FPSAC 2010, San Francisco, USA DMTCS proc. **AN**, 2010, 413–424

Zonotopes, toric arrangements, and generalized Tutte polynomials

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Abstract. We introduce a multiplicity Tutte polynomial $M(x, y)$, which generalizes the ordinary one and has applications to zonotopes and toric arrangements. We prove that $M(x, y)$ satisfies a deletion-restriction recurrence and has positive coefficients. The characteristic polynomial and the Poincaré polynomial of a toric arrangement are shown to be specializations of the associated polynomial $M(x, y)$, likewise the corresponding polynomials for a hyperplane arrangement are specializations of the ordinary Tutte polynomial. Furthermore, $M(1, y)$ is the Hilbert series of the related discrete Dahmen-Micchelli space, while $M(x, 1)$ computes the volume and the number of integral points of the associated zonotope.

Résumé. On introduit un polynôme de Tutte avec multiplicité $M(x, y)$, qui généralise le polynôme de Tutte ordinaire et a des applications aux zonotopes et aux arrangements toriques. Nous prouvons que $M(x, y)$ satisfait une récurrence de "deletion-restriction" et a des coefficients positifs. Le polynôme caractéristique et le polynôme de Poincaré d'un arrangement torique sont des spécialisations du polynôme associé $M(x, y)$, de même que les polynômes correspondants pour un arrangement d'hyperplans sont des spécialisations du polynôme de Tutte ordinaire. En outre, $M(1, y)$ est la série de Hilbert de l'espace discret de Dahmen-Micchelli associé, et $M(x, 1)$ calcule le volume et le nombre de points entiers du zonotope associé.

Keywords: Tutte polynomial, zonotope, integral points, toric arrangement, characteristic polynomial, Dahmen-Micchelli, partition function

1 Introduction

The Tutte polynomial is an invariant naturally associated to a matroid and encoding many of its features, such as the number of bases and their *internal and external activity* ([21], [3], [6]). If the matroid is defined by a finite list of vectors, it is natural to consider the arrangement obtained by taking the hyperplane orthogonal to each vector. To the poset of the intersections of the hyperplanes one associates its *characteristic polynomial*, which provides a rich combinatorial and topological description of the arrangement ([19], [22]). This polynomial can be obtained as a specialization of the Tutte polynomial.

Let T be a complex torus (i.e., a multiplicative group $(\mathbb{C}^*)^n$ of n-tuples of nonzero complex numbers) and take a finite list of characters: $X \subset Hom(T, \mathbb{C}^*)$. Then we consider the arrangement of hypersurfaces in T obtained by taking the kernel of each element of the list X. To understand the geometry of this *toric arrangement* one needs to describe the poset $\mathcal{C}(X)$ of the *layers*, i.e. connected components of the intersections of the hypersurfaces ([5], [9], [15], [18]). Clearly this poset depends also on the arithmetics of X, and not only on its linear algebra: for example, the kernel of the identity character λ of $\mathbb{C}^*$ is the point $t = 1$, but the kernel of 2λ has equation $t^2 = 1$, hence is made of two points. Therefore we have no chance to get the characteristic polynomial of $\mathcal{C}(X)$ as a specialization of the ordinary Tutte polynomial $T(x, y)$ of X. In this paper we define a polynomial $M(x, y)$ that specializes to the characteristic polynomial of $\mathcal{C}(X)$ (Theorem 5.5) and to the Poincaré polynomial of the complement $\mathcal{R}_X$ of the toric arrangement (Theorem 5.6). In particular $M(1, 0)$ equals the Euler characteristic of $\mathcal{R}_X$, and also the number of connected components of the complement of the arrangement in the *compact torus* $\overline{T} = (\mathbb{S}^1)^n$.

We call $M(x, y)$ the *multiplicity Tutte polynomial* of X, since it coincides with $T(x, y)$ when X is unimodular, and in general it satisfies the same *deletion-restriction* recurrence that holds for $T(x, y)$. By this formula (Theorem 3.3) we prove that $M(x, y)$ has positive coefficients (Theorem 3.4).

Actually a similar polynomial can be defined more generally for matroids, if we enrich their structure in order to encode some "arithmetic data"; we call such objects *multiplicity matroids*. We hope to develop in a future paper an axiomatic theory of these matroids, as well as applications to graph theory. In the present paper the focus is on the case of a list X of vectors in $\mathbb{Z}^n$. Given such a list, we consider two finite dimensional vector spaces: a space of polynomials $D(X)$, defined by differential equations, and a space of quasipolynomials $DM(X)$, defined by difference equations. These spaces were introduced by Dahmen and Micchelli to study respectively *box splines* and *partition functions*, and are deeply related respectively with the hyperplane arrangement and the toric arrangement defined by X, as explained in the forthcoming book [6]. In particular, $T(1, y)$ is known to be the Hilbert series of $D(X)$; then we prove that $M(1, y)$ is the Hilbert series of $DM(X)$ (Theorem 6.3).

On the other hand, by Theorem 4.1 the coefficients of $M(x, 1)$ count integral points in some faces of a convex polytope, the *zonotope* defined by X. The relations between arrangements, zonotopes and Dahmen-Micchelli spaces is being studied intensively in the very last years: see for example [6], [10], [7], [1], [11], . In particular $M(1, 1)$ equals the volume of the zonotope (Proposition 2.1), while $M(2, 1)$ is the number of its integral points (Proposition 4.2).

Finally we focus on the case in which X is a root system: then we show some connections with the theory of Weyl groups (see for instance Corollary 7.3).

Remark 1.1 *This paper is an extended abstract of [17], which contains more details and all the proofs, which are omitted here.*

2 Multiplicity matroids and multiplicity Tutte polynomials

We start recalling the notions we are going to generalize.

A *matroid* $\mathfrak{M}$ is a pair (X, I), where X is a finite set and I is a family of subsets of X (called the *independent sets*) with the following properties:

1. The empty set is independent;
2. Every subset of an independent set is independent;
3. Let A and B be two independent sets and assume that A has more elements than B. Then there exists an element $a \in A \setminus B$ such that $B \cup \{a\}$ is still independent.

A maximal independent set is called a *basis*. The last axiom implies that all bases have the same cardinality, which is called the *rank* of the matroid. Every $A \subseteq X$ has a natural structure of matroid, defined by considering a subset of A independent if and only if it is in I. Then each $A \subseteq X$ has a rank which we denote by $r(A)$.

The *Tutte polynomial* of the matroid is then defined as

$$T(x,y) \doteq \sum_{A\subseteq X}(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)}.$$

From the definition it is clear that $T(1,1)$ equals the number of bases of the matroid.

In the next sections we will recall the main example of matroid and some properties of its Tutte polynomial.

We now introduce the following definitions.

A *multiplicity matroid* $\mathfrak{M}$ is a triple (X, I, m), where (X, I) is a matroid and m is a function (called *multiplicity*) from the family of all subsets of X to the positive integers.

We say that m is the *trivial multiplicity* if it is identically equal to 1.

We define the *multiplicity Tutte polynomial* of a multiplicity matroid as

$$M(x,y) \doteq \sum_{A\subseteq X} m(A)(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)}.$$

Let us remark that we can endow every matroid with the trivial multiplicity, and then $M(x,y) = T(x,y)$.

Let X be a finite list of vectors spanning a real vector space U, and I be the family of its linearly independent subsets; then (X, I) is a matroid, and the rank of a subset A is just the dimension of the spanned subspace. We denote by $T_X(x,y)$ the associated Tutte polynomial.

We associate to the list X a *zonotope*, that is a convex polytope in U defined as follows:

$$\mathcal{Z}(X) \doteq \left\{ \sum_{x\in X} t_x x, 0 \leq t_x \leq 1 \right\}.$$

Zonotopes play an important role in the theory of hyperplane arrangements, and also in that of *splines*, a class of functions studied in Approximation Theory. (see [6]).

We recall that a *lattice* Λ of rank n is a discrete subgroup of $\mathbb{R}^n$ which spans the real vector space $\mathbb{R}^n$. Every such Λ can be generated from some basis of the vector space by forming all linear combinations with integral coefficients; hence the group Λ is isomorphic to $\mathbb{Z}^n$. We will use the word *lattice* always with this meaning, and not in the combinatorial sense (poset with *join* and *meet*).

Then let X be a finite list of elements in a lattice Λ, and let I and r be as above. We denote by $\langle A\rangle_{\mathbb{Z}}$ and $\langle A\rangle_{\mathbb{R}}$ respectively the sublattice of Λ and the subspace of $\Lambda \otimes \mathbb{R}$ spanned by A. Let us define $\Lambda_A \doteq \Lambda \cap \langle A\rangle_{\mathbb{R}}$: this is the largest sublattice of Λ in which $\langle A\rangle_{\mathbb{Z}}$ has finite index. Then we define m as this index:

$$m(A) \doteq [\Lambda_A : \langle A\rangle_{\mathbb{Z}}].$$

This defines a multiplicity matroid and then a multiplicity Tutte polynomial $M_X(x,y)$, which is the main subject of this paper. We start by showing the relations with the zonotope $\mathcal{Z}(X)$ generated by X in $U \doteq \Lambda \otimes \mathbb{R}$.

We already observed that $T_X(1,1)$ equals the number of bases that can be extracted from X; on the other hand we have:

Proposition 2.1 $M_X(1,1)$ *equals the volume of the zonotope* $\mathcal{Z}(X)$.

Further relations between the polynomial $M_X(x,y)$ and the zonotope $\mathcal{Z}(X)$ will be shown in Section 4.

3 Deletion-restriction formula and positivity

The central idea that inspired Tutte in defining the polynomial $T(x,y)$, was to find the most general invariant satisfying a recurrence known as *deletion-restriction*. Such recurrence allows to reduce the computation of the Tutte polynomial to some trivial cases. We will explain this algorithm in the case above, i.e. when the matroid is defined by a list of vectors, and we will show that in this case also the polynomial $M(x,y)$ satisfies a similar recursion.

3.1 Lists of vectors

Let X be a finite list of elements spanning a vector space U, and let $v \in X$ be a nonzero element. We define two new lists: the list $X_1 \doteq X \setminus \{v\}$ of elements of U and the list X_2 of elements of $U/\langle v \rangle$ obtained by reducing X_1 modulo v. Assume that v is dependent in X, i.e. $v \in \langle X_1 \rangle_{\mathbb{R}}$. Then we have the following well-known formula:

Theorem 3.1

$$T_X(x,y) = T_{X_1}(x,y) + T_{X_2}(x,y)$$

It is now clear why we defined X as a list, and not as a set: even if we start with X made of (nonzero) distinct elements, in X_2 some vector may appear many times (and some vector may be zero).

By this recurrence we get:

Theorem 3.2 $T_X(x,y)$ *is a polynomial with positive coefficients.*

3.2 Lists of elements in finitely generated abelian groups.

We now want to show a similar recursion for the polynomial $M_X(x,y)$. Inspired by [8], we notice that in order to do this, we need to work in a larger category. Indeed, whereas the quotient of a vector space by a subspace is still a vector space, the quotient of a lattice by a sublattice is not a lattice, but a *finitely generated abelian group*. For example in the 1-dimensional case, the quotient of $\mathbb{Z}$ by $m\mathbb{Z}$ is the cyclic group of order m.

Then let Γ be a finitely generated abelian group. For every subset S of Γ we denote by $\langle S \rangle$ the generated subgroup. We recall that Γ is isomorphic to the direct product of a lattice Λ and of a finite group Γ_t, which is called the *torsion subgroup* of Γ. We denote by π the projection $\pi : \Gamma \to \Lambda$.

Let X be a finite subset of Γ; for every $A \subseteq X$ we set $\Lambda_A \doteq \Lambda \cap \left\langle \pi(A) \right\rangle_{\mathbb{R}}$ and $\Gamma_A \doteq \Lambda_A \times \Gamma_t$. In other words, Γ_A is the largest subgroup of Γ in which $\langle A \rangle$ has finite index.

Then we define $m(A) \doteq \left[\Gamma_A : \langle A \rangle\right]$. We also define $r(A)$ as the rank of $\pi(A)$. In this way we defined a multiplicity matroid, to which is associated a multiplicity Tutte polynomial:

$$M_X(x,y) \doteq \sum_{A \subseteq X} m(A)(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)}.$$

It is clear that if Γ is a lattice, these definitions coincide with the ones given in the previous sections.

If on the opposite hand Γ is a finite group, $M(x,y)$ is a polynomial in which only the variable y appears; furthermore this polynomial, evaluated at $y=1$, gives the order of Γ. Indeed the only summand that does not vanish is the contribution of the empty set, which generates the trivial subgroup.

Now let $\lambda \in X$ be a nonzero element such that $\pi(\lambda) \in \big\langle \pi\big(X \setminus \{\lambda\}\big)\big\rangle_{\mathbb{R}}$. We set $X_1 \doteq X \setminus \{\lambda\} \subset \Gamma$ and we denote by $\overline{A}$ the image of every $A \subseteq X$ under the natural projection $\Gamma \longrightarrow \Gamma/\langle\lambda\rangle$. We denote by X_2 the subset $\overline{X_1}$ of $\Gamma/\langle\lambda\rangle$. Then we have the following deletion-restriction formula.

Theorem 3.3

$$M_X(x,y) = M_{X_1}(x,y) + M_{X_2}(x,y).$$

By this recurrence we prove:

Theorem 3.4 *$M_X(x,y)$ is a polynomial with positive coefficients.*

4 Integral points in zonotopes

Let X be a finite list of vectors contained in a lattice Λ and generating the vector space $U = \Lambda \otimes \mathbb{R}$. We say that a point of U is *integral* if it is contained in Λ. In this section we prove that $M_X(2,1)$ equals the number of integral points of the zonotope $\mathcal{Z}(X)$. Moreover we compare this number with the volume. In order to do that, we have to move the zonotope to a "generic position"; we proceed as follows. Following [6, Section 1.3], we define the *cut-locus* of the couple (Λ, X) as the union of all hyperplanes in U that are translations, under elements of Λ, of the linear hyperplanes spanned by subsets of X. Then let $\underline{\varepsilon}$ be a vector of U which does not lie in the cut-locus and has length $\varepsilon << 0$. Let $\mathcal{Z}(X) - \underline{\varepsilon}$ be the polytope obtained translating $\mathcal{Z}(X)$ by $-\underline{\varepsilon}$, and let $\mathfrak{I}(X)$ be the set of its integral points:

$$\mathfrak{I}(X) \doteq (\mathcal{Z}(X) - \underline{\varepsilon}) \cap \Lambda.$$

It is intuitive (and proved in [6, Prop 2.50]) that this number equals the volume:

$$|\mathfrak{I}(X)| = vol\,(\mathcal{Z}(X)) = M_X(1,1)$$

by Proposition 2.1. We now prove a stronger result. Let us choose $\underline{\varepsilon}$ so that $\mathcal{Z}(X) - \underline{\varepsilon}$ contanins the origin $\underline{0}$. We partition $\mathfrak{I}(X)$ as follows: set $\mathfrak{I}_n(X) = \{\underline{0}\}$, and for every $k = n-1, \ldots, 0$, let $\mathfrak{I}_k(X)$ be the set of elements of $\mathfrak{I}(X)$ that are contained in some $k-$codimensional face of $\mathcal{Z}(X)$ and that are not contained in $\mathfrak{I}_h(X)$ for $h > k$.

Then we have:

Theorem 4.1

$$M_X(x,1) = \sum_{k=0}^{n} |\mathfrak{I}_k(X)|\; x^k.$$

Furthermore we prove:

Proposition 4.2

$$M_X(2,1) = |\mathcal{Z}(X) \cap \Lambda|$$

Example 4.3 *Consider the list in* $\mathbb{Z}^2$

$$X = \{(3,3),(1,-1),(2,0)\}.$$

Then

$$M_X(x,y) = (x-1)^2 + (3+1+2)(x-1) + (6+6+2) + 2(y-1).$$

Hence

$$M_X(x,1) = x^2 + 4x + 9$$

and $M_X(2,1) = 21$. *Indeed the zonotope* $\mathcal{Z}(X)$ *has volume* 14 *and contains* 21 *integral points, 14 of which lying in* $\mathcal{Z}(X) - \underline{\varepsilon}$. *The sets* $\mathfrak{I}_2(X)$, $\mathfrak{I}_1(X)$, *and* $\mathfrak{I}_0(X)$ *contain 1, 4 and 9 points respectively.*

5 Application to arrangements

In this Section we describe some geometrical objects related to the lists considered in Section 2.2, and show that many of their features are encoded in the polynomials $T_X(x,y)$ and $M_X(x,y)$.

5.1 Recall on hyperplane arrangements

Let X be a finite list of elements of a vector space U. Then in the dual space $V = U^*$ a *hyperplane arrangement* $\mathcal{H}(X)$ is defined by taking the orthogonal hyperplane of each element of X. Conversely, given an arrangement of hyperplanes in a vector space V, let us choose for each hyperplane a nonzero vector in V^* orthogonal to it; let X be the list of such vectors. Since every element of X is determined up to scalar multiples, the matroid associated to X is well defined; in this way a Tutte polynomial is naturally associated to the hyperplane arrangement.

The importance of the Tutte polynomial in the theory of hyperplane arrangements is well known. Here we just recall some results that we generalize in the next sections.

To every sublist $A \subseteq X$ is associated the subspace $A^\perp$ of V that is the intersection of the corresponding hyperplanes of $\mathcal{H}(X)$; in other words, $A^\perp$ is the subspace of vectors that are orthogonal to every element of A. Let $\mathcal{L}(X)$ be the set of such subspaces, partially ordered by reverse inclusion, and having as minimal element $\mathbf{0}$ the whole space $V = \emptyset^\perp$. $\mathcal{L}(X)$ is called the *intersection poset* of the arrangement, and is "the most important combinatorial object associated to a hyperplane arrangement" (R. Stanley).

We also recall that to every finite poset $\mathcal{P}$ is associated a *Moebius function* $\mu : \mathcal{P} \times \mathcal{P} \to \mathbb{Z}$, recursively defined as follows:

$$\mu(L,M) = \begin{cases} 0 & \text{if } L > M \\ 1 & \text{if } L = M \\ -\sum_{L \le N < M} \mu(L,N) & \text{if } L < M. \end{cases}$$

Notice that the poset $\mathcal{L}(X)$ is *ranked* by the dimension of the subspaces; then we define *characteristic polynomial* of the poset as

$$\chi(q) \doteq \sum_{L \in \mathcal{L}(X)} \mu(\mathbf{0}, L) q^{dim(L)}.$$

This is an important invariant of $\mathcal{H}(X)$. Indeed, let $\mathcal{M}_X$ be the complement in V of the union of the hyperplanes of $\mathcal{H}(X)$. Let $P(q)$ be *Poincaré polynomial* of $\mathcal{M}_X$, i.e. the polynomial having as coefficient of q^k the $k-$th Betti number of $\mathcal{M}_X$. Then if V is a complex vector space, by [19] we have the following theorem.

Theorem 5.1

$$P(q) = (-q)^n \chi(-1/q).$$

If on the other hand V is a real vector space, by [22] the number $Ch(X)$ of *chambers* (i.e., connected components of $\mathcal{M}_X$) is:

Theorem 5.2

$$Ch(X) = (-1)^n \chi(-1).$$

The Tutte polynomial $T_X(x, y)$ turns out to be a stronger invariant, in the following sense. Assume that $\underline{0} \notin X$; then

Theorem 5.3

$$(-1)^n T_X(1-q, 0) = \chi(q).$$

The proof of these theorems can be found for example in [6, Theorems 10.5, 2.34 and 2.33].

5.2 Toric arrangements and their generalizations

Let $\Gamma = \Lambda \times \Gamma_t$ be a finitely generated abelian group, and define $T_\Gamma \doteq Hom(\Gamma, \mathbb{C}^*)$. T_Γ has a natural structure of abelian linear algebraic group: indeed it is the direct product of a complex torus T_Λ of the same rank as Λ and of the finite group ${\Gamma_t}^*$ dual to Γ_t (and isomorphic to it).

Moreover Γ is identified with the group of characters of T_Γ: indeed given $\lambda \in \Lambda$ and $t \in T_\Gamma$ we can take any representative $\varphi_t \in Hom(\Gamma, \mathbb{C})$ of t and set $\lambda(t) \doteq e^{2\pi i \varphi_t(\lambda)}$. When this is not ambiguous we will denote T_Γ by T.

Let $X \subset \Lambda$ be a finite subset spanning a sublattice of Λ of finite index. The kernel of every character $\chi \in X$ is a (non-connected) hypersurface in T:

$$H_\chi \doteq \big\{ t \in T | \chi(t) = 1 \big\}.$$

The collection $\mathcal{T}(X) = \{H_\chi, \chi \in X\}$ is called the *generalized toric arrangement* defined by X on T.

We denote by $\mathcal{R}_X$ the complement of the arrangement:

$$\mathcal{R}_X \doteq T \setminus \bigcup_{\chi \in X} H_\chi$$

and by $\mathcal{C}_X$ the set of all the connected components of all the intersections of the hypersurfaces H_χ, ordered by reverse inclusion and having as minimal elements the connected components of T.

Since $rank(\Lambda) = dim(T)$, the maximal elements of $\mathcal{C}(X)$ are 0-dimensional, hence (since they are connected) they are points. We denote by $\mathcal{C}_0(X)$ the set of such layers, which we call the *points* of the arrangement.

Given $A \subseteq X$ let us define $H_A \doteq \bigcap_{\lambda \in A} H_\lambda$. Then we have:

Lemma 5.4 *$m(A)$ equals the number of connected components of H_A.*

In particular, when Γ is a lattice, T is a torus and $\mathcal{T}(X)$ is called the *toric arrangement* defined by X. Such arrangements have been studied for example in [14], [5], [15], [18]; see [6] for a complete reference. In particular, the complement $\mathcal{R}_X$ has been described topologically and geometrically. In this description the poset $\mathcal{C}(X)$ plays a major role, for many aspects analogous to that of the intersection poset for hyperplane arrangements (see [5], [18]).

We will now explain the importance in this framework of the polynomial $M_X(x, y)$ defined in Section 3.3.

5.3 Characteristic polynomial and Poincaré polynomial

Let μ be the Moebius function of $\mathcal{C}(X)$; notice that we have a natural rank function given by the dimension of the layers. For every $C \in \mathcal{C}(X)$, let T_C be the connected component of T that contains C. Then we define the characteristic polynomial of $\mathcal{C}(X)$:

$$\chi(q) \doteq \sum_{C \in \mathcal{C}(X)} \mu(T_C, C) q^{dim(C)}.$$

This polynomial is a specialization of the multiplicity Tutte polynomial:

Theorem 5.5

$$(-1)^n M_X(1-q, 0) = \chi(q)$$

Furthermore, by applying our results to a theorem proved in [5, Theor. 4.2] (or [6, 14.1.5]), we give a formula for the Poincare' polynomial $P(q)$ of $\mathcal{R}_X$:

Theorem 5.6

$$P(q) = q^n M_X\left(\frac{2q+1}{q}, 0\right).$$

Therefore, by comparing Theorem 5.5 and Theorem 5.6, we get the following formula, which relates the combinatorics of $\mathcal{C}(X)$ with the topology of $\mathcal{R}_X$, and is the "toric" analogue of Theorem 4.1.

Corollary 5.7

$$P(q) = (-q)^n \chi\left(-\frac{q+1}{q}\right).$$

We recall that the *Euler characteristic* of a space can be defined as the evaluation at -1 of its Poincaré polynomial. Hence by Theorem 5.6 we have:

Corollary 5.8 $(-1)^n M_X(1,0)$ *equals the Euler characteristic of* $\mathcal{R}_X$.

Example 5.9 *Take* $T = (\mathbb{C}^*)^2$ *with coordinates* (t, s) *and*

$$X = \{(2,0), (0,2), (1,1), (1,-1)\}$$

defining equations:

$$t^2 = 1, s^2 = 1, ts = 1, ts^{-1} = 1.$$

It is easily seen (see [17] for details) that this arrangement has six $1-$*dimensional layers and four* $0-$*dimensional layers, and that*

$$\chi(q) = q^2 - 6q + 8.$$

The polynomial $M_X(x, y)$ *is composed by the following summands:*

- $(x-1)^2$*, corresponding to the empty set;*
- $6(x-1)$*, corresponding to the 4 singletons, each giving contribution* $(x-1)$ *or* $2(x-1)$*;*
- 14*, corresponding to the 6 pairs: indeed, the basis* $X = \{(2,0), (0,2)\}$ *spans a sublattice of index* 4*, while the other bases span sublattices of index* 2*;*

- $8(y-1)$*, corresponding to the 4 triples, each contributing with* $2(y-1)$*;*
- $2(y-1)^2$*, corresponding to the whole set* X*.*

Hence

$$M_X(x,y) = x^2 + 2y^2 + 4x + 4y + 3.$$

Notice that

$$M_X(1-q,0) = q^2 - 6q + 8 = \chi(q)$$

as claimed in Theorem 5.5. Furthermore Theorem 5.6 (or Corollary 4.12) implies that

$$P(q) = 15q^2 + 8q + 1$$

and hence the Euler characteristic is $P(-1) = 8 = M_X(1,0)$*. Notice that this is the toric arrangement arising from the root system of type* C_2 *(see Section 7).*

5.4 Number of regions of the compact torus

In this section we consider the compact abelian group dual to Γ $\overline{T} \doteq Hom(\Gamma, \mathbb{S}^1)$, where we set $\mathbb{S}^1 \doteq \{z \in \mathbb{C} \mid |z| = 1\} \simeq \mathbb{R}/\mathbb{Z}$.

We assume for simplicity Γ to be a lattice; then $\overline{T}$ is a *compact torus*, i.e. it is isomorphic to $(\mathbb{S}^1)^n$, and in it every $\chi \in X$ defines a hypersurface $\overline{H_\chi} \doteq \left\{t \in \overline{T} | \chi(t) = 1\right\}$. We denote by $\overline{\mathcal{T}(X)}$ this arrangement; clearly its poset of layers is the same as for the arrangement $\mathcal{T}(X)$ defined in the complex torus T. We denote by $\overline{\mathcal{R}_X}$ the complement

$$\overline{\mathcal{R}_X} \doteq \overline{T} \setminus \bigcup_{\chi \in X} \overline{H_\chi}.$$

The compact toric arrangement $\overline{\mathcal{T}(X)}$ has been studied in [9]; in particular the number $R(X)$ of *region*s (i.e. of connected components) of $\overline{\mathcal{R}_X}$ is proved to be a specialization of the characteristic polynomial $\chi(q)$:

Theorem 5.10

$$R(X) = (-1)^n \chi(0).$$

By comparing this result with Theorem 5.5 we get the following

Corollary 5.11

$$R(X) = M_X(1,0)$$

6 Dahmen-Micchelli spaces

Until now we considered evaluations of $T_X(x,y)$ and $M_X(x,y)$ at $y = 0$ and $y = 1$. However, there is another remarkable specialization of the Tutte polynomial: $T_X(1,y)$, which is called the *polynomial of the external activity* of X. It is related with the corresponding specialization of $M_X(x,y)$ in a simple way:

Lemma 6.1

$$M_X(1,y) = \sum_{p \in \mathcal{C}_0(X)} T_{X_p}(1,y).$$

The previous lemma has an interesting consequence. In [4] to every finite set $X \subset V$ is associated a space $D(X)$ of functions $V \to \mathbb{C}$, and to every finite set $X \subset \Lambda$ is associated a space $DM(X)$ of functions $\Lambda \to \mathbb{C}$. Such spaces are defined as the solutions of a system, respectively of differential equations and of difference equations, in the following way.

For every $\lambda \in X$, let ∂_λ be the usual directional derivative $\partial_\lambda f(x) \doteq \partial f/\partial\lambda(x)$ and let ∇_λ be the difference operator $\nabla_\lambda f(x) \doteq f(x) - f(x-\lambda)$.

Then for every $A \subset X$ we define the differential operator $\partial_A \doteq \prod_{\lambda\in A} \partial_\lambda$ and the difference operator $\nabla_A \doteq \prod_{\lambda\in A} \nabla_\lambda$. We can now define define the *differentiable Dahmen-Micchelli space*

$$D(X) \doteq \{f : V \to \mathbb{C} \mid \partial_A f = 0\ \forall A \text{ such that } r(X \setminus A) < n\}$$

and the *discrete Dahmen-Micchelli space*

$$DM(X) \doteq \{f : \Lambda \to \mathbb{C} \mid \nabla_A f = 0\ \forall A \text{ such that } r(X \setminus A) < n\}.$$

The space $D(X)$ is a space of polynomials, which was introduced in order to study the *box spline*. This is a piecewise-polynomial function studied in Approximation Theory; its local pieces, together with their derivatives, span $D(X)$. On the other hand, $DM(X)$ is a space of quasipolynomials which arises in the study of the *partition function*. This is the function that counts in how many ways an element of Λ can be written as a linear combination with positive integer coefficients of elements of X. This function is piecewise-quasipolynomial, and its local pieces, together with their translates, span $DM(X)$. In the recent book [6] the spaces $D(X)$ and $DM(X)$ are shown to be deeply related respectively with the hyperlane arrangement and with the toric arrangement defined by X.

In order to compare these two spaces, we consider the elements of $D(X)$ as functions $\Lambda \to \mathbb{C}$ by restricting them to the lattice Λ. Since the elements of $DM(X)$ are polynomial functions, they are determined by their restriction. For every $p \in \mathcal{C}(X)^0$, let us define $\varphi_p : \Lambda \to \mathbb{C}$ as the map $\lambda \mapsto \lambda(p)$. (see Section 2.4.2). In [4] (see also [6, Formula 16.1]) the following result is proved.

Theorem 6.2

$$DM(X) = \bigoplus_{p\in\mathcal{C}_0(X)} \varphi_p D(X_p).$$

Since every $D(X_p)$ is defined by homogeneous differential equations, it is naturally graded, the degree of every element being just its degree as a polynomial. The Hilbert series of $D(X_p)$ is known to be $T_{X_p}(1, y)$; in other words, the coefficients of this polynomial equal the dimensions of the graded parts (see [2] or [6, Theorem 11.8]). Then, by the theorem above, also the space $DM(X)$ is graded, and by Lemma 6.1 we have:

Theorem 6.3 *$M_X(1, y)$ is the Hilbert series of $DM(X)$.*

By comparing this theorem with Proposition 2.1 we recover the following known result, which can be found for example in ([6, Chapter 13]) :

Corollary 6.4 *The dimension of $DM(X)$ equals the volume of the zonotope $\mathcal{Z}(X)$.*

7 The case of root systems

This section is devoted to describe a remarkable class of examples. We will assume standard notions about root systems, Lie algebras and algebraic groups, which are exposed for example in [13] and [12].

Let Φ be a root system, $\langle \Phi^\vee \rangle$ be the lattice spanned by the coroots, and Λ be its dual lattice (which is called the *cocharacters* lattice). Then we define as in Section 4.2 a torus $T = T_\Lambda$ having Λ as group of characters. In other words, if $\mathfrak{g}$ is the semisimple complex Lie algebra associated to Φ and $\mathfrak{h}$ is a Cartan subalgebra, T is defined as the quotient $T \doteq \mathfrak{h}/\langle \Phi^\vee \rangle$.

Each root α takes integer values on $\langle \Phi^\vee \rangle$, so it induces a character $e^\alpha : T \to \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$. Let X be the set of this characters; more precisely, since α and $-\alpha$ define the same hypersurface, we set

$$X \doteq \left\{ e^\alpha,\ \alpha \in \Phi^+ \right\}.$$

In this way to every root system Φ is associated a toric arrangement. These arrangements have been studied in [15]; we now show two applications to the present work. Let W be the (finite) Weyl group of Φ, and let $\widetilde{W}$ be the associated affine Weyl group. We denote by $s_0, \dots, s_n$ its generators, and by W_k the subgroup of $\widetilde{W}$ generated by all the elements s_i but s_k. Let $\Phi_k \subset \Phi$ be the root system of W_k, and denote by X_k the corresponding sublist of X. Then we have:

Corollary 7.1

$$M_X(1, y) = \sum_{k=0}^{n} \frac{|W|}{|W_k|} T_{X_k}(1, y).$$

Furthermore, in [15] the following theorem is proved. Let W be the Weyl group of Φ.

Theorem 7.2 *The Euler characteristic of $\mathcal{R}_X$ is equal to* $(-1)^n |W|$.

By comparing this statement with Corollary 5.8, we get the following

Corollary 7.3

$$M_X(1, 0) = |W|.$$

It would be interesting to have a more direct proof of this fact.

Acknowledgements

I am grateful to my advisor Corrado De Concini for many important suggestions. I also wish to thank Emanuele Delucchi, Olga Holtz, Mario Marietti, Claudio Procesi and Michèle Vergne for stimulating discussions and remarks. .

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