FPSAC 2010, San Francisco, USA DMTCS proc. **AN**, 2010, 533–544

Combinatorial aspects of Escher tilings

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Abstract. In the late 30's, Maurits Cornelis Escher astonished the artistic world by producing some puzzling drawings. In particular, the tesselations of the plane obtained by using a single tile appear to be a major concern in his work, drawing attention from the mathematical community. Since a tile in the continuous world can be approximated by a path on a sufficiently small square grid - a widely used method in applications using computer displays - the natural combinatorial object that models the tiles is the polyomino. As polyominoes are encoded by paths on a four letter alphabet coding their contours, the use of combinatorics on words for the study of tiling properties becomes relevant. In this paper we present several results, ranging from recognition of these tiles to their generation, leading also to some surprising links with the well-known sequences of Fibonacci and Pell.

Résumé. Lorsque Maurits Cornelis Escher commença à la fin des années 30 à produire des pavages du plan avec des tuiles, il étonna le monde artistique par la singularité de ses dessins. En particulier, les pavages du plan obtenus avec des copies d'une seule tuile apparaissent souvent dans son oeuvre et ont attiré peu à peu l'attention de la communauté mathématique. Puisqu'une tuile dans le monde continu peut être approximée par un chemin sur un réseau carré suffisemment fin - une méthode universellement utilisée dans les applications utilisant des écrans graphiques - l'object combinatoire qui modèle adéquatement la tuile est le polyomino. Comme ceux-ci sont naturellement codés par des chemins sur un alphabet de quatre lettres, l'utilisation de la combinatoire des mots devient pertinente pour l'étude des propriétés des tuiles pavantes. Nous présentons dans ce papier plusieurs résultats, allant de la reconnaissance de ces tuiles à leur génération, conduisant à des liens surprenants avec les célèbres suites de Fibonacci et de Pell.

Keywords: Tesselations, tilings, polyomino, Fibonacci, Pell.

1 Introduction

We study here a special class of periodic tilings consisting of translated copies of a single tile, and we refer the reader to Grünbaum and Shephard (1987) for a more general presentation of tilings, and to Ardila and Stanley (2005) for an introduction to combinatorial problems related with tilings. For instance, consider the problem of tiling the plane with an infinite number of copies of a single tile. While it is not known whether it admits a periodic tiling of the plane, the situation is easier with translations of a polyomino.

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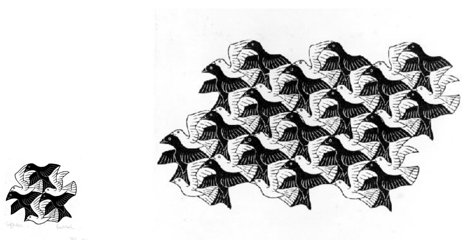
The problem of deciding if a given polyomino tiles the plane by translation goes back to Wijshoff and van Leeuven (1984) who coined the term *exact polyomino* for these. Up to our knowledge, Beauquier and Nivat (1991) were the first to provide a characterization stating that the boundary $b(P)$ of an exact polyomino P satisfies the following (not necessarily unique) *Beauquier-Nivat factorization*

$$b(P) = A \cdot B \cdot C \cdot \widehat{A} \cdot \widehat{B} \cdot \widehat{C} \tag{1}$$

where at most one variable may be empty. Hereafter, this condition is referred as the BN-factorization,

Polyominoes having a BN-factorization (where BN stands for Beauquier-Nivat) with A, B and C nonempty were called *pseudo-hexagons*. For sake of simplicity, we call them *hexagons*. The example on the right shows a hexagonal tiling. Indeed the basic tile is composed of 6 sides, each one corresponding to one side of a hexagon.

If one of the variables in Equation (1) is empty, they are called *squares*. A tiling may have both features as shown below.



Indeed, in the tiling on the left, we have two basic ways of decomposing the tesselation: into squares (a pair of white and black birds) and hegaxons (3 white and 3 black). Even more, a tile may have both hexagon and square factorizations, as a polyomino consisting of k contiguous unit squares shows.

In this paper we present a combinatorial approach for understanding the structure of the polyominoes that tile the plane by translation. The combinatorics on words point of view is powerful for a number of decision problems such as deciding if a polyomino tiles the plane by translation or checking if a tile is digitally convex. Enumeration of such tiles is a challenging problem, and we have exhibited new classes of polyominoes having surprising properties.

Indeed there are square tiles that can be assembled in exactly two different ways, defining two sets of distinct translations: we call them *double squares*. On the other hand we did not find a square having 3 distinct square factorizations, confirming a conjecture due to Provençal (2008). In particular, we describe two infinite families of squares linked to the Christoffel words and to the Fibonacci sequence.

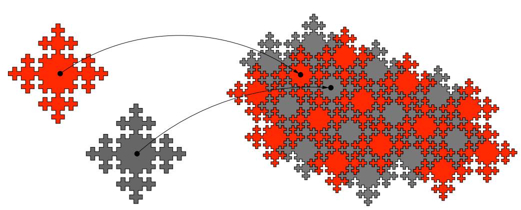


Fig. 1: Tiling with a Fibonacci polyomino.

Proofs of the results are based on combinatorics on words techniques and are omitted due to lack of space.

2 Preliminaries

The usual terminology and notation on words is from Lothaire (1997). An *alphabet* Σ is a finite set whose elements are called *letters*. A finite word w is a sequence of letters, that is, a function $w : \{1, 2, \ldots, n\} \to \Sigma$, where w_i is the i-th letter, $1 \leq i \leq n$. The *length* of w, denoted by $|w|$, is given by the integer n. The unique word of length 0 is denoted ε, and the set of all finite words over Σ is denoted Σ^*. The set of n-length word is Σ^n, and $\Sigma^{\geq k}$ denotes those of length at least k. The *reversal* $\widetilde{w}$ of $w = w_1 w_2 \cdots w_n$ is the word $\widetilde{w} = w_n w_{n-1} \cdots w_1$. Words p satisfying $p = \widetilde{p}$ are called *palindromes*. The set of all palindromes over Σ is denoted $\mathrm{Pal}(\Sigma^*)$. A word u is a *factor* of another word w if there exist $x, y \in \Sigma^*$ such that $w = xuy$. We denote by $|w|_u$ the number of occurrences of u in w. Two words u and v are *conjugate* if there are words x and y such that $u = xy$ and $v = yx$. In that case, we write $u \equiv v$. Clearly, $\equiv$ is an equivalence relation. Given two alphabets Σ_1 and Σ_2, a *morphism* is a function $\varphi : \Sigma_1^* \to \Sigma_2^*$ compatible with concatenation, that is, $\varphi(uv) = \varphi(u)\varphi(v)$ for any $u, v \in \Sigma_1^*$. It is clear that a morphism is completely defined by its action on the letters of Σ_1.

Paths on the square lattice. The notation of this section is partially adapted from Brlek et al. (2006b). A path in the square lattice, identified as $\mathbb{Z} \times \mathbb{Z}$, is a polygonal path made of the elementary unit translations

$$a = (1, 0), \overline{a} = (-1, 0), b = (0, 1), \overline{b} = (0, -1).$$

A finite path w is therefore a word on the alphabet $\mathcal{F} = \{a, \overline{a}, b, \overline{b}\}$. Furthermore, we say that a path w is *closed* if it satisfies $|w|_a = |w|_{\overline{a}}$ and $|w|_b = |w|_{\overline{b}}$. A *simple path* is a word w such that none of its proper factors is a closed path. A *boundary word* is a closed path such that none of its proper factors is closed. Finally, a *polyomino* is the subset of $\mathbb{Z}^2$ contained in some boundary word. On the square grid, a path can be encoded by a sequence of basic movements in the left (L), right (R), forward (F) and backward (B) directions, so that there is a map $\mathcal{D} : \mathcal{F}^2 \to \mathcal{R} = \{\mathrm{L}, \mathrm{R}, \mathrm{F}, \mathrm{B}\}$ defined by

$$\mathcal{D}(u) = \begin{cases} \mathrm{L} & \text{if } u \in V_{\mathrm{L}} = \{ab, b\overline{a}, \overline{a}\overline{b}, \overline{b}a\}, \\ \mathrm{R} & \text{if } u \in V_{\mathrm{R}} = \{ba, a\overline{b}, \overline{b}\overline{a}, \overline{a}b\}, \\ \mathrm{F} & \text{if } u \in V_{\mathrm{F}} = \{aa, \overline{a}\overline{a}, \overline{b}\overline{b}, bb\}, \\ \mathrm{B} & \text{if } u \in V_{\mathrm{B}} = \{a\overline{a}, \overline{a}a, b\overline{b}, \overline{b}b\}. \end{cases}$$

It is extended to a function on arbitrary words, denoted by the same letter $\mathcal{D} : \mathcal{F}^{\geq 1} \to \mathcal{R}^*$, by setting

$$\mathcal{D}(w) = \begin{cases} \varepsilon & \text{if } |w| = 1, \\ \prod_{i=2}^{n} \mathcal{D}(w_{i-1} w_i) & \text{if } |w| \geq 2, \end{cases}$$

where $|w| = n$ and the product is the concatenation. For example, $\mathcal{D}(bab\overline{a}baa\overline{b}) = \mathrm{RLLRRFR}$. Notice that each path $w \in \mathcal{F}^{\geq 1}$ is completely determined, up to translation, by its initial step $\alpha \in \mathcal{F}$ and a word y on the alphabet $\mathcal{R}$. Therefore, for each $\alpha \in \mathcal{F}$ there is a function $\mathcal{D}_\alpha^- : \mathcal{R}^* \to \mathcal{F}^{\geq 1}$ defined recursively by

$$\mathcal{D}_\alpha^{-1}(y) = \begin{cases} \alpha & \text{if } |y| = 0, \\ \alpha \mathcal{D}_\beta^{-1}(y') & \text{if } |y| \geq 1, \end{cases}$$

where $\beta \in \mathcal{F}$ is the letter such that $\alpha\beta \in V_x$ and $y = xy'$ with $x \in \mathcal{R}$. For example, if $y = \mathrm{RLLRRFR}$, then $\mathcal{D}_b^-(y) = bab\overline{a}baa\overline{b}$, while $\mathcal{D}_a^-(y) = a\overline{b}aba\overline{b}\overline{b}\overline{a}$. The next lemma gives some easily established statements and shows how both functions $\mathcal{D}$ and $\mathcal{D}^-$ behave.

Lemma 1 *Let $w, w' \in \mathcal{F}^*$, $y, y' \in \mathcal{R}^*$, $\alpha \in \mathcal{F}$ and $x \in \mathcal{R}$. Then*

(i) $\mathcal{D}^{-}_{w_1}\mathcal{D}(w) = w$ *and* $\mathcal{D} \circ \mathcal{D}^{-}_{\alpha}(y) = y$, *where w_1 is the first letter of w;*

(ii) $\mathcal{D}(ww') = \mathcal{D}(w) \cdot \mathcal{D}(w_n w'_1) \cdot \mathcal{D}(w')$ *and* $\mathcal{D}^{-}_{\alpha}(yxy') = \mathcal{D}^{-}_{\alpha}(y)\mathcal{D}^{-}_{\beta}(y')$, *where w_n is the last letter of w, w'_1 is the first letter of w' and β is the last letter of $\mathcal{D}^{-}_{\alpha}(yx)$.* □

In Brlek et al. (2006b), the authors introduced the *winding number*, the valuation Δ defined on $\mathcal{R}^*$ by $\Delta(y) = |y|_{\text{L}} - |y|_{\text{R}} + 2|y|_{\text{B}}$ as well as on $\mathcal{F}^{\geq 1}$ by setting $\Delta(w) = \Delta(\mathcal{D}w)$.

Transformations. Some useful transformations on $\mathcal{F}^*$ are rotations by an angle $k\pi/2$ and reflections with respect to axes of angles $k\pi/4$, where $k \in \mathbb{N}$. The rotation of angle $\pi/2$ translates merely in $\mathcal{F}$ by the morphism $\rho : a \mapsto b, b \mapsto \overline{a}, \overline{a} \mapsto \overline{b}, \overline{b} \mapsto a$. We denote the other rotations by ρ^2 and ρ^3 according to the usual notation. The rotation ρ^2 is also noted $\overline{\cdot}$ since it can be seen as the *complement* morphism defined by the relations $\overline{\overline{a}} = a$ and $\overline{\overline{b}} = b$. Similarly, for $k \in \{0, 1, 2, 3\}$, σ_k is the reflection defined by the axis containing the origin and having an angle of $k\pi/4$ with the abscissa. It may be seen as a morphism on $\mathcal{F}^*$ as well:

$$\sigma_0 : a \mapsto \overline{a}, \overline{a} \mapsto a, b \mapsto b, \overline{b} \mapsto \overline{b} \;\text{ and }\; \sigma_1 : a \mapsto b, b \mapsto a, \overline{a} \mapsto \overline{b}, \overline{b} \mapsto \overline{a}.$$

The two other reflections are $\sigma_2 = \sigma_0 \circ \rho^2$ and $\sigma_3 = \sigma_1 \circ \rho^2$. Another useful map is the antimorphism $\widehat{\cdot} = \overline{\cdot} \circ \widetilde{\cdot}$ defined on $\mathcal{F}^*$: $\widehat{w}$ is the path traversed in the opposite direction. The behaviour of the operators $\widehat{\cdot}$, $\widetilde{\cdot}$ and $\overline{\cdot}$ is illustrated in Figure 2.

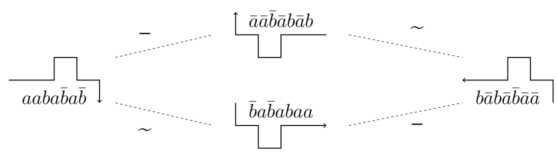


Fig. 2: Effect of the operators $\widehat{\cdot}$, $\widetilde{\cdot}$ and $\overline{\cdot}$ on $\mathcal{F}^*$.

On the alphabet $\mathcal{R}$, we define an involution $\imath : \text{L} \mapsto \text{R}, \text{R} \mapsto \text{L}, \text{F} \mapsto \text{F}, \text{B} \mapsto \text{B}$. This function $\imath$ extends to $\mathcal{R}^*$ as a morphism, so that the map $\widehat{\cdot}$ extends as well to $\widehat{\cdot} : \mathcal{R}^* \to \mathcal{R}^*$ by setting $\widehat{\cdot} = \imath \circ \widetilde{\cdot}$. All these operations are closely related as shown in the lemmas hereafter. The proofs are left to the reader.

Lemma 2 *Let $w \in \mathcal{F}^*$, $y \in \mathcal{R}^*$ and $\alpha \in \mathcal{F}$. The following properties hold:*

(i) $\mathcal{D}(w) = \mathcal{D}(\rho^i(w))$ *for all* $i \in \{1, 2, 3\}$,
(ii) $\imath(\mathcal{D}(w)) = \mathcal{D}(\sigma_i(w))$ *for all* $i \in \{0, 1, 2, 3\}$,
(iii) $\mathcal{D}(\widehat{w}) = \widehat{\mathcal{D}(w)} = \mathcal{D}(\widetilde{w})$,
(iv) $\rho^i(\mathcal{D}^{-}_{\alpha}(y)) = \mathcal{D}^{-}_{\rho^i(\alpha)}(y)$ *for all* $i \in \{1, 2, 3\}$,
(v) $\sigma_i(\mathcal{D}^{-}_{\alpha}(y)) = \mathcal{D}^{-}_{\sigma_i(\alpha)}\imath(y)$ *for all* $i \in \{0, 1, 2, 3\}$,
(vi) $\widetilde{\mathcal{D}^{-}_{\alpha}(y)} = \mathcal{D}^{-}_{\beta}\widehat{y}$ *where β is the last letter of $\mathcal{D}^{-}_{\alpha}(y)$,*
(vii) $\widehat{\mathcal{D}^{-}_{\alpha}(y)} = \mathcal{D}^{-}_{\overline{\beta}}(\widehat{y})$ *where β is the last letter of $\mathcal{D}^{-}_{\alpha}(y)$,*
(viii) *If β is the last letter of $\mathcal{D}^{-}_{\alpha}y$, then $\beta = \rho^i(\alpha)$ where $i = \Delta(y)$.* □

For the rest of the paper, the words w of $\mathcal{F}^*$ and $\mathcal{R}^*$ satisfying $\widehat{w} = w$ are called *antipalindromes*.

Lemma 3 *Let $w \in \mathcal{F}^*$. Then the following statements are equivalent.*

(i) $\widehat{w} = \rho^2(w)$
(ii) *w is a palindrome*
(iii) *$\mathcal{D}(w)$ is an antipalindrome.*

Finally, reflections on $\mathcal{F}^*$ are easily described on $\mathcal{R}^*$.

Lemma 4 *Let $w \in \mathcal{F}^*$. There exists $i \in \{0, 1, 2, 3\}$ such that $\widehat{w} = \sigma_i(w)$ if and only if $\mathcal{D}(w)$ is a palindrome.* □

Square Tilings. Let P be a polyomino having W for boundary word, and Q a square having $V = AB\widehat{A}\widehat{B}$ as a BN-factorization. Then *the product of P and Q*, denoted by $P \circ Q$, is the polyomino whose boundary word is given by $\gamma(W)$, where $\gamma : \mathcal{F}^* \to \mathcal{F}^*$ is the morphism defined by

$$\gamma(a) = A, \gamma(\overline{a}) = \widehat{A}, \gamma(b) = B, \gamma(\overline{b}) = \widehat{B}.$$

The assumption for Q to be a square is essential in order to glue together the tiles. Here is an illustration of the composition where P is a tetramino, which is an hexagon but not a square.

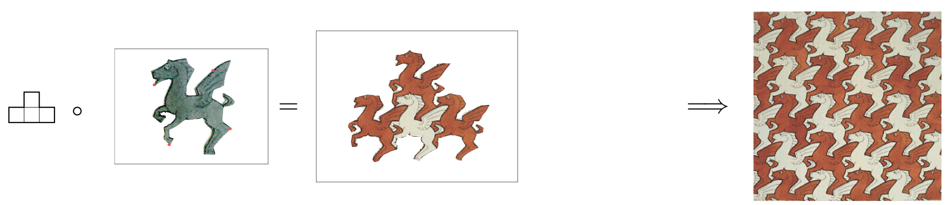


Fig. 3: Composition of tiles and the resulting tiling

Of particular interest are the constructions yielding double squares, discovered by Provençal (2008).

Proposition 5 *Let P be a double square and Q a square. Then the following properties hold:*

(i) *the BN-factorizations of P must overlap, i.e. no factor of a BN-factorization may be included in a factor of the other one.*

(ii) *$P \circ Q$ is a double square.* □

The composition of tiles lead naturally to the notion of primality, and a polyomino R is called *prime* if the relation $R = P \circ Q$ implies that either $R = P$ or $R = Q$. Of course, every square with prime area is prime. In Figure 3, the winged horse is a prime square (!).

Lemma 6 *Let P be a square with boundary word W, A and B be words such that $W \equiv AB\widehat{A}\widehat{B}$. Then $A, B \in \mathrm{Pal}(\mathcal{F}^*)$ if and only if $W = w\overline{w}$ for some word w.*

For more reading on square tilings see Brlek and Provençal (2006); Brlek et al. (2009b).

3 Recognition of tiles

Wijshoff and van Leeuven (1984) provided a naive $\mathcal{O}(n^4)$ algorithm for recognizing exact polyominoes. Later, using the BN-factorization of Beauquier and Nivat, Gambini and Vuillon (2007) exhibited a general $\mathcal{O}(n^2)$ algorithm.

Brlek and Provençal (2006) designed a linear algorithm for recognizing squares. It uses all the power of combinatorics on words as accounted in Lothaire (2005). The main idea is to choose a position p in $W = AB\widehat{A}\widehat{B}$, and then, to list all the candidate factors A that overlap this fixed position p. To achieve this, the auxiliary functions Longest-Common-Right-Extension (LCRE) and Longest-Common-Left-Extension (LCLE) of W and $\widehat{W}$ at some respective positions i and j are essential : their computation is performed in constant time thanks to a pre-processing in linear time (!). (see Lothaire (2005) for more details)

Nevertheless, there is still a gap to close for completely solving the recognition problem. In the case of hexagons the solution is not complete: if the polyominoes do not have too long square factors then the algorithm is still linear (Brlek et al. (2009b)). A general algorithm in $\mathcal{O}(n(\log(n)^3)$ also appears in the thesis of Provençal (2008). Nevertheless, we conjecture that a linear algorithm exists.

It has been shown in Provençal (2008) that there exist polyominoes admitting a linear number of distinct non trivial factorizations as hexagons. The case is different for squares. Indeed, based on exhaustive computation of tiles of small length, showing no square with 3 distinct square factorizations the following result, conjectured in the thesis of Provençal (2008), holds.

Proposition 7 (Blondin Massé et al. (2010a)) *The number of distinct BN-factorizations of a square is at most* 2*.*

4 Generation of tiles

The greedy algorithm, consisting in computing for each even n all polyominoes of perimeter, does not allow to produce large size candidates. Therefore, we used the following approach, which is based on the generation of self avoiding walks.

1. Generate 2 self avoiding walks A, B of length n, m. Each self avoiding walk can be built in two steps:
   (a) generate randomly a word w of length n : this takes $\mathcal{O}(n)$;
   (b) check if w intersects itself, which amounts to check if a point appears twice in the walk. This step can be achieved in $\mathcal{O}(n)$ thanks to a sequential algorithm we provided in Brlek et al. (2009a)). The key point is that there is no need to sort the points (which requires a $\log n$ factor) thanks to the combination of two data structures: a radix-tree for storing the visited points on the square grid is enriched with a quad-tree structure encoding the neighborhood relation of points.

   Note : both steps can be combined in a single pass of Step 1(b). It suffices to substitute the sequential reading with the sequential random generation of a letter.

2. Check if the final word $AB\widehat{A}\widehat{B}$ does not intersect itself by extending the nonintersection verification in Step 1 (b) with the factor $\widehat{A}\widehat{B}$.

The resulting algorithm is clearly linear in the perimeter size.

Some double squares are displayed in Figure 4. In fact, there is an infinite number of these and we are able to describe some infinite families.

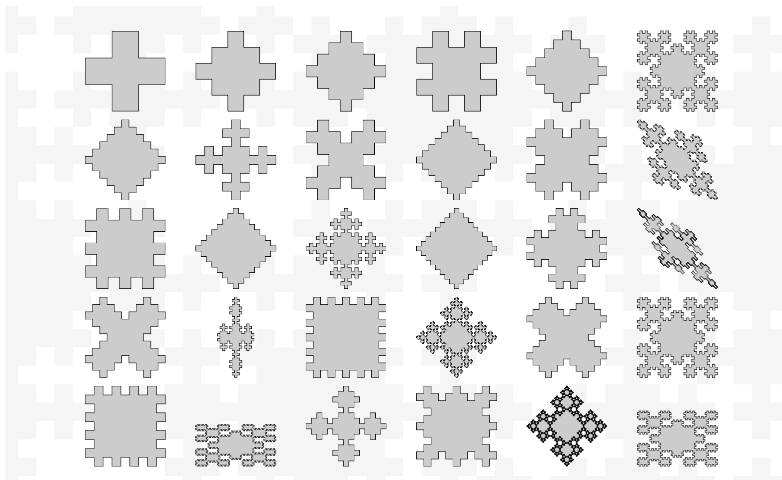


Fig. 4: Some double squares

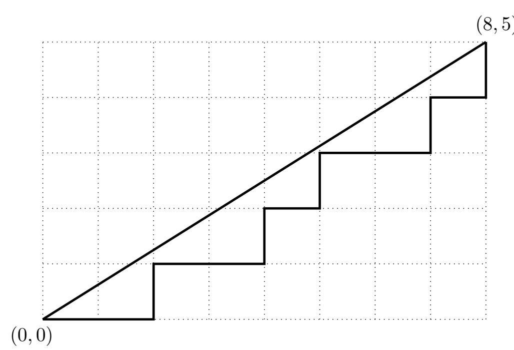
The first diagonal in Figure 4 contains what we call the Fibonacci tiles. Two special classes of double squares are described now.

4.1 Christoffel Tiles

Recall that Christoffel words are finite Sturmian words, that is, they are obtained by discretizing a segment in the plane. Let $(p,q) \in \mathbb{N}^2$ with $\gcd(p,q) = 1$, and let S be the segment with endpoints $(0,0)$ and (p,q).

The word w is a *lower Christoffel word* if the path induced by w is under S and if they both delimit a polygon with no integral interior point. An *upper Christoffel word* is defined similarly. A *Christoffel word* is either a lower Christoffel word or an upper Christoffel word. On the right is illustrated the lower one corresponding to

$$w = aabaababaabab.$$



It is well known that if w and w' are respectively the lower and upper Christoffel words associated to (p,q), then $w' = \widetilde{w}$. Moreover, we have $w = amb$ and $w' = bma$, where m is a palindrome and a, b are letters. The word m is called *cutting word*. They have been widely studied in the literature (see e.g. Berstel et al. (2008)), where they are also called *central words*.

Let $\mathcal{B} = \{a, b\}$. Consider the morphism $\lambda : \mathcal{B}^* \to \mathcal{F}^*$ by $\lambda(a) = a\bar{b}ab$ and $\lambda(b) = ab$, which can be seen as a "crenelation" of the steps *east* and *north-east*.

Theorem 8 (Blondin Massé et al. (2009)) *Let $w = amb$ where a and b are letters.*

(i) *If m is a palindrome, then $\lambda(w\overline{w})$ is a square tile.*

(ii) *$\lambda(w\overline{w})$ is a double square if and only if w is a Christoffel word.*

We say that a crenelated tile $\lambda(w\overline{w})$ obtained from a lower Christoffel word w is a *basic Christoffel tile* while a *Christoffel tile* is a polyomino isometric to a basic Christoffel tile under some rotations ρ and symmetries σ_i (see Figure 5).

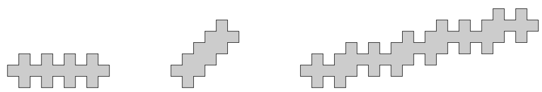


Fig. 5: Basic Christoffel tiles: (a) $w = aaaab$ (b) $w = abbbb$ and (c) $w = aabaababaabab$.

Theorem 9 (Blondin Massé et al. (2009)) *Let P be a crenelated tile. Then P is a double square if and only if it is obtained from a Christoffel word.* □

It can also be shown in view of Lemma 6, that each Christoffel tile is highly symmetrical.

Proposition 10 *If $AB\widehat{A}\widehat{B}$ is a BN-factorization of a Christoffel tile, then A and B are palindromes.*

Moreover, one verifies easily the following facts.

Proposition 11 *Let T be a Christoffel tile obtained from the (p, q) Christoffel word, where p and q are relatively prime. Then the perimeter and the area of T are given respectively by $\mathbf{P}(T) = 8p + 4q$ and $\mathbf{A}(T) = 4p + 3q - 2$.* □

4.2 Fibonacci Tiles

In this section, in order to simplify the notation, we redefine the operator $\overline{\cdot}$ on $\mathcal{R}^*$ by setting $\overline{y} = \iota(y)$, where ι is the involution $\iota : \mathtt{R} \leftrightarrow \mathtt{L}, \mathtt{F} \mapsto \mathtt{F}, \mathtt{B} \mapsto \mathtt{B}$. We define a sequence $(q_n)_{n\in\mathbb{N}}$ in $\mathcal{R}^*$ by setting $q_0 = \varepsilon$, $q_1 = \mathtt{R}$ and

$$q_n = \begin{cases} q_{n-1}q_{n-2} & \text{if } n \equiv 2 \bmod 3, \\ q_{n-1}\overline{q_{n-2}} & \text{if } n \equiv 0, 1 \bmod 3. \end{cases}$$

whenever $n \geq 2$. The first terms of $(q_n)_{n\in\mathbb{N}}$ are

$$\begin{array}{lll lll lll} q_0 & = & \varepsilon & q_3 & = & \mathtt{RL} & q_6 & = & \mathtt{RLLRLLRR} \\ q_1 & = & \mathtt{R} & q_4 & = & \mathtt{RLL} & q_7 & = & \mathtt{RLLRLLRRLRRLR} \\ q_2 & = & \mathtt{R} & q_5 & = & \mathtt{RLLRL} & q_8 & = & \mathtt{RLLRLLRRLRRLRRLLRLLRR} \end{array}$$

Note that $|q_n| = F_n$ is the n-th Fibonacci number. Moreover, given $\alpha \in \mathcal{F}$, the path $\mathcal{D}_\alpha^-(q_n)$ presents strong symmetric properties. The next two lemmas are from Blondin Massé et al. (2010b).

Lemma 12 *Let $n \in \mathbb{N}$. Then $q_{3n+1} = p\alpha$, $q_{3n+2} = q\alpha$ and $q_{3n+3} = r\overline{\alpha}$ for some antipalindrome p, and some palindromes q, r and some letter $\alpha \in \{\mathtt{L}, \mathtt{R}\}$.*

Lemma 13 *Let $n \in \mathbb{N}$ and $\alpha \in \mathcal{F}$. Then $\mathcal{D}_\alpha^-(q_n)$ is a simple path and $\mathcal{D}_\alpha^-(q_{3n+1})^4$ is the boundary word of a polyomino.*

A Fibonacci tile of order n is a polyomino having $\mathcal{D}_\alpha^-(q_{3n+1})^4$ as a boundary word, where $n \in \mathbb{N}$. They are somehow related to the Fibonacci fractals found in Monnerot-Dumaine. The first Fibonacci tiles are illustrated in Figure 6.

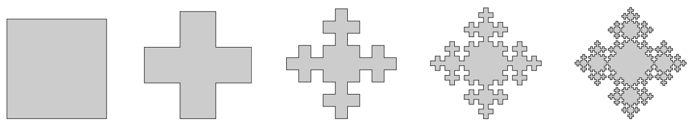


Fig. 6: Fibonacci tiles of order $n = 0, 1, 2, 3, 4$.

Theorem 14 *Fibonacci tiles are double squares.*

As for Christoffel tiles, Fibonacci tiles also suggest that the conjecture of Provençal (2008) for double squares is true as stated in the next result.

Corollary 15 *If $AB\widehat{A}\widehat{B}$ is a BN-factorisation of a Fibonacci tile, then A and B are palindromes.*

We have established in Blondin Massé et al. (2010b) that the perimeter of the Fibonacci tiles is given by $4F(3n+1)$ while their area $A(n)$ satisfies the recurrence formulas

$$\begin{aligned} A(0) &= 1, \quad A(1) = 5; \\ A(n) &= 6A(n-1) - A(n-2), \text{ for } n \geq 2, \end{aligned}$$

whose first terms are $1, 5, 29, 169, 985, 5741, 33461, 195025, 1136689, 6625109, 38613965, \ldots$ This sequence is the subsequence of odd index Pell numbers.

We end this section by presenting four families of double squares, a variant of the Fibonacci tiles whose areas satisfy the same recurrence. Indeed, consider the sequence $(r_{d,m,n})_{(d,m,n)\in\mathbb{N}^3}$ satisfying the following recurrence, for $d \geq 2$,

$$r_{d,m,n} = \begin{cases} r_{d-1,n,m}\overline{r_{d-2,n,m}} & \text{if } d \equiv 0 \bmod 3 \\ r_{d-1,n,m}r_{d-2,n,m} & \text{if } d \equiv 1 \bmod 3 \\ r_{d-1,m,n}\overline{r_{d-2,m,n}} & \text{if } d \equiv 2 \bmod 3 \end{cases}$$

Using similar arguments as in the Fibonacci tiles case, one shows that both families obtained respectively with seed values

$$\begin{aligned} r_{0,m,n} &= (\text{RLLR})^m\text{RLR}, & r_{1,m,n} &= (\text{RLLR})^n\text{R}, \\ r_{0,m,n} &= (\text{RL})^m\text{RLR}, & r_{1,m,n} &= (\text{RL})^n\text{RL} \end{aligned}$$

are such that $\mathcal{D}_\alpha^-(r_{3d,m,n}r_{3d,n,m})^2$, where $\alpha \in \mathcal{F}$, is a boundary word whose associated polyomino is a double square (see Figure 7). Their level of fractality increases with d so that one could say that they are crenelated versions of the Fibonacci Tiles.

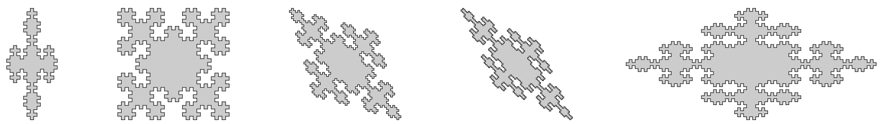


Fig. 7: Tiles obtained with different seeds: from $r_{2,0,1}$, from $r_{3,m,0}$ for $m = 0, 1, 2$, from $r_{3,1,0}$.

Similarly, let $(s_{d,m,n})_{(d,m,n)\in\mathbb{N}^3}$ be a sequence satisfying for $d \geq 2$ the recurrence

$$s_{d,m,n} = \begin{cases} s_{d-1,n,m}\overline{s_{d-2,n,m}} & \text{if } d \equiv 0, 2 \bmod 3, \\ s_{d-1,m,n}s_{d-2,m,n} & \text{if } d \equiv 1 \bmod 3. \end{cases}$$

Then the families obtained with seed values

$$\begin{array}{rclcrcl} s_{0,m,n} & = & (\text{RLLR})^m\text{RLR}, & & s_{1,m,n} & = & \text{RL}, \\ s_{0,m,n} & = & (\text{RL})^m\text{RLR}, & & s_{1,m,n} & = & \text{R} \end{array}$$

yield double squares $\mathcal{D}^-_\alpha(s_{3d,m,n}s_{3d,n,m})^2$ as well (see Figure 8). One may verify that $r_{d,0,0} = s_{d,0,0}$ for any $d \in \mathbb{N}$ for some conveniently chosen seed values.

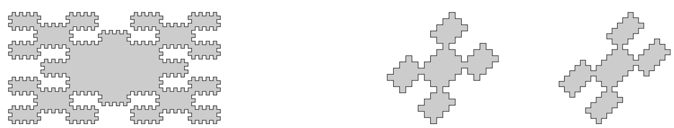


Fig. 8: Tile obtained from $s_{3,2,0}$, and from $s_{2,0,n}$ for $n = 1, 2$.

The area of the tiles $\mathcal{D}^-(r_{3d,m,n}r_{3d,n,m})^2$ and $\mathcal{D}^-(s_{3d,m,n}s_{3d,n,m})^2$ for each values of d, m and n share particular properties. In fact, all the sequences are obtained by the same recurrence (see the first values in Table 1), and we have the following proposition.

Proposition 16 *Let $m, n \in \mathbb{N}$ be fixed. The sequence of areas indexed by $d \in \mathbb{N}$ of the four families of generalized Fibonacci tiles satisfy the recurrence $A(d) = 6A(d-1) - A(d-2)$ for $d \geq 2$.* □

5 Concluding remarks

The study of double squares suggests some interesting and challenging problems. For instance, there is a conjecture of Provençal (2008) stating that if $AB\widehat{A}\widehat{B}$ is a BN-factorization of a prime double square, then A and B are palindromes, for which, despite a lot of computation time, we have not been able to provide any counter-example. Another problem is to prove that Christoffel and Fibonacci tiles are prime, that is, they are not obtained by composition of smaller squares. This leads to a number of questions on the "arithmetics" of tilings, such as the unique decomposition, distribution of prime tiles, and their enumeration. Partial results may be found in Brlek et al. (2006a)

It is also appealing to conjecture that a prime double square is either of Christoffel type or of Fibonacci type. However, that is not the case, as illustrated by Figure 9. This begs for a thorough study in order to exhibit a complete zoology of such tilings.

m	n	d							
		0	1	2	3	4	5	6	7
0	0	5	13	73	425	2477	14437	84145	490433
1	0	9	29	165	961	5601	32645	190269	1108969
0	1	9	29	165	961	5601	32645	190269	1108969
2	0	13	45	257	1497	8725	50853	296393	1727505
1	1	17	65	373	2173	12665	73817	430237	2507605
0	2	13	45	257	1497	8725	50853	296393	1727505
3	0	17	61	349	2033	11849	69061	402517	2346041
2	1	25	101	581	3385	19729	114989	670205	3906241
1	2	25	101	581	3385	19729	114989	670205	3906241
0	3	17	61	349	2033	11849	69061	402517	2346041
4	0	21	77	441	2569	14973	87269	508641	2964577
3	1	33	137	789	4597	26793	156161	910173	5304877
2	2	37	157	905	5273	30733	179125	1044017	6084977
1	3	33	137	789	4597	26793	156161	910173	5304877
0	4	21	77	441	2569	14973	87269	508641	2964577

Tab. 1: Area of the tile $\mathcal{D}^-(r_{3d,m,n}r_{3d,n,m})^2$ with seed values $r_{0,m,n} = (\text{RLLR})^m\text{RLR}$ and $r_{1,m,n} = (\text{RLLR})^n\text{R}$.

The fractal nature of the Fibonacci tiles strongly suggests that Lindemayer systems (L-systems) may be used for their construction Rozenberg and Salomaa (1980). The formal grammars used for describing them have been widely studied, and their impact in biology, computer graphics Rozenberg and Salomaa (2001) and modeling of plants is significant Prusinkiewicz and Lindenmayer (1990). A number of designs including snowflakes fall into this category.

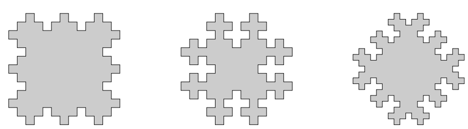


Fig. 9: Three double squares not in the Christoffel and Fibonacci tiles families.

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