an explicit description of the combinatorial R-matrix (combinatorial R for short) and energy function on $B^{r,s}\otimes B^{r',s'}$. To begin with we define few terminologies about Young tableaux. Denote rows of a Young tableaux Y by $y_1,y_2,\dots y_r$ from top to bottom. Then row word $row(Y)$ is defined by concatenating rows as $row(Y)=y_ry_{r-1}\dots y_1$. Let $x=(x_1,x_2,\dots)$ and $y=(y_1,y_2,\dots)$ be two partitions. We define concatenation of x and y by the partition $(x_1+y_1,x_2+y_2,\dots)$.

Proposition 2.1 ([27]) *$b\otimes b'\in B^{r,s}\otimes B^{r',s'}$ is mapped to $\tilde{b}'\otimes\tilde{b}\in B^{r',s'}\otimes B^{r,s}$ under the combinatorial R, i.e.,*

$$b\otimes b'\overset{R}{\simeq}\tilde{b}'\otimes\tilde{b}, \tag{2}$$

if and only if

$$(b'\leftarrow row(b))=(\tilde{b}\leftarrow row(\tilde{b}')). \tag{3}$$

Moreover, the energy function $H(b\otimes b')$ is given by the number of nodes of $(b'\leftarrow row(b))$ outside the concatenation of partitions (s^r) and $(s'^{r'})$.

For special cases of $B^{1,s}\otimes B^{1,s'}$, the function H is called unwinding number in [22]. Explicit values for the case $b\otimes b'\in B^{1,1}\otimes B^{1,1}$ are given by $H(b\otimes b')=\chi(b<b')$ where $\chi(\mathrm{True})=1$ and $\chi(\mathrm{False})=0$.

In order to describe the algorithm for finding $\tilde{b}$ and $\tilde{b}'$ from the data $(b'\leftarrow row(b))$, we introduce a terminology. Let Y be a tableau, and Y' be a subset of Y such that Y' is also a tableau. Consider the set theoretic subtraction $\theta=Y\setminus Y'$. If the number of nodes contained in θ is r and if the number of nodes of θ contained in each row is always 0 or 1, then θ is called vertical r-strip.

Given a tableau $Y=(b'\leftarrow row(b))$, let Y' be the upper left part of Y whose shape is (s^r). We assign numbers from 1 to $r's'$ for each node contained in $\theta=Y\setminus Y'$ by the following procedure. Let θ_1 be the vertical r'-strip of θ as upper as possible. For each node in θ_1, we assign numbers 1 through r' from the bottom to top. Next we consider $\theta \setminus \theta_1$, and find the vertical r' strip θ_2 by the same way. Continue this procedure until all nodes of θ are assigned numbers up to $r's'$. Then we apply inverse bumping procedure according to the labeling of nodes in θ. Denote by u_1 the integer which is ejected when we apply inverse bumping procedure starting from the node with label 1. Denote by Y_1 the tableau such that $(Y_1 \leftarrow u_1) = Y$. Next we apply inverse bumping procedure starting from the node of Y_1 labeled by 2, and obtain the integer u_2 and tableau Y_2. We do this procedure until we obtain $u_{r's'}$ and $Y_{r's'}$. Finally, we have

$$\tilde{b}' = (\emptyset \leftarrow u_{r's'}u_{r's'-1}\cdots u_1), \qquad \tilde{b} = Y_{r's'}. \tag{4}$$

3 Energy statistics and its generalizations on the set of paths

For a path $b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B^{r_1,s_1} \otimes B^{r_2,s_2} \otimes \cdots \otimes B^{r_L,s_L}$, let us define elements $b_j^{(i)} \in B^{r_j,s_j}$ for $i < j$ by the following isomorphisms of the combinatorial R;

$$\begin{aligned} & b_1 \otimes b_2 \otimes \cdots \otimes b_{i-1} \otimes b_i \otimes \cdots \otimes b_{j-1} \otimes b_j \otimes \cdots \\ \simeq\ & b_1 \otimes b_2 \otimes \cdots \otimes b_{i-1} \otimes b_i \otimes \cdots \otimes b_j^{(j-1)} \otimes b'_{j-1} \otimes \cdots \\ \simeq\ & \cdots \\ \simeq\ & b_1 \otimes b_2 \otimes \cdots \otimes b_{i-1} \otimes b_j^{(i)} \otimes \cdots \otimes b'_{j-2} \otimes b'_{j-1} \otimes \cdots , \end{aligned} \tag{5}$$

where we have written $b_k \otimes b_j^{(k+1)} \simeq b_j^{(k)} \otimes b'_k$ assuming that $b_j^{(j)} = b_j$.

Define the statistics $\mathrm{maj}(p)$ by

$$\mathrm{maj}(p) = \sum_{i<j} H(b_i \otimes b_j^{(i+1)}). \tag{6}$$

For example, consider a path $a = a_1 \otimes a_2 \otimes \cdots \otimes a_L \in (B^{1,1})^{\otimes L}$. In this case, we have $a_j^{(i)} = a_i$, since the combinatorial R act on $B^{1,1} \otimes B^{1,1}$ as identity. Therefore, we have

$$\mathrm{maj}(a) = \sum_{i=1}^{L-1} (L-i)\chi(a_i < a_{i+1}). \tag{7}$$

Define another statistics *tau* as follows.

Definition 3.1 *For the path $p \in B^{r_1,s_1} \otimes B^{r_2,s_2} \otimes \cdots \otimes B^{r_L,s_L}$, define $\tau^{r,s}$ by*

$$\tau^{r,s}(p) = \mathrm{maj}(u_s^{(r)} \otimes p), \tag{8}$$

where $u_s^{(r)}$ is the highest element of $B^{r,s}$.

Here the highest element $u_s^{(r)} \in B^{r,s}$ is the tableau whose i-th row is occupied by integers i. For example, $u_4^{(3)} = \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&2&2&2\\\hline 3&3&3&3\\\hline\end{array}$. In particular, the statistics $\tau^{r,1}$ on $B^{1,1}$ type paths $a \in (B^{1,1})^{\otimes L}$ has the following form;

$$\tau^{r,1}(a) = L \cdot \chi(r < a_1) + \sum_{i=1}^{L-1} (L-i)\chi(a_i < a_{i+1}), \tag{9}$$

where a_1 denotes the first letter of the path a. Note that $\tau^{1,1}$ is a special case of the tau functions for the box-ball systems [20, 24] which originates as an ultradiscrete limit of the tau functions for the KP hierarchy [9].

Definition 3.2 *For composition $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$, write $\mu_{[i]} = \sum_{j=1}^{i} \mu_j$ with convention $\mu_{[0]} = 0$. Then we define a generalization of $\tau^{r,1}$ by*

$$\tau_{\mu}^{r,1}(a) = \sum_{i=1}^{n} \tau^{r,1}(a_{[i]}), \tag{10}$$

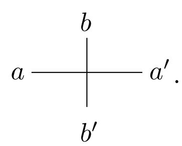
where

$$a_{[i]} = a_{\mu_{[i-1]}+1} \otimes a_{\mu_{[i-1]}+2} \otimes \cdots \otimes a_{\mu_{[i]}} \in (B^{1,1})^{\otimes \mu_i}. \tag{11}$$

Note that we have $a = a_{[1]} \otimes a_{[2]} \otimes \cdots \otimes a_{[n]}$, i.e., the path a is partitioned according to μ.

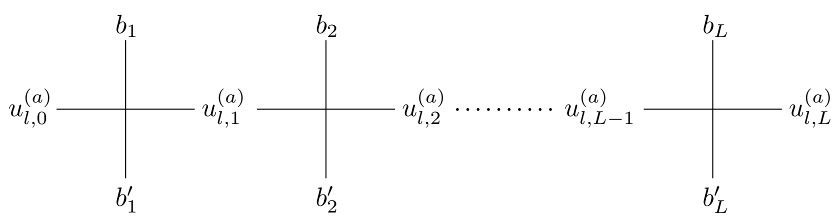
4 Box-ball system

In this section, we summarize basic facts about the box-ball system in order to explain physical origin of $\tau^{1,1}$. For our purpose, it is convenient to express the isomorphism of the combinatorial R: $a \otimes b \simeq b' \otimes a'$ by the following vertex diagram:



Successive applications of the combinatorial R is depicted by concatenating these vertices.

Following [7, 2], we define time evolution of the box-ball system $T_l^{(a)}$. Let $u_{l,0}^{(a)} = u_l^{(a)} \in B^{a,l}$ be the highest element and $b_i \in B^{r_i,s_i}$. Define $u_{l,j}^{(a)}$ and $b_i' \in B^{r_i,s_i}$ by the following diagram.



(12)

$u_{l,j}^{(a)}$ are usually called *carrier* and we set $u_{l,0}^{(a)} := u_l^{(a)}$. Then we define operator $T_l^{(a)}$ by

$$T_l^{(a)}(b) = b' = b_1' \otimes b_2' \otimes \cdots \otimes b_L'. \tag{13}$$

Recently [25], operators $T_l^{(a)}$ have used to derive crystal theoretical meaning of the rigged configuration bijection.

It is known ([19] Theorem 2.7) that there exists some $l \in \mathbb{Z}_{>0}$ such that

$$T_l^{(a)} = T_{l+1}^{(a)} = T_{l+2}^{(a)} = \cdots (=: T_{\infty}^{(a)}). \tag{14}$$

If the corresponding path is $b \in (B^{1,1})^{\otimes L}$, we have the following combinatorial description of the box-ball system [29, 28]. We regard $\boxed{1} \in B^{1,1}$ as an empty box of capacity 1, and $\boxed{i} \in B^{1,1}$ as a ball of label (or internal degree of freedom) i contained in the box. Then we have:

Proposition 4.1 ([7]) *For a path $b \in (B^{1,1})^{\otimes L}$ of type $A_n^{(1)}$, $T_\infty^{(1)}(b)$ is given by the following procedure.*

1. *Move every ball only once.*
2. *Move the leftmost ball with label $n+1$ to the nearest right empty box.*
3. *Move the leftmost ball with label $n+1$ among the rest to its nearest right empty box.*
4. *Repeat this procedure until all of the balls with label $n+1$ are moved.*
5. *Do the same procedure 2–4 for the balls with label n.*
6. *Repeat this procedure successively until all of the balls with label 2 are moved.*

There are extensions of this box and ball algorithm corresponding to generalizations of the box-ball systems with respect to each affine Lie algebra, see e.g., [8]. Using this box and ball interpretation, our statistics $\tau^{1,1}(b)$ admits the following interpretation.

Theorem 4.2 ([20] Theorem 7.4) *For a path $b \in (B^{1,1})^{\otimes L}$ of type $A_n^{(1)}$, $\tau^{1,1}(b)$ coincides with number of all balls $2, \cdots, n+1$ contained in paths b, $T_\infty^{(1)}(b)$, $\cdots$, $(T_\infty^{(1)})^{L-1}(b)$.*

Example 4.3 Consider the path $p = a \otimes b$ where $a = 4311211111$, $b = 4321111111$. Note that we omit all frames of tableaux of $B^{1,1}$ and symbols for tensor product. We compute $\tau_{(10,10)}(p)$ by using Theorem 4.2. According to Proposition 4.1, the time evolutions of the paths a and b are as follows:

4	3	1	1	2	1	1	1	1	1
1	1	4	3	1	2	1	1	1	1
1	1	1	1	4	1	3	2	1	1
1	1	1	1	1	4	1	1	3	2
1	1	1	1	1	1	4	1	1	1
1	1	1	1	1	1	1	4	1	1
1	1	1	1	1	1	1	1	4	1
1	1	1	1	1	1	1	1	1	4

4	3	2	1	1	1	1	1	1	1
1	1	1	4	3	2	1	1	1	1
1	1	1	1	1	1	4	3	2	1
1	1	1	1	1	1	1	1	1	4
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1

Here the left and right tables correspond to a and b, respectively. Rows of left (resp. right) table represent a, $T_\infty^{(1)}(a)$, $\cdots$, $(T_\infty^{(1)})^L(a)$ (resp., those for b) from top to bottom. Counting letters 2, 3 and 4 in each table, we have $\tau^{1,1}(a) = 16$, $\tau^{1,1}(b) = 10$ and we get $\tau^{1,1}_{(10,10)}(p) = 16 + 10 = 26$, which coincides with the computation by Eq.(9). Meanings of the above two dynamics corresponding to paths a and b are summarized as follows:

(a) Dynamics of the path a. In the first two rows, there are two solitons (length two soliton 43 and length one soliton 2), and in the lower rows, there are also two solitons (length one soliton 4 and length two soliton 32). This is scattering of two solitons. After the scattering, soliton 4 propagates at velocity one and soliton 32 propagates at velocity two without scattering.

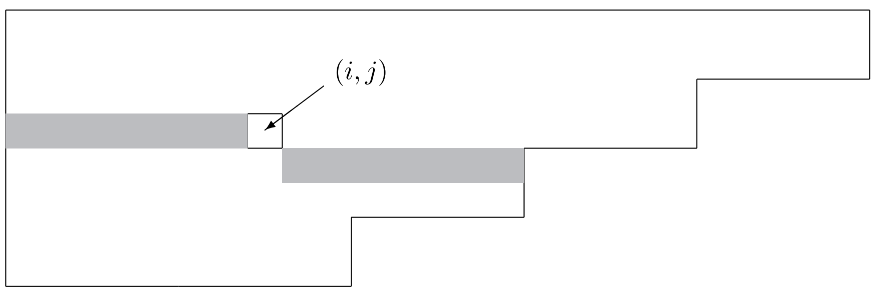
(b) Dynamics of the path b. This shows free propagation of one soliton of length three 432 at velocity three.

5 Haglund's statistics

Tableaux language description For a given path $a = a_1 \otimes a_2 \otimes \cdots \otimes a_L \in (B^{1,1})^{\otimes L}$, associate tabloid t of shape μ whose reading word coincides with a. For example, to path $p = abcdefgh$ and the composition $\mu = (3, 2, 3)$ one associates the tabloid

$$\begin{array}{|c|c|c|} \hline c & b & a \\ \hline e & d & \\ \cline{1-2} h & g & f \\ \hline \end{array} \ . \tag{15}$$

Denote the cell at the i-th row, j-th column (we denote the coordinate by (i, j)) of the tabloid t by t_{ij}. Attacking region of the cell at (i, j) is all cells (i, k) with $k < j$ or $(i + 1, k)$ with $k > j$. In the following diagram, gray zonal regions are the attacking regions of the cell (i, j).



Follow [5], define $|\mathrm{Inv}_{ij}|$ by

$$|\mathrm{Inv}_{ij}| = \#\{(k, l) \in \text{ attacking region for } (i, j) \,|\, t_{kl} > t_{ij}\}. \tag{16}$$

Then we define

$$|\mathrm{Inv}_\mu(a)| = \sum_{(i,j)\in\mu} |\mathrm{Inv}_{ij}|. \tag{17}$$

If we have $t_{(i-1)j} < t_{ij}$, then the cell (i, j) is called by *descent*. Then define

$$\mathrm{Des}_\mu(a) = \sum_{\text{all descent } (i,j)} (\mu_i - j). \tag{18}$$

Note that $(\mu_i - j)$ is the arm length of the cell (i, j).

Path language description Consider two paths $a^{(1)}, a^{(2)} \in (B^{1,1})^{\otimes\mu}$. We denote by $a^{(1)} \otimes a^{(2)} = a_1 \otimes a_2 \otimes \cdots \otimes a_{2\mu}$. Then we define

$$\mathrm{Inv}_{(\mu,\mu)}(a^{(1)}, a^{(2)}) = \sum_{k=1}^{\mu} \sum_{i=k+1}^{k+\mu-1} \chi(a_k < a_i). \tag{19}$$

For more general cases $a^{(1)} \in (B^{1,1})^{\otimes \mu_1}$ and $a^{(2)} \in (B^{1,1})^{\otimes \mu_2}$ satisfying $\mu_1 > \mu_2$, we define

$$\mathrm{Inv}_{(\mu_1,\mu_2)}(a^{(1)}, a^{(2)}) := \mathrm{Inv}_{(\mu_1,\mu_1)}(a^{(1)}, 1^{\otimes(\mu_1-\mu_2)} \otimes a^{(2)}). \quad (20)$$

Then the above definition of $|\mathrm{Inv}_\mu(a)|$ is equivalent to

$$|\mathrm{Inv}_\mu(a)| = \sum_{i=1}^{n-1} \mathrm{Inv}_{(\mu_i,\mu_{i+1})}. \quad (21)$$

Consider two paths $a^{(1)} \in (B^{1,1})^{\otimes \mu_1}$ and $a^{(2)} \in (B^{1,1})^{\otimes \mu_2}$ satisfying $\mu_1 \geq \mu_2$. Denote $a = a^{(1)} \otimes a^{(2)}$. Then define

$$\mathrm{Des}_{(\mu_1,\mu_2)}(a) = \sum_{k=\mu_1-\mu_2+1}^{\mu_1} (k - (\mu_1 - \mu_2) - 1)\chi(a_k < a_{k+\mu_2}). \quad (22)$$

For the tableau T of shape μ corresponding to the path a, we define

$$\mathrm{Des}_\mu(T) = \sum_{i=1}^{n} \mathrm{Des}_{(\mu_i,\mu_{i+1})}(a_{[i]} \otimes a_{[i+1]}). \quad (23)$$

Definition 5.1 ([4]) *For a path a, statistics maj_μ is defined by*

$$\mathrm{maj}_\mu(a) = \sum_{i=1}^{\mu_1} \mathrm{maj}(t_{1,i} \otimes t_{2,i} \otimes \cdots \otimes t_{\mu'_i,i}). \quad (24)$$

and $\mathrm{inv}_\mu(a)$ is defined by

$$\mathrm{inv}_\mu(a) = |\mathrm{Inv}_\mu(a)| - \mathrm{Des}_\mu(a). \quad (25)$$

If we associate to a given path $p \in \mathcal{P}(\lambda)$ with the shape μ tabloid T, we sometimes write $\mathrm{maj}_\mu(p) = \mathrm{maj}(T)$ and $\mathrm{inv}_\mu(p) = \mathrm{inv}(T)$.

6 Haglund–Haiman–Loehr formula

Let $\tilde{H}_\mu(x; q, t)$ be the (integral form) modified Macdonald polynomials where x stands for infinitely many variables $x_1, x_2, \cdots$. Here $\tilde{H}_\mu(x; q, t)$ is obtained by simple plethystic substitution (see, e.g., section 2 of [6]) from the original definition of the Macdonald polynomials [21]. Schur function expansion of $\tilde{H}_\mu(x; q, t)$ is given by

$$\tilde{H}_\mu(x; q, t) = \sum_\lambda \tilde{K}_{\lambda,\mu}(q, t) s_\lambda(x), \quad (26)$$

where $\tilde{K}_{\lambda,\mu}(q, t)$ stands for the following transformation of the Kostka–Macdonald polynomials:

$$\tilde{K}_{\lambda,\mu}(q, t) = t^{n(\mu)} K_{\lambda,\mu}(q, t^{-1}). \quad (27)$$

Here we have used notation $n(\mu) = \sum_i (i-1)\mu_i$. Then the celebrated Haglund–Haiman–Loehr (HHL) formula is as follows.

Theorem 6.1 ([5]) *Let $\sigma : \mu \to \mathbb{Z}_{>0}$ be the filling of the Young diagram μ by positive integers $\mathbb{Z}_{>0}$, and define $x^\sigma = \prod_{u\in\mu} x_{\sigma(u)}$. Then the Macdonald polynomial $\tilde{H}_\mu(x;q,t)$ have the following explicit formula:*

$$\tilde{H}_\mu(x;q,t) = \sum_{\sigma:\mu\to\mathbb{Z}_{>0}} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma. \tag{28}$$

From the HHL formula, we can show the following formula.

Proposition 6.2 *For any partition μ and composition α of the same size, one has*

$$\sum_{p\in\mathcal{P}(\alpha)} q^{\mathrm{inv}_\mu(p)} t^{\mathrm{maj}_\mu(p)} = \sum_{\eta\vdash|\mu|} K_{\eta,\alpha} \tilde{K}_{\eta,\mu}(q,t), \tag{29}$$

where $\mathcal{P}(\alpha)$ stands for the set of type $B^{1,1}$ paths of weight $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ and η runs over all partitions of size $|\mu|$.

Corollary 6.3 *The (modified) Macdonald polynomial $\tilde{H}_\mu(x;q,t)$ have the following expansion in terms of the monomial symmetric functions $m_\lambda(x)$:*

$$\tilde{H}_\mu(x;q,t) = \sum_{\lambda\vdash|\mu|} \left(\sum_{p\in\mathcal{P}(\lambda)} q^{\mathrm{inv}_\mu(p)} t^{\mathrm{maj}_\mu(p)} \right) m_\lambda(x), \tag{30}$$

where λ runs over all partitions of size $|\mu|$.

To find combinatorial interpretation of the Kostka–Macdonald polynomials $\tilde{K}_{\lambda,\mu}(q,t)$ remains significant open problem. Among many important partial results about this problem, we would like to mention the following theorem also due to Haglund–Haiman–Loehr:

Theorem 6.4 ([5] Proposition 9.2) *If $\mu_1 \le 2$, we have*

$$\tilde{K}_{\lambda,\mu}(q,t) = \sum_{p\in\mathcal{P}_+(\lambda)} q^{\mathrm{inv}_\mu(p)} t^{\mathrm{maj}_\mu(p)}, \tag{31}$$

where $\mathcal{P}_+(\lambda)$ is the set of all highest weight elements of $\mathcal{P}(\lambda)$ according to the reading order explained in Eq.(15).

It is interesting to compare this formula with the formula obtained by S. Fishel [1], see also [14], [18].

Concerning validity of the formula Eq.(31), we state the following conjecture.

Conjecture 6.5 *Explicit formula for the Kostka–Macdonald polynomials*

$$\tilde{K}_{\lambda,\mu}(q,t) = \sum_{p\in\mathcal{P}_+(\lambda)} q^{\mathrm{inv}_\mu(p)} t^{\mathrm{maj}_\mu(p)}. \tag{32}$$

is valid if and only if at least one of the following two conditions is satisfied.

(i) $\mu_1 \le 3$ *and* $\mu_2 \le 2$.

(ii) λ *is a hook shape.*

7 Generating function of tau functions

In [17], we give an elementary proof for special case $t = 1$ of the formula Eq.(29) in the following form.

Theorem 7.1 *Let α be a composition and μ be a partition of the same size. Then,*

$$\sum_{p\in\mathcal{P}(\alpha)} q^{\mathrm{maj}_{\mu'}(p)} = \sum_{\eta\vdash|\mu|} K_{\eta,\alpha}\, K_{\eta,\mu}(q,1). \tag{33}$$

Conjecture 7.2 *Let α be a composition and μ be a partition of the same size. Then,*

$$q^{-\sum_{i>r}\alpha_i} \sum_{p\in\mathcal{P}(\alpha)} q^{\tau_\mu^{r,1}(p)} = \sum_{\eta\vdash|\mu|} K_{\eta,\alpha}\tilde{K}_{\eta,\mu}(q,1). \tag{34}$$

This conjecture contains Conjecture 5.8 of [17] and Theorem 7.1 above as special cases $r = 1$ and $r = \infty$, respectively. Also, extensions for paths of more general representations without partition μ are discussed in Section 5.3 of [17].

Example 7.3 Let us consider case $\alpha = (4,1,1)$ and $\mu = (4,2)$. The following is a list of paths p and the corresponding value of tau function $\tau^{2,1}_{(4,2)}(p)$. For example, the top left corner $\boxed{111123 \quad 1}$ means $p = \boxed{1}\otimes\boxed{1}\otimes\boxed{1}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{3}$ and $\tau^{2,1}_{(4,2)}(p) = 1$.

111123	1	111132	2	111213	2	111231	3	111312	2	111321	1
112113	3	112131	4	112311	3	113112	3	113121	2	113211	2
121113	4	121131	5	121311	4	123111	5	131112	4	131121	3
131211	4	132111	3	211113	1	211131	2	211311	1	213111	2
231111	3	311112	5	311121	4	311211	5	312111	6	321111	4

Summing up, LHS of Eq.(34) is

$$q^{-1} \sum_{p\in\mathcal{P}((4,1,1))} q^{\tau^{2,1}_{(4,2)}(p)} = q^5 + 4q^4 + 7q^3 + 7q^2 + 7q + 4$$

which coincides with the RHS of Eq.(34). Compare this with $\tau^{1,1}_{(4,2)}$ data for the same set of paths at Example 5.9 of [17].

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