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Models and refined models for involutory reflection groups and classical Weyl groups

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Abstract. A finite subgroup G of $GL(n, \mathbb{C})$ is involutory if the sum of the dimensions of its irreducible complex representations is given by the number of absolute involutions in the group, i.e. elements $g \in G$ such that $g\bar{g} = 1$, where the bar denotes complex conjugation. A uniform combinatorial model is constructed for all non-exceptional irreducible complex reflection groups which are involutory including, in particular, all infinite families of finite irreducible Coxeter groups. If G is a classical Weyl group this result is much refined in a way which is compatible with the Robinson-Schensted correspondence on involutions.

Résumé. Un sousgroupe fini G de $GL(n, \mathbb{C})$ est dit involutoire si la somme des dimensions de ses representations irréductibles complexes est donné par le nombre de involutions absolues dans le groupe, c'est-a-dire le nombre de éléments $g \in G$ tels que $g\bar{g} = 1$, où le bar dénotes la conjugaison complexe. Un model combinatoire uniform est construit pour tous les groupes de réflexions complexes irréductibles qui sont involutoires, en comprenant, toutes les familles de groupes de Coxeter finis irreductibles. Si G est un groupe de Weyl ce resultat peut se raffiner dans une manière compatible avec la correspondence de Robinson-Schensted sur les involutions.

Keywords: Complex reflection groups, Gelfand models, Classical Weyl groups

1 Introduction

In their paper [7] Bernstein, Gelfand and Gelfand introduced the problem of the construction of a model of a group G, i.e. a representation which is the direct sum of all irreducible complex representations of G with multiplicity one. We can find several constructions of models in the literature for the symmetric group [2, 3, 13, 15, 16, 17] and for some other special classes of complex reflection groups [1, 4, 5, 6].
A complex reflection group, or simply a reflection group, is a subgroup of $GL(V)$, where V is a finite dimensional complex vector space, generated by reflections, i.e. by elements of finite order which fix a hyperplane pointwise. There is a well-known classification of irreducible reflection groups due to Shephard-Todd [20] including an infinite family $G(r, p, n)$ depending on 3 parameters together with 34 exceptional cases. As mentioned above one can find in the literature models for some reflection groups such as the wreath product groups $G(r, 1, n)$ as well as the groups $G(2, 2, n)$, which are better known as the Weyl groups of type D.
If G is a finite subgroup of $GL(n, \mathbb{C})$, a specialization of a theorem of Bump and Ginzburg [8] gives a

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combinatorial description of the character of a model of the group G if its dimension is given by the number of absolute involutions of G (i.e. elements $g \in G$ such that $g\bar{g} = 1$). We say that a group satisfying this condition is involutory. It turns out that a complex reflection group $G(r, p, n)$ is involutory if and only if $\mathrm{GCD}(p, n) = 1, 2$ and that one can construct an explicit model for all these groups in a uniform way. This construction involves in a crucial way the theory of projective reflection groups developed in [9]. Indeed a byproduct of this construction is also a model for some related projective reflection groups.

The model of the group G considered in this paper has a basis indexed by the absolute involutions of the dual group G^* (see §2) and it is clear from the definition that the subspace spanned by the basis elements indexed by the absolute involutions in a symmetric conjugacy class is a submodule. If G is a classical Weyl group we show that any such submodule is given by the sum of all irreducible representations indexed by the shapes corresponding to the indexing involutions by means of the projective Robinson-Schensted correspondence. This decomposition becomes particularly interesting for Weyl groups of type D with respect to the so-called split representations.

The paper is organized as follows. In §2 we collect the notation and the preliminary results which are needed. In §3 we classify all projective reflection groups of the form $G(r, p, q, n)$ (see §2 for the definition) which are involutory. In §4 we show an explicit model for all involutory reflection groups. In §5 a first decomposition is given for the model of the generic involutory reflection group $G(r, p, n)$, which reflects the existence of the split representations. In §6 and §7 a finer decomposition is given for the groups of type B_n and D_n.

2 Notation and preliminaries

In this section we collect the notations that are used in this paper as well as the preliminary results that are needed.

We let $\mathbb{Z}$ be the set of integer numbers and $\mathbb{N}$ be the set of nonnegative integer numbers. For $a, b \in \mathbb{Z}$, with $a \leq b$ we let $[a, b] \stackrel{\mathrm{def}}{=} \{a, a+1, \ldots, b\}$ and, for $n \in \mathbb{N}$ we let $[n] \stackrel{\mathrm{def}}{=} [1, n]$. For $r \in \mathbb{N}$ we let $\mathbb{Z}_r \stackrel{\mathrm{def}}{=} \mathbb{Z}/r\mathbb{Z}$. If $r \in \mathbb{N}$, $r > 0$, we denote by ζ_r the primitive r-th root of unity $\zeta_r \stackrel{\mathrm{def}}{=} e^{\frac{2\pi i}{r}}$.

The main subject of this work are the complex reflection groups, or simply reflection groups, with particular attention to their combinatorial representation theory. The most important example of a complex reflection group is the group of permutations of $[n]$, known as the symmetric group, that we denote by S_n. We know by the work of Shephard-Todd [20] that all but a finite number of irreducible reflection groups are the groups $G(r, p, n)$ that we are going to describe. If A is a matrix with complex entries we denote by $|A|$ the real matrix whose entries are the absolute values of the entries of A. The *wreath product* groups $G(r, n) = G(r, 1, n)$ are given by all $n \times n$ matrices satysfying the following conditions:

- the non-zero entries are r-th roots of unity;
- there is exactly one non-zero entry in every row and every column (i.e. $|A|$ is a permutation matrix).

If p divides r then the reflection group $G(r, p, n)$ is the subgroup of $G(r, n)$ given by all matrices $A \in G(r, n)$ such that $\frac{\det A}{\det |A|}$ is a $\frac{r}{p}$-th root of unity.
Following [9], a *projective reflection group* is a quotient of a reflection group by a scalar subgroup. Observe that a scalar subgroup of $G(r, n)$ is necessarily a cyclic group of the form $C_q =< \zeta_q I >$ of order q, for some $q|r$.

It is also easy to characterize all possible scalar subgroups of the groups $G(r,p,n)$: in fact the scalar matrix $\zeta_q I$ belongs to $G(r,p,n)$ if and only if $q|r$ and $pq|rn$. In this case we let $G(r,p,q,n) \stackrel{\text{def}}{=} G(r,p,n)/C_q$. If $G = G(r,p,q,n)$ then the projective reflection group $G^* \stackrel{\text{def}}{=} G(r,q,p,n)$, where the roles of the parameters p and q are interchanged, is always well-defined. We say that G^* is the *dual* of G and we refer the reader to [9] for the main properties of this duality. In this work we will see another important occurrence of the relationship between a group G and its dual G^*.

If the non-zero entry in the i-th row of $g \in G(r,n)$ is $\zeta_r^{z_i}$ we let $z_i(g) \stackrel{\text{def}}{=} z_i \in \mathbb{Z}_r$ and say that $z_1(g),\dots,z_n(g)$ are the *colors* of g. We can also note that g belongs to $G(r,p,n)$ if and only if $z(g) \stackrel{\text{def}}{=} \sum z_i(g) \equiv 0 \mod p$.

For $g \in G(r,n)$ we let $|g| \in S_n$ be the permutation defined by $|g|(i) = j$ if $g_{i,j} \neq 0$. We may observe that an element $g \in G(r,n)$ is uniquely determined by the permutation $|g|$ and by its colors $z_i(g)$ for all $i \in [n]$.

If $g \in G(r,n)$ we let $\bar{g} \in G(r,n)$ be the complex conjugate of g. We can also observe that $\bar{g}$ is determined by the conditions $|\bar{g}| = |g|$ and $z_i(\bar{g}) = -z_i(g)$ for all $i \in [n]$. Since the bar operator stabilizes the cyclic subgroup $C_q =< \zeta_q I >$ it is well-defined also on the projective reflection groups $G(r,p,q,n)$.

In [9] we can find a parametrization of the irreducible representations of the groups $G(r,p,q,n)$, that we briefly recall for the reader's convenience. Given a partition $\lambda = (\lambda_1,\dots,\lambda_l)$ of n, the *Ferrers diagram of shape* λ is a collection of boxes, arranged in left-justified rows, with λ_i boxes in row i. We denote by $\mathrm{Fer}(r,n)$ the set of r-tuples $(\lambda^{(0)},\dots,\lambda^{(r-1)})$ of Ferrers diagrams such that $\sum |\lambda^{(i)}| = n$. If $\mu \in \mathrm{Fer}(r,n)$ we define the *color* of μ by $z(\mu) = \sum_i i|\lambda^{(i)}|$ and, if $p|r$ we let $\mathrm{Fer}(r,p,n) \stackrel{\text{def}}{=} \{\mu \in \mathrm{Fer}(r,n) : z(\mu) \equiv 0 \mod p\}$. If $q \in \mathbb{N}$ is such that $q|r$ and $pq|nr$ then the cyclic group C_q acts on $\mathrm{Fer}(r,p,n)$ by a shift of r/q positions of its elements (see [9, Lemma 6.1]). Paralleling the definition for the projective reflection groups we denote the corresponding quotient set by $\mathrm{Fer}(r,p,q,n)$. We denote by $(C_p)_\mu$ the stabilizer of μ in C_p. For example, if

$$\mu = \left[\yng(2,1), \yng(1), \yng(2,1), \yng(1)\right] \text{ and } \mu' = \left[\yng(1), \yng(2,1), \yng(1), \yng(2,1)\right],$$

then μ and μ' are elements in $\mathrm{Fer}(4,2,8)$ which represent the same class in $\mathrm{Fer}(4,2,4,8)$. We also observe that in this case the stabilizer in $C_4(\mu) = C_4(\mu')$ is the cyclic group C_2 of order 2.

Proposition 2.1 *The irreducible complex representations of $G(r,p,q,n)$ can be parametrized by pairs (μ,ρ), where $\mu \in \mathrm{Fer}(r,q,p,n)$ and $\rho \in (C_p)_\mu$, where $(C_p)_\mu$ is the stabilizer of any element in the class μ by the action of C_p.*

If $\mu \in \mathrm{Fer}(r,n)$ we denote by $\mathcal{ST}_\mu$ the set of all possible fillings of the boxes in μ with all the numbers from 1 to n appearing once, in such way that rows are increasing from left to right and columns are incresing from top to bottom in every single Ferrers diagram of μ. We let $\mathcal{ST}(r,n) \stackrel{\text{def}}{=} \cup_{\mu\in\mathrm{Fer}(r,n)}\mathcal{ST}_\mu$ and we define $\mathcal{ST}(r,p,n)$ and $\mathcal{ST}(r,p,q,n)$ as already done for Ferrers diagrams. For example, the two elements

$$T = \left[\young(28,4), \young(1), \young(35,7), \young(6)\right] \text{ and } T' = \left[\young(6), \young(28,4), \young(1), \young(35,7)\right]$$

belong to $\mathcal{ST}(4,2,8)$ and represent the same class in $\mathcal{ST}(4,2,4,8)$.

The classical Robinson-Schensted correspondence [22, §7.11] for the symmetric groups was generalized to the Stanton-White correspondence [23] for the wreath products $G(r,n)$. A further generalization of the correspondence, which is valid for all projective reflection groups $G(r,p,q,n)$, is explicitely shown in [9, §10]. We refer to this correspondence as the *projective Robinson-Schensted correspondence*. We do not describe this correspondence explicitly, but we briefly state it for future reference.

Theorem 2.2 *There exists an explicit map*

$$\begin{array}{rcl} G(r,p,q,n) & \longrightarrow & \mathcal{ST}(r,p,q,n) \times \mathcal{ST}(r,p,q,n) \\ g & \longmapsto & [P(g), Q(g)], \end{array}$$

satisfying the following properties:

1. *$P(g)$ and $Q(g)$ have the same shape in $\mathrm{Fer}(r,p,q,n)$ for all $g \in G(r,p,q,n)$;*
2. *if $P, Q \in \mathcal{ST}(r,p,q,n)$ have the same shape $\mu \in \mathrm{Fer}(r,p,q,n)$ then*

$$|\{g \in G(r,p,q,n) : P(g) = P \text{ and } Q(g) = Q\}| = |(C_q)_\mu|,$$

$(C_q)_\mu$ being, as above, the stabilizer in C_q of any element in the class μ.

If G is a finite group we let $\mathrm{Irr}(G)$ be the set of irreducible complex representations of G. If M is a complex vector space and $\rho : G \to GL(M)$ is a representation of G we say that the pair (M, ρ) is a G-*model* if the character χ_ρ is the sum of the characters of all irreducible representations of G over $\mathbb{C}$, i.e. M is isomorphic as a G-module to the direct sum of all irreducible modules of G with multiplicity one. Sometimes we simply say that M is a G-model if we do not need to know the map ρ explicitly or if it is clear from the context. It is clear that two G-models are always isomorphic as G-modules, and so we can also speak about "the" G-model. The last result in this section is a beautiful theorem of Bump and Ginzburg, which generalizes a classical theorem of Frobenius and Schur [11], and allows us in some cases to determine the character of the model of a finite group if we know its dimension.

Theorem 2.3 ([8], Theorem 7) *Let G be a finite group, $\tau \in Aut(G)$ with $\tau^2 = 1$ and M be a G-model. Assume that*

$$\dim(M) = \#\{g \in G : g\tau(g) = z\},$$

where z is a central element in G such that $z^2 = 1$. Then

$$\chi_M(g) = \#\{u \in G : u\tau(u) = gz\}.$$

3 Involutory projective reflection groups

In this section we start the investigation of a model for the projective reflection groups $G(r,p,q,n)$. The main result here is the characterization of the groups $G(r,p,q,n)$ such that the dimension of a $G(r,p,q,n)$-model is equal to the number of absolute involutions in $G(r,p,q,n)$. In these groups we can directly apply Theorem 2.3 to obtain a combinatorial description of the character of the model. The next result relates the dimension of a model with the projective Robinson-Schensted correspondence.

Proposition 3.1 *Let $G = G(r,p,q,n)$. The dimension of a G-model is equal to the number of elements g in the dual group G^* which correspond by means of the projective Robinson-Schensted correspondence to pairs of the form $[P,P]$, for some $P \in \mathcal{ST}(r,q,p,n)$.*

The next target is to show that absolute involutions in G^* correspond to pairs of the form $[P, P]$ under the projective Robinson-Schensted correspondence, and then to characterize those groups for which the converse holds, i.e. the groups where the fact that $v \mapsto [P, P]$ implies that v is an absolute involution.

If $g \in G(r,p,q,n)$, we say that g is a *symmetric* element if any (equivalently every) lift of g in $G(r,n)$ is a symmetric matrix. We similarly define *antisymmetric* elements in $G(r,p,q,n)$. Observe that we can have antisymmetric elements only if r is even. The following result is a characterization of absolute involutions in $G(r,p,q,n)$.

Lemma 3.2 *Let $g \in G(r,p,q,n)$. Then g is an absolute involution, i.e. $g\bar{g} = 1$, if and only if either g is symmetric or q is even and g is antisymmetric.*

We denote by $I(r,p,q,n)$ the set of absolute involutions in $G(r,p,q,n)$.

Theorem 3.3 *Let $G = G(r,p,q,n)$. Then*

$$\sum_{\phi \in \mathrm{Irr}(G)} \dim \phi \geq |I(r,q,p,n)|$$

and equality holds if and only if either $\mathrm{GCD}(p,n) = 1, 2$*, or* $\mathrm{GCD}(p,n) = 4$ *and* $r \equiv p \equiv q \equiv n \equiv 4 \mod 8$.

We conclude this section by observing that a projective reflection group $G = G(r,p,q,n)$ and its dual group G^* always have the same number of absolute involutions. This fact will be the keypoint in the description of the character of the model for the groups satisfying the conditions of Theorem 3.3.

Proposition 3.4 *We always have $|I(r,p,q,n)| = |I(r,q,p,n)|$.*

The proof of this proposition is by direct computation. A "nice" bijective proof is desirable.

We say that a projective reflection group $G = G(r,p,q,n)$ is *involutory* if the dimension of a model of G is equal to the number of absolute involutions in G. By Proposition 3.4 we have that $G(r,p,q,n)$ is involutory if and only if it satisfies the conditions in Theorem 3.3.

If we restrict our attention to standard reflection groups we may note that a group $G(r,p,n)$ is involutory if and only if $\mathrm{GCD}(p,n) = 1, 2$. In particular all infinite families of finite irreducible Coxeter groups (these are $A_n = G(1,1,n)$, $B_n = G(2,1,n)$, $D_n = G(2,2,n)$, $I_2(r) = G(r,r,2)$) are involutory. In the next section we establish a unified construction of a model for all involutory reflection groups (and the corresponding quotients).

The fact that $G(r,p,n)$ is involutory if $\mathrm{GCD}(p,n) = 1, 2$ can also be deduced from known results in the following alternative way. From the characterization of automorphism of complex reflection groups appearing in [18, §1] one can deduce that, under these hypothesis, any irreducible representation ϕ of $G(r,p,n)$ can be realized by a matrix representation $\phi : G(r,p,n) \to GL_n(\mathbb{C})$ satisfying $\phi(\bar{g}) = \overline{\phi(g)}$. Then a straightforward application of the twisted Schur-Frobenius theory developed in [14] implies that $G(r,p,n)$ is involutory.

4 Models

From the results of the previous section we have that the dimension of the model of an involutory reflection group G, is equal to the number of absolute involutions of G and also to the number of absolute involutions of G^*. In this section we show how we can give the structure of a G-model to the formal vector space having a basis indexed by the absolute involutions in G^*.

Unless otherwise stated, we let $G = G(r, p, n)$ be an involutory reflection group, i.e. such that $\mathrm{GCD}(p, n) = 1, 2$. By Theorem 2.3 we have that the character χ of a G-model is given by

$$\chi(g) = |\{u \in G : u\bar{u} = g\}|.$$

Once we have an algebraic-combinatorial description of the dimension and of the character of a model for $G(r, p, n)$ we have two of the main ingredients of the proof of our main result. Before stating it, we need some more definitions. If $\sigma, \tau \in S_n$ with $\tau^2 = 1$ we let

$$\mathrm{Inv}(\sigma) = \{\{i, j\} : (j - i)(\sigma(j) - \sigma(i)) < 0\} \quad \text{and} \quad \mathrm{Pair}(\tau) = \{\{i, j\} : \tau(i) = j \neq i\}.$$

If $g \in G(r, p, n)$ and $v \in I(r, p, n)^*$ we let

$$\begin{aligned} s(g, v) &= \# \left(\mathrm{Inv}(|g|) \cap \mathrm{Pair}(|v|)\right) \\ a(g, v) &= z_1(\tilde{v}) - z_{|g|^{-1}(1)}(\tilde{v}) \in \mathbb{Z}_r \end{aligned}$$

where $\tilde{v}$ is any lift of v in $G(r, n)$. Note that since $a(g, v)$ is the difference of two colors of $\tilde{v}$ it is well-defined. Furthermore, given $g, g' \in G(r, n)$, we let

$$< g, g' >= \sum_i z_i(g) z_i(g') \in \mathbb{Z}_r.$$

Also, it is easy to see that, given $g \in G = G(r, p, n)$, the function of the dual group $G^* = G(r, 1, p, n)$

$$\begin{aligned} T_g : G(r, 1, p, n) &\to \mathbb{Z}_r \\ g' &\mapsto < g, g' > \end{aligned}$$

is well defined, i.e., taken any two lifts $\bar{g}$ and $\hat{g}$ of g' in $G(r, n)$, we have $< g, \bar{g} > \equiv < g, \hat{g} > \mod r$. We denote by $I(r, p, n)^* = I(r, 1, p, n)$ the set of absolute involutions in G^* and we recall (Lemma 3.2) that these elements can be either symmetric or antisymmetric.

Theorem 4.1 *Let* $\mathrm{GCD}(p, n) = 1, 2$ *and let*

$$M(r, p, n)^* \stackrel{\text{def}}{=} \bigoplus_{v \in I(r,p,n)^*} \mathbb{C} C_v$$

and $\varrho : G(r, p, n) \to GL(M(r, p, n)^*)$ *be defined by*

$$\varrho(g)(C_v) \stackrel{\text{def}}{=} \begin{cases} \zeta_r^{<g,v>} (-1)^{s(g,v)} C_{|g|v|g|^{-1}} & \text{if } v \text{ is symmetric} \\ \zeta_r^{<g,v>} \ \zeta_r^{a(g,v)} \ C_{|g|v|g|^{-1}} & \text{if } v \text{ is antisymmetric}. \end{cases} \tag{1}$$

Then $(M(r, p, n)^*, \varrho)$ *is a* $G(r, p, n)$*-model.*

The proof of this theorem consists in the explicit and rather involved computation of the character of this representation, and in verifying that this character agrees with the character described in Theorem 2.3.

If $q|r$ and $pq|rn$ (i.e. the group $G(r, p, q, n)$ is defined) we can consider the submodule $M(r, q, p, n) \subseteq M(r, p, n)^*$ spanned by all elements C_v such that $v \in I(r, q, p, n)$. The next result shows that $M(r, q, p, n)$ is the sum of all irreducible representations of $G(r, p, n)$ indexed by elements $\mu \in \mathrm{Fer}(r, q, p, n)$.

Corollary 4.2 *Let* $\mathrm{GCD}(p, n) = 1, 2$*. Then the pair* $(M(r, q, p, n), \varrho)$*, where*

$$\varrho : G(r, p, q, n) \to GL(M(r, q, p, n))$$

is defined as in Theorem 4.1, is a $G(r, p, q, n)$*-model.*

We will see in the following sections several important generalizations of these results if the group G is a classical Weyl group.

5 Splitting split representations

If $\mathrm{GCD}(p, n) = 2$, there is another natural decomposition of $M(r, p, n)^*$ into two $G(r, p, n)$-submodules. The submodule $\mathrm{Sym}(r, p, n)^*$ spanned by symmetric elements and the submodule $\mathrm{Asym}(r, p, n)^*$ spanned by antisymmetric elements. Recall from Proposition 2.1 that an irreducible representation μ of $G(r, n)$ when restricted to $G(r, p, n)$ either remains irreducible if the stabilizer $(C_p)_\mu$ is trivial, or splits into two irreducible representations of $G(r, p, n)$ if $(C_p)_\mu$ has two elements (note that there are no other possibilities since $\mathrm{GCD}(p, n) = 2$), and that all irreducible representations of $G(r, p, n)$ are obtained in this way.

Theorem 5.1 *Let* χ *be the character of* $\mathrm{Sym}(r, p, n)^*$ *and* ϕ *be an irreducible representation of* $G(r, n)$*. If* ϕ *does not split in* $G(r, p, n)$ *then* $< \chi, \chi_\phi >= 1$*. If* ϕ *splits into two irreducible representations* ϕ^+, ϕ^- *of* $G(r, p, n)$ *then*

$$< \chi, \chi_{\phi^+} >= 1 \iff < \chi, \chi_{\phi^-} >= 0.$$

If we restrict our attention to the case of Weyl groups $D_n = G(2, 2, n)$, the proof of this result is based on the following observation which is a direct consequence of the explicit formulas for the split characters of the groups D_n (see [21, 19]).

Proposition 5.2 *Let* $g \in S_n$ *be of cycle-type* 2α*. Then one can label the split representations of* D_n *by* $(\lambda, \lambda)^+$ *and* $(\lambda, \lambda)^-$ *so that*

$$\sum_{\lambda \vdash n/2} (\chi_{(\lambda,\lambda)^+} - \chi_{(\lambda,\lambda)^-})(g) = 2^{\ell(\alpha)} \chi_M(\alpha),$$

where χ_M *is the character of the model for* $S_{n/2}$*.*

Consider now the two representations of D_n $(\mathrm{Asym}(2, 2, n)^*, \rho^+)$ and $(\mathrm{Asym}(2, 2, n)^*, \rho^-)$, given by

$$\rho^+(g)(C_v) \stackrel{\text{def}}{=} (-1)^{<g,v>} C_{|g|v|g|^{-1}}, \qquad \rho^-(g)(C_v) \stackrel{\text{def}}{=} (-1)^{<g,v>} (-1)^{a(g,v)} C_{|g|v|g|^{-1}}$$

(notice that $\rho^-(g) = \varrho(g)|_{\mathrm{Asym}(2,2,n)^*}$). An explicit computation of the characters of the representations ρ^+ and ρ^- and Proposition 5.2 show that

$$\sum_{\lambda \vdash n/2} \chi_{(\lambda,\lambda)^+}(g) - \sum_{\lambda \vdash n/2} \chi_{(\lambda,\lambda)^-}(g) = \chi_{\rho^+}(g) - \chi_{\rho^-}(g) \qquad \forall\, g \in D_n.$$

Comparing the dimensions of the representations involved, and recalling the linear independence of characters, we can conclude that

$$\chi_{\rho^+}(g) = \sum_{\lambda \vdash n/2} \chi_{(\lambda,\lambda)^+}(g) \quad \text{and} \quad \chi_{\rho^-}(g) = \sum_{\lambda \vdash n/2} \chi_{(\lambda,\lambda)^-}(g) :$$

this means that $(\mathrm{Asym}(2, 2, n)^*, \varrho) \cong \bigoplus_{\lambda \vdash n/2} (\lambda, \lambda)^-$, as claimed.

6 Refinement for B_n

Let us have a closer look at the model (M, ϱ) for $G = G(r, n)$. There is an immediate decomposition of M into submodules that we are going to describe.

Let $g, h \in G(r, n)$. We say that g and h are S_n-conjugate if there exists $\sigma \in S_n$ such that $g = \sigma h \sigma^{-1}$. If c is an S_n-conjugacy class of absolute involutions in G we denote by $M(c)$ the subspace of M spanned by the elements in c, and it is clear that

$$M = \bigoplus_c M(c) \quad \text{as } G\text{-modules},$$

where the sum runs through all S_n-conjugacy classes of absolute involutions. It is natural to ask if we can describe the irreducible decomposition of the submodules $M(c)$. This decomposition is known if G is the symmetric group S_n (see [1, 13]). We will focus on the case of B_n and we show that the irreducible decompositions of these submodules are well behaved with respect to the RS correspondences, a problem which was raised in [2]. The meaning of 'well behaved with respect to the RS correspondence' will be clarified in Theorem 6.1.

Let v be an involution of B_n. We denote by $R(v)$ the element of $\mathrm{Fer}(2, n)$ which is the shape of the tableaux of the image of v via the Robinson-Schensted correspondence. Namely $R(v) \stackrel{\text{def}}{=} (\lambda, \mu)$, where

$$v \xrightarrow{RS} [P, P], \quad P \in \mathcal{ST}(2, n), \quad P \text{ of shape } (\lambda, \mu).$$

For notational convenience we let $R(c) = \cup_{v \in c} R(v)$. The main goal of this section is the following result.

Theorem 6.1 *Let c be an S_n-conjugacy class of involutions in B_n. Then the following decomposition holds:*

$$M(c) \cong \bigoplus_{(\lambda,\mu) \in R(c)} \rho_{\lambda,\mu}.$$

In order to prove Theorem 6.1, first of all we need to parametrize the S_n-conjugacy classes of involutions explicitly. With this purpose we let

- $\mathrm{fix}(v) \stackrel{\text{def}}{=} \#\{i : i > 0 \text{ and } v(i) = i\}$
- $\mathrm{fix}^-(v) \stackrel{\text{def}}{=} \#\{i : i > 0 \text{ and } v(i) = -i\}$
- $\mathrm{pair}(v) \stackrel{\text{def}}{=} \#\{(i, j) : 0 < i < j, \ v(i) = j \text{ and } v(j) = i\}$
- $\mathrm{pair}^-(v) \stackrel{\text{def}}{=} \#\{(i, j) : 0 < i < j, \ v(i) = -j \text{ and } v(j) = -i\}$.

Proposition 6.2 *Two involutions v, w in B_n are S_n-conjugate if and only if*

$$\mathrm{fix}(v) = \mathrm{fix}(w), \quad \mathrm{pair}(v) = \mathrm{pair}(w),$$

$$\mathrm{fix}^-(v) = \mathrm{fix}^-(w), \quad \mathrm{pair}^-(v) = pair^-(w).$$

Furthermore, given an involution v in B_n, let $R(v) = (\lambda, \mu)$. Then λ has $\mathrm{fix}(v)$ *odd columns and* $\mathrm{fix}(v) + 2\,\mathrm{pair}(v)$ *boxes, while μ has* $\mathrm{fix}^-(v)$ *odd columns and* $\mathrm{fix}^-(v) + 2\,\mathrm{pair}^-(v)$ *boxes.*

We can thus name the S_n-conjugacy classes of the involutions of B_n in this way:

$$c_{f_0,f_1,p_0,p_1} \stackrel{\text{def}}{=} \{v : \text{fix}(v) = f_0;\ \text{fix}^-(v) = f_1;\ \text{pair}(v) = p_0;\ \text{pair}^-(v) = p_1\}.$$

The description given for the S_n-conjugacy classes ensures that the subspace M_0 of M generated by the involutions $v \in B_n$ with $\text{fix}(v) = \text{fix}^-(v) = 0$, is a B_n-submodule. The crucial step in the proof of Theorem 6.1 is the following partial result regarding this submodule (we observe that in this case n is necessarily even, $n = 2m$): M_0 is the direct sum of all the irreducible representations of B_{2m} indexed by pairs of diagrams whose columns have an even number of boxes, each of such representations occurring once. To show this we need the following argument which generalizes an idea appearing in [13].

Lemma 6.3 *Let Π_m be representations of B_{2m}, m ranging in $\mathbb{N}$. Then the following are equivalent:*

a) for every m, Π_m is the direct sum of all the irreducible representations of B_{2m} indexed by pairs of diagrams whose columns have an even number of boxes, each of such representations occurring once;

b) for every m,

(b0) Π_0 is unidimensional;

(b1) the following isomorphism holds:

$$\Pi_m \downarrow_{B_{2m-1}} \cong \Pi_{m-1} \uparrow^{B_{2m-1}}; \tag{2}$$

(b2) the module Π_m contains all the irreducible representations of B_{2m} indexed by the pairs of diagrams $(1^{2j}, 1^{2(m-j)})$, $j \in [0, m]$, where 1^k is the single Ferrers diagram with one column of length k.

This lemma can be proved constructively by means of a generalization to B_n of the branching rule (see [12]). The implication b)⇒ a) of the preceding lemma can be applied to the case $\Pi_m = M_0$.

The group B_0 is the identity group so property b0) is trivially verified.

Let us denote by N_0 the B_{2m-2}-module constructed in the same way. To check property b1), we have to show that

$$M_0 \downarrow_{B_{2m-1}} \cong N_0 \uparrow^{B_{2m-1}} . \tag{3}$$

The following argument is used. Let M_0^h be the submodule of M_0 generated by the involutions v satisfying $\text{fix}(v) = \text{fix}^-(v) = 0$, $\text{pair}(v) = h$ and $\text{pair}^-(v) = m - h$. Each M_0^h, once restricted to B_{2m-1}, splits into two submodules according to the color of $2m$. We denote by $M_0^{h,+}$ the submodule of M_0^h containing involutions v such that $z_{2m}(v) = 0$, and similarly for $M_0^{h,-}$. So we have

$$M_0 \downarrow_{B_{2m-1}} = \bigoplus_{h=0}^{m} \left(M_0^{h,+} \oplus M_0^{h,-} \right).$$

One checks that $N_0^h \uparrow^{B_{2m-1}} \cong M_0^{h+1,+} \oplus M_0^{h,-}$ and property (b1) follows.

As for property (b2) one can proceed as follows. For any $S \subseteq [2m]$ let $C_S = \sum C_v$, where the sum is over all involutions $v \in B_{2m}$ with $\text{fix}(v) = \text{fix}^-(v) = 0$ and such that $z_i(v) = 0$ if and only if

$i \in S$. Then one can check that the subspace spanned by all C_S with $|S| = 2h$ affords the representation parametrized by the single-rowed diagrams $(2h, 2(n-h))$. From this it is possible to derive the representation $(1^{2h}, 1^{2(m-h)})$.

Let us now turn to the case of the general submodule $M(c)$. For every $k \in [0, n]$, let f_0, f_1, p_0, p_1 be nonnegative integers such that $f_0 + f_1 = k$, $2(p_0 + p_1) = n - k$. By means of Proposition 6.2 we have to show that

$$M(c_{f_0,f_1,p_0,p_1}) \cong \bigoplus_{(\lambda,\mu)\in R(c_{f_0,f_1,p_0,p_1})} \varrho_{\lambda,\mu},$$

where

$$R(c_{f_0,f_1,p_0,p_1}) = \{(\lambda,\mu) \text{ such that } \lambda \vdash f_0 + 2p_0,\ \mu \vdash f_1 + 2p_1, \\ \lambda \text{ has } f_0 \text{ odd columns, } \mu \text{ has } f_1 \text{ odd columns}\}.$$

Generalizing the ideas developed for M_0, one shows that

$$M(c_{f_0,f_1,p_0,p_1}) \cong \mathrm{Ind}_{B_{n-(f_0+f_1)}\times B_{f_0+f_1}}^{B^n}(M_0 \otimes \varrho_{\iota_{f_0},\iota_{f_1}}),$$

where M_0 is the $B_{n-(f_0+f_1)}$-module constructed as above, and ι_k is the single-rowed Ferrers diagram of length k. This isomorphism can be achieved by standard representation theory, while the rest of the proof can be carried out by applying the partial result obtained on M_0 and a generalization of the Littlewood-Richardson rule to the case of B_n.

Example 6.4 *Let $v \in B_6$ given by $|v| = [6,4,3,2,5,1]$ and $z(v) = [1,0,0,0,1,1]$. Then $f_0 = f_1 = p_0 = p_1 = 1$ and the S_n-conjugacy class c of v has 180 elements. Then the B_6-module $M(c)$ is given by the sum of the irreducible representations indexed by $(\lambda,\mu) \in \mathrm{Fer}(2,6)$ such that both λ and μ are partitions of 3 and have exactly one column of odd length. In particular*

$$M(c) \cong \rho_{\left(\ydiagram{2,1},\,\ydiagram{1,1,1}\right)} \oplus \rho_{\left(\ydiagram{1,1,1},\,\ydiagram{2,1}\right)} \oplus \rho_{\left(\ydiagram{2,1},\,\ydiagram{2,1}\right)} \oplus \rho_{\left(\ydiagram{1,1,1},\,\ydiagram{1,1,1}\right)}.$$

7 Refinement for D_n

We have already seen that in an involutory reflection group $G(r,p,n)$ the submodule generated by the antisymmetric absolute involutions $\mathrm{Asym}(r,p,n)^*$ is isomorphic to the multiplicity-free sum of all the irreducible representations $\rho_{(\lambda,\lambda)^-}$, while all the other irreducible representations of $G(r,p,n)$ are afforded by $\mathrm{Sym}(r,p,n)^*$. We will make use of what was proved for B_n to give a finer decomposition of $\mathrm{Sym}(2,2,n)^*$ for the groups D_n.

Let $\bar{v}$ be a symmetric involution of $D_n^* = B_n/\pm I$ and v and $-v$ be its lifts in B_n. We also denote by $\bar{c}$ the S_n-conjugacy class of $\bar{v}$ in D_n^* and by c and c' the S_n-conjugacy classes of v and $-v$ in B_n. Generalizing the notation used in §6 we let $R(\bar{v})$ be the element of $\mathrm{Fer}(2,1,2,n)$ which is the shape of the tableaux of the image of $\bar{v}$ via the projective Robinson-Schensted correspondence. Namely $R(\bar{v}) \stackrel{\text{def}}{=} (\lambda,\mu)$, where

$$v \xrightarrow{RS} [P,P], \quad P \in \mathcal{ST}(2,1,2,n), \quad P \text{ of shape } (\lambda,\mu).$$

We also let

$$R(\bar{c}) = \bigcup_{\bar{w}\in\bar{c}} R(\bar{w}).$$

One can verify that the restrictions of the B_n-modules $M(c)$ and $M(c')$ to D_n are isomorphic. If v and $-v$ are not S_n-conjugate then a direct application of Theorem 6.1 provides

$$M(\bar{c}) \cong \bigoplus_{(\lambda,\mu)\in R(\bar{c})} \rho_{\lambda,\mu}.$$

Note that in this case we obtain unsplit representations only since $R(v) = (\lambda, \mu)$ implies $R(-v) = (\mu, \lambda)$. If v and $-v$ are S_n-conjugate, using Theorems 6.1 and 5.1 we can conclude that

$$M(\bar{c}) \cong \bigoplus_{\substack{(\lambda,\mu)\in R(\bar{c}):\\ \lambda\neq\mu}} \rho_{\lambda,\mu} \oplus \bigoplus_{(\lambda,\lambda)\in R(\bar{c})} \rho_{(\lambda,\lambda)^+}.$$

Example 7.1 *Let $v \in B_6$ given by $|v| = [6, 4, 3, 2, 5, 1]$ and $z(v) = [1, 0, 0, 0, 1, 1]$. Then $\bar{c}$, the S_n-conjugacy class of $\bar{v}$, has 90 elements and the decomposition of the D_n-module $M(\bar{c})$ is given by all representations indexed by $(\lambda, \mu) \in \mathrm{Fer}(2, 1, 2, 6)$ where both λ and μ are partitions of 3 and have exactly one column of odd length, with the additional condition that if $\lambda = \mu$ the split representation to be considered is $(\lambda, \lambda)^+$. Therefore*

$$M(\bar{c}) \cong \rho_{\left(\yng(2,1),\, \yng(1,1,1)\right)} \oplus \rho_{\left(\yng(2,1),\, \yng(2,1)\right)^+} \oplus \rho_{\left(\yng(1,1,1),\, \yng(1,1,1)\right)^+}.$$

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