The Murnaghan–Nakayama rule for k-Schur functions

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Abstract. We prove a Murnaghan–Nakayama rule for k-Schur functions of Lapointe and Morse. That is, we give an explicit formula for the expansion of the product of a power sum symmetric function and a k-Schur function in terms of k-Schur functions. This is proved using the noncommutative k-Schur functions in terms of the nilCoxeter algebra introduced by Lam and the affine analogue of noncommutative symmetric functions of Fomin and Greene.

Résumé. Nous prouvons une règle de Murnaghan-Nakayama pour les fonctions de k-Schur de Lapointe et Morse, c'est-à-dire que nous donnons une formule explicite pour le développement du produit d'une fonction symétrique "somme de puissances" et d'une fonction de k-Schur en termes de fonctions k-Schur. Ceci est prouvé en utilisant les fonctions non commutatives k-Schur en termes d'algèbre nilCoxeter introduite par Lam et l'analogue affine des fonctions symétriques non commutatives de Fomin et Greene.

Keywords: Murnaghan–Nayakama rule, symmetric functions, noncommutative symmetric functions, k-Schur functions

1 Introduction

The Murnaghan–Nakayama rule [11, 14, 15] is a combinatorial formula for the characters $\chi_\lambda(\mu)$ of the symmetric group in terms of ribbon tableaux. Under the Frobenius characteristic map, there exists an analogous statement on the level of symmetric functions, which follows directly from the formula

$$p_r s_\lambda = \sum_\mu (-1)^{\mathrm{ht}(\mu/\lambda)} s_\mu. \tag{1}$$

Here p_r is the r-th power sum symmetric function, s_λ is the Schur function labeled by partition λ, and the sum is over all partitions $\lambda \subseteq \mu$ for which μ/λ is a border strip of size r. Recall that a border strip is a connected skew shape without any 2×2 squares. The height $\mathrm{ht}(\mu/\lambda)$ of a border strip μ/λ is one less than the number of rows.

In [4], Fomin and Greene develop the theory of Schur functions in noncommuting variables. In particular, they derive a noncommutative version of the Murnaghan–Nakayama rule [4, Theorem 1.3] for the nilCoxeter algebra (or more generally the local plactic algebra)

$$\mathbf{p}_r \mathbf{s}_\lambda = \sum_w (-1)^{\mathrm{asc}(w)} w \mathbf{s}_\lambda \ , \tag{2}$$

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where w is a hook word of length r. Here $\mathbf{p}_r$ and $\mathbf{s}_\lambda$ are the noncommutative analogues of the power sum symmetric function and the Schur function (introduced in Section 2). The word w is a hook word if $w = b_l b_{l-1} \dots b_1 a_1 a_2 \dots a_m$ where

$$b_l > b_{l-1} > \cdots > b_1 > a_1 \leq a_2 \leq \cdots \leq a_m \tag{3}$$

and $\mathrm{asc}(w) = m - 1$ is the number of ascents in w. Actually, by [4, Theorem 5.1] it can further be assumed that the support of w is an interval.

In this paper, we derive a (noncommutative) Murnaghan–Nakayama rule for the k-Schur functions of Lapointe and Morse [10]. k-Schur functions form a basis for the ring $\Lambda_{(k)} = \mathbb{Z}[h_1, \dots, h_k]$ spanned by the first k complete homogeneous symmetric functions h_r, which is a subring of the ring of symmetric functions Λ. Lapointe and Morse [10] gave a formula for a homogeneous symmetric function h_r times a k-Schur function (at $t = 1$) as

$$h_r s_\lambda^{(k)} = \sum_{\mu \in \mathcal{P}^{(k)}} s_\mu^{(k)}, \tag{4}$$

where the sum is over all k-bounded partitions $\mu \in \mathcal{P}^{(k)}$ such that μ/λ is a horizontal r-strip and $\mu^{(k)}/\lambda^{(k)}$ is a vertical r-strip. Here $\lambda^{(k)}$ denotes the k-conjugate of λ. Equation (4) is a simple analogue of the Pieri rule for usual Schur functions, called the k-Pieri rule. This formula can in fact be taken as the definition of k-Schur functions from which many of their properties can be derived. Conjecturally, the k-Pieri definition of the k-Schur functions is equivalent to the original definition by Lapointe, Lascoux, and Morse [6] in terms of atoms.

Lam [5] defined a noncommutative version of the k-Schur functions in the affine nilCoxeter algebra as the dual of the affine Stanley symmetric functions

$$F_w(X) = \sum_{a=(a_1,\dots,a_t)} \langle \mathbf{h}_{a_t}(u)\mathbf{h}_{a_{t-1}}(u) \cdots \mathbf{h}_{a_1}(u) \cdot 1, w \rangle \, x_1^{a_1} \cdots x_t^{a_t}, \tag{5}$$

where the sum is over all compositions of the length of w satisfying $a_i \in [0, k]$. Here

$$\mathbf{h}_r(u) = \sum_A u_A^{\mathrm{dec}}$$

are the analogues of homogeneous symmetric functions in noncommutative variables, where the sum is over all r-subsets A of $[0, k]$ and u_A^{dec} is the product of the generators of the affine nilCoxeter algebra in cyclically decreasing order with indices appearing in A. We denote the noncommutative analogue of $\Lambda_{(k)}$ by $\mathbf{\Lambda}_{(k)}$ as the subalgebra of the affine nilCoxeter algebra generated by these analogues of homogeneous symmetric functions. See Section 2.3 for further details.

Denote by $\mathbf{s}_\lambda^{(k)}$ the noncommutative k-Schur function labeled by the k-bounded partition λ and $\mathbf{p}_r$ the noncommutative power sum symmetric function in the affine nilCoxeter algebra. There is a natural bijection from k-bounded partitions λ to $(k + 1)$-cores, denoted $\mathrm{core}_{k+1}(\lambda)$ (see Section 2.1). We define a vertical domino in a skew-partition to be a pair of cells in the diagram, with one sitting directly above the other. For the skew of two k-bounded partitions $\lambda \subseteq \mu$ we define the height as

$$\mathrm{ht}(\mu/\lambda) = \text{number of vertical dominos in } \mu/\lambda\, . \tag{6}$$

For ribbons, that is skew shapes without any 2×2 squares, the definition of height can be restated as the number of occupied rows minus the number of connected components. Notice that is compatible with the usual definition of the height of a border strip.

All notation and definitions regarding our main Definition 1.1 and Theorem 1.2 are given in Section 2 below.

Definition 1.1 *The skew of two k-bounded partitions, μ/λ, is called a k*-ribbon of size r *if μ and λ satisfy the following properties:*

(0) (containment condition) $\lambda \subseteq \mu$ and $\lambda^{(k)} \subseteq \mu^{(k)}$;

(1) (size condition) $|\mu/\lambda| = r$;

(2) (ribbon condition) $\mathrm{core}_{k+1}(\mu)/\mathrm{core}_{k+1}(\lambda)$ is a ribbon;

(3) (connectedness condition) $\mathrm{core}_{k+1}(\mu)/\mathrm{core}_{k+1}(\lambda)$ is k-connected (see Definition 2.3);

(4) (height statistics condition) $\mathrm{ht}(\mu/\lambda) + \mathrm{ht}(\mu^{(k)}/\lambda^{(k)}) = r - 1$.

Our main result is the following theorem.

Theorem 1.2 *For $1 \leq r \leq k$ and λ a k-bounded partition, we have*

$$\mathbf{p}_r \mathbf{s}_\lambda^{(k)} = \sum_\mu (-1)^{\mathrm{ht}(\mu/\lambda)} \mathbf{s}_\mu^{(k)},$$

where the sum is over all k-bounded partitions μ such that μ/λ is a k-ribbon of size r.

Let λ, ν be k-bounded partitions of the same size and ℓ the length of ν. A k-ribbon tableau of shape λ and type ν is a filling, T, of the cells of λ with the labels $\{1, 2, \ldots, \ell\}$ which satisfies the following conditions for all $1 \leq i \leq \ell$:

(i) the shape of the restriction of T to the cells labeled $1, \ldots, i$ is a partition, and

(ii) the skew shape r_i, which is the restriction of T to the cells labeled i, is a k-ribbon of size ν_i.

We also define

$$\chi_{\lambda,\nu}^{(k)} = \sum_T \left(\prod_{i=1}^{\ell} (-1)^{\mathrm{ht}(r_i)} \right) ,$$

where the sum is over all k-ribbon tableaux T of shape λ and type ν.

Iterating Theorem 1.2 gives the following corollary. We remark that this formula may also be considered as a definition of the k-Schur functions.

Corollary 1.3 *For ν a k-bounded partition, we have*

$$\mathbf{p}_\nu = \sum_{\lambda \in \mathcal{P}^{(k)}} \chi_{\lambda,\nu}^{(k)} \, \mathbf{s}_\lambda^{(k)}.$$

In Section 2 we will see that there is a ring isomorphism

$$\iota : \mathbf{\Lambda}_{(k)} \to \Lambda_{(k)}$$

sending the noncommutative symmetric functions to their symmetric function counterpart. This leads us to the following corollary.

Corollary 1.4 *Theorem 1.2 and Corollary 1.3 also hold when replacing $\mathbf{p}_r$ by the power sum symmetric function p_r, and $\mathbf{s}_\lambda^{(k)}$ by the k-Schur function $s_\lambda^{(k)}$.*

Dual k-Schur functions $\mathfrak{S}_\lambda^{(k)}$ indexed by k-bounded partitions λ form a basis of the quotient space $\Lambda^{(k)} = \Lambda/\langle p_r \mid r > k\rangle = \Lambda/\langle m_\lambda \mid \lambda_1 > k\rangle$ (they correspond to the affine Stanley symmetric functions indexed by Grassmannian elements). The Hall inner product $\langle \cdot,\cdot\rangle : \Lambda\times\Lambda \to \mathbb{Q}$ defined by $\langle h_\lambda, m_\mu\rangle = \langle s_\lambda, s_\mu\rangle = \delta_{\lambda,\mu}$, can be restricted to $\langle\cdot,\cdot\rangle : \Lambda_{(k)} \times \Lambda^{(k)} \to \mathbb{Q}$, so that $s_\lambda^{(k)}$ and $\mathfrak{S}_\mu^{(k)}$ form dual bases $\langle s_\lambda^{(k)}, \mathfrak{S}_\mu^{(k)}\rangle = \delta_{\lambda,\mu}$. Let z_λ be the size of the centralizer of any permutation of cycle type λ. Then $\langle p_\lambda, p_\mu\rangle = z_\lambda \delta_{\lambda,\mu}$.

Corollary 1.5 *For ν a k-bounded partition, we have*

$$\mathfrak{S}_\nu^{(k)} = \sum_{\lambda\in\mathcal{P}^{(k)}} \frac{1}{z_\lambda} \chi_{\nu,\lambda}^{(k)}\, p_\lambda\ .$$

Since the product of two k-bounded power symmetric functions is again a k-bounded power symmetric function, the expansion of the dual k-Schur functions in terms of p_λ of Corollary 1.5 is better suited for multiplication than the expansion in terms of monomial symmetric functions. The product of two k-bounded monomial symmetric functions is a sum of monomial symmetric functions which are not necessarily k-bounded.

The paper is organized as follows. In Section 2 we introduce all notation and definitions. In particular, we define the various noncommutative symmetric functions. In Section 3 we prove Theorem 3.1, which is the analogue of Theorem 1.2 formulated in terms of the nilCoxeter algebra. We conclude in Section 4 with some related open questions.

A long version of this paper containing a proof that Theorems 1.2 and 3.1 are equivalent is available as a preprint [1].

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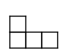
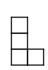
2 Notation

In this section we give all necessary definitions.

2.1 Partitions and cores

A sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a partition if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. We say that ℓ is the length of λ and $|\lambda| = \lambda_1 + \cdots + \lambda_\ell$ is its size. A partition λ is k-bounded if $\lambda_1 \leq k$. We denote by $\mathcal{P}^{(k)}$ the set of all k-bounded partitions.

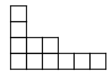
One may represent a partition λ by its partition diagram, which contains λ_i boxes in row i. The conjugate λ^t corresponds to the diagram with rows and columns interchanged. We use French convention and label rows in decreasing order from bottom to top. For example

 and 

correspond to the partition $(3,1)$ and its conjugate $(2,1,1)$, respectively.

For two partitions λ and μ whose diagrams are contained, that is $\lambda \subseteq \mu$, we denote by μ/λ the skew partition consisting of the boxes in μ not contained in λ. A ribbon is a skew shape which does not contain any 2×2 squares. An r-border strip is a connected ribbon with r boxes.

A partition λ is an r-core if no r-border strip can be removed from λ and still results in a partition. For example



(7)

is a 4-core. We denote the set of all r-cores by $\mathcal{C}_r$.

For a cell $c = (i, j) \in \lambda$ in row i and column j we define its hook length to be the number of cells in row i of λ to the right of c plus the number of cells in column j of λ weakly above c (including c). An alternative definition of an r-core is a partition without any cells of hook length equal to a multiple of r [13, Ch. 1, Ex. 8]. The content of cell $c = (i, j)$ is given by $j - i \pmod r$.

There exists a bijection [9]

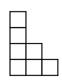
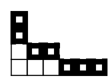
$$\mathrm{core}_{k+1} : \mathcal{P}^{(k)} \to \mathcal{C}_{k+1} \tag{8}$$

from k-bounded partitions to $(k+1)$-cores defined as follows. Let $\lambda \in \mathcal{P}^{(k)}$ considered as a set of cells. Starting from the smallest row, check whether there are any cells of hook length greater than k. If so, slide the row and all those in the rows below to the right by the minimal amount so that none of cells in that row have a hook length greater than k. Then continue the procedure with the rows below. The positions of the cells define a skew partition and the outer partition is a $(k+1)$-core.

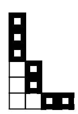
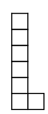
The inverse map $\mathrm{core}_{k+1}^{-1} : \mathcal{C}_{k+1} \to \mathcal{P}^{(k)}$ is slightly easier to compute. The partition $\mathrm{core}_{k+1}^{-1}(\kappa)$ is of the same length as the $(k+1)$-core κ and the i^{th} entry of the partition is the number of cells in the the i^{th} row of κ which have a hook smaller or equal to k.

Let $\lambda \in \mathcal{P}^{(k)}$. Then the k-conjugate $\lambda^{(k)}$ of λ is defined as $\mathrm{core}_{k+1}^{-1}(\mathrm{core}_{k+1}(\lambda)^t)$.

Example 2.1 *For $k = 3$, take $\lambda = (3, 2, 1, 1) \in \mathcal{P}^{(k)}$ so that*

$\mathrm{core}_4 :$  $\mapsto$ 

which is the 4-core in (7) *(where we have drawn the original boxes of λ in bold). To obtain the k-conjugate $\lambda^{(3)}$ of λ we calculate*

$\mathrm{core}_4^{-1} :$  $\mapsto$  .

2.2 Affine nilCoxeter algebra

The affine nilCoxeter algebra $\mathcal{A}_k$ is the algebra over $\mathbb{Z}$ generated by $u_0, u_1, \ldots, u_k$ satisfying

$$\begin{aligned} &u_i^2 = 0 && \text{for } i \in [0, k], \\ &u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} && \text{for } i \in [0, k], \\ &u_i u_j = u_j u_i && \text{for } i, j \in [0, k] \text{ such that } |i - j| \geq 2, \end{aligned} \tag{9}$$

where all indices are taken modulo $k + 1$. We view the indices $i \in [0, k]$ as living on a circle, with node i being adjacent to nodes $i - 1$ and $i + 1$ (modulo $k + 1$). As with Coxeter groups, we have a notion of reduced words of elements $u \in \mathcal{A}_k$ as the shortest expressions in the generators. If $u = u_{i_1} \cdots u_{i_m}$ is a reduced expression, we call $\{i_1, \ldots, i_m\}$ the support of u denoted $\mathrm{supp}(u)$ (which is independent of the reduced word and only depends on u itself). Also, $i_1 \ldots i_m$ is the corresponding reduced word and $\mathrm{len}(u) = m$ is the length of u.

A word w in the letters $[0,k]$ is cyclically decreasing (resp. increasing) if the length of w is at most k, every letter appears at most once, and if $i, i-1 \in w$ then i occurs before (resp. after) $i-1$. Note that since u_i and u_j commute if i is not adjacent to j, all cyclically decreasing (resp. increasing) words w with the same support give rise to the same affine nilCoxeter group element $\prod_{i\in w} u_i$. For a proper subset $A \subsetneq [0,k]$ we define $u_A^{\mathrm{dec}} \in \mathcal{A}_k$ (resp. $u_A^{\mathrm{inc}} \in \mathcal{A}_k$) to be the element corresponding to cyclically decreasing (resp. increasing) words with support A.

Example 2.2 *Take* $k = 6$ *and* $A = \{0,2,3,4,6\}$. *Then* $u_A^{\mathrm{dec}} = (u_0u_6)(u_4u_3u_2) = (u_4u_3u_2)(u_0u_6)$ *and* $u_A^{\mathrm{inc}} = (u_6u_0)(u_2u_3u_4) = (u_2u_3u_4)(u_6u_0)$.

If $u \in \mathcal{A}_k$ is supported on a proper subset S of $[0,k]$, then we specify a canonical interval I_S which contains the subset S. Identify the smallest element a (from the numbers 0 through k with the integer order) which does not appear in S. Then the canonical cyclic interval which we choose orders the elements

$$a+1 < a+2 < \cdots < k < 0 < 1 < \cdots < a-1,$$

(where we identify k and -1 when necessary).

Definition 2.3 *An element* $u \in \mathcal{A}_k$ *(resp. word* w*) is* k*-connected if its support* S *is an interval in* I_S.

Example 2.4 *For* $k = 6$*, the word* $w = 0605$ *is* k*-connected, whereas* $w = 06052$ *is not.*

Suppose $u \in \mathcal{A}_k$ has support $S \subsetneq [0,k]$. We say that u corresponds to a hook word if it has a reduced word w of the form of Equation (3) with respect to the canonical order I_S. In this case we denote by $\mathrm{asc}(u)$ or $\mathrm{asc}(w)$ the number of ascents $\mathrm{asc}_{I_S}(w)$ in the canonical order.

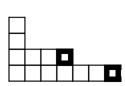
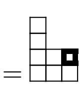
Example 2.5 *Take* $u = u_3u_2u_6u_0u_4 \in \mathcal{A}_6$. *In this case* $S = \{0,2,3,4,6\}$ *and* I_S *is given by* $2 < 3 < 4 < 5 < 6 < 0$. *The word* $w = (3)(2460)$ *is a hook word with respect to* I_S *and* $\mathrm{asc}(u) = 3$.

The generators u_i in the nilCoxeter algebra $\mathcal{A}_k$ act on a $(k+1)$-core $\nu \in \mathcal{C}_{k+1}$ by

$$u_i \cdot \nu = \begin{cases} \nu \text{ with all corner cells of content } i \text{ added if they exist,} \\ 0 \text{ otherwise.} \end{cases} \tag{10}$$

This action is extended to the rest of the algebra $\mathcal{A}_k$ and can be shown to be consistent with the relations of the generators. Under the bijection core_{k+1}^{-1} to k-bounded partitions only the topmost box added to diagram survives. The action of u_i on a k-bounded partition λ under core_{k+1} is denoted $u_i \cdot \lambda$.

Example 2.6 *Taking* $\nu = \mathrm{core}_4(\lambda)$ *from Example 2.1 we obtain*

$u_2 \cdot \nu =$  *and* $\mathrm{core}_4^{-1}(u_2 \cdot \nu) =$ 

where the boxes added by u_2 *of content 2 are indicated in bold.*

2.3 Noncommutative symmetric functions

We now give the definition of the noncommutative symmetric functions $\mathbf{e}_r$, $\mathbf{h}_r$, $\mathbf{s}_{(r-i,1^i)}$, $\mathbf{p}_r$, and $\mathbf{s}_\lambda^{(k)}$ in terms of the affine nilCoxeter algebra.

Following Lam [5], for $r = 1, \ldots, k$, we define the noncommutative homogeneous symmetric functions

$$\mathbf{h}_r = \sum_{A \in \binom{[0,k]}{r}} u_A^{\mathrm{dec}} \ ,$$

where u_A^{dec} is a cyclically decreasing element with support A as defined in Section 2.2. We take as a defining relation for the elements $\mathbf{e}_r$ the equation $\sum_{i=0}^{r}(-1)^i \mathbf{e}_{r-i}\mathbf{h}_i = 0$. It can be shown [5, Proposition 16] that then

$$\mathbf{e}_r = \sum_{A \in \binom{[0,k]}{r}} u_A^{\mathrm{inc}} \ ,$$

where u_A^{inc} is a cyclically increasing element with support A. More generally, the hook Schur functions for $r \le k$ are given by

$$\mathbf{s}_{(r-i,1^i)} = \mathbf{h}_{r-i}\mathbf{e}_i - \mathbf{h}_{r-i+1}\mathbf{e}_{i-1} + \cdots + (-1)^i \mathbf{h}_r$$

and we will demonstrate in Corollary 3.5 (below) that these elements may also be expressed as a sum over certain words.

The noncommutative power sum symmetric functions for $1 \le r \le k$ are defined through the analogue of a classical identity with ribbon Schur functions

$$\mathbf{p}_r = \sum_{i=0}^{r-1} (-1)^i \mathbf{s}_{(r-i,1^i)}.$$

Lam [5, Proposition 8] proved that, even though the variables u_i do not commute, the elements $\mathbf{h}_r$ for $1 \le r \le k$ commute and consequently, so do the other elements $\mathbf{e}_r$, $\mathbf{p}_r$, $\mathbf{s}_{(r-i,1^i)}$ we have defined in terms of the $\mathbf{h}_r$. We define $\mathbf{\Lambda}_{(k)} = \mathbb{Z}[\mathbf{h}_1, \ldots, \mathbf{h}_k]$ to be the noncommutative analogue of $\Lambda_{(k)} = \mathbb{Z}[h_1, \ldots, h_k]$.

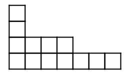
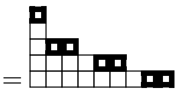
We define the noncommutative k-Schur functions $\mathbf{s}_\lambda^{(k)}$ by the noncommutative analogue of the k-Pieri rule (4). Let us denote by $\mathcal{H}_r^{(k)}$ the set of all pairs (μ, λ) of k-bounded partitions μ, λ such that μ/λ is a horizontal r-strip and $\mu^{(k)}/\lambda^{(k)}$ is a vertical r-strip (which describes the summation in the k-Pieri rule). Then for a k-bounded partition λ we require that

$$\mathbf{h}_r \mathbf{s}_\lambda^{(k)} = \sum_{\mu : (\mu,\lambda) \in \mathcal{H}_r^{(k)}} \mathbf{s}_\mu^{(k)}. \tag{11}$$

This definition can be used to expand the $\mathbf{h}_\mu$ elements in terms of the elements $\mathbf{s}_\lambda^{(k)}$. The transition matrix is described by the number of k-tableaux of given shape and weight (see [9]). Since this matrix is unitriangular, this system of relations can be inverted over the integers and hence $\{\mathbf{s}_\lambda^{(k)} \mid \lambda \in \mathcal{P}^{(k)}\}$ forms a basis of $\mathbf{\Lambda}_{(k)}$.

As shown in [9, 7], for $1 \le r \le k$, we have if $(\mu, \lambda) \in \mathcal{H}_r^{(k)}$, then there is a cyclically decreasing element $u \in \mathcal{A}_k$ of length r such that $\mu = u \cdot \lambda$. Moreover, if $u \in \mathcal{A}_k$ is cyclically decreasing and $\mu = u \cdot \lambda \neq 0$, then $(\mu, \lambda) \in \mathcal{H}_r^{(k)}$.

Example 2.7 *Take* $\lambda = (3,3,1,1) \in \mathcal{P}^{(3)}$ *and* $u = u_0 u_3$. *Then*

$\mathrm{core}_4(\lambda) =$  *and* $u \cdot \mathrm{core}_4(\lambda) =$ 

so that $((3,3,2,1,1),(3,3,1,1)) \in \mathcal{H}_2^{(3)}$.

Hence, we may rewrite (11) as

$$\mathbf{h}_r \mathbf{s}_\lambda^{(k)} = \sum_{\mu:(\mu,\lambda)\in\mathcal{H}_r^{(k)}} \mathbf{s}_\mu^{(k)} = \sum_{A\in\binom{[0,k]}{r}} \mathbf{s}^{(k)}_{u_A^{\mathrm{dec}}\cdot\lambda} \,,$$

where we assume $\mathbf{s}^{(k)}_{u_A^{\mathrm{dec}}\cdot\lambda} = 0$ if $u_A^{\mathrm{dec}} \cdot \lambda = 0$. The elements $\mathbf{h}_r = \sum_{A\in\binom{[0,k]}{r}} u_A^{\mathrm{dec}}$ generate $\mathbf{\Lambda}_{(k)}$, and therefore more generally for any element $\mathbf{f} = \sum_u c_u u \in \mathbf{\Lambda}_{(k)}$ with $u \in \mathcal{A}_k$ and $c_u \in \mathbb{Z}$

$$\mathbf{f}\ \mathbf{s}_\lambda^{(k)} = \sum_u c_u \mathbf{s}^{(k)}_{u\cdot\lambda} \,. \tag{12}$$

Since all of the noncommutative symmetric functions in this section commute and satisfy the same defining relations as their commutative counterparts, there is a ring isomorphism

$$\iota : \mathbf{\Lambda}_{(k)} \to \Lambda_{(k)}$$

sending $\mathbf{h}_r \mapsto h_r$, $\mathbf{e}_r \mapsto e_r$, $\mathbf{p}_r \mapsto p_r$, $\mathbf{s}_\lambda^{(k)} \mapsto s_\lambda^{(k)}$.

3 Main result: Murnaghan–Nakayama rule in terms of words

We now restate Theorem 1.2 in terms of the action of words. This result is proved in the remainder of this section.

Theorem 3.1 *For* $1 \le r \le k$ *and* λ *a* k*-bounded partition, we have*

$$\mathbf{p}_r \mathbf{s}_\lambda^{(k)} = \sum_{(w,\mu)} (-1)^{\mathrm{asc}(w)} \mathbf{s}_\mu^{(k)}, \tag{13}$$

where the sum is over all pairs (w,μ) *of reduced words* w *in the affine nilCoxeter algebra* $\mathcal{A}_k$ *and* k*-bounded partitions* μ *satisfying*

- *(1′) (size condition)* $\mathrm{len}(w) = r$*;*
- *(2′) (ribbon condition)* w *is a hook word;*
- *(3′) (connectedness condition)* w *is* k*-connected;*
- *(4′) (weak order condition)* $\mu = w \cdot \lambda$.

The proof of Theorem 3.1 essentially amounts to computing an expression for $\mathbf{p}_r$ in terms of words. Since all words involved will be of length $\le k$, there will be a canonical order on the support as introduced in Section 2.2. The statistic $\mathrm{asc}(w)$, and the property of being a hook word, will always be in terms of this canonical ordering.

Lemma 3.2 *For $0 \leq i \leq r \leq k$,*

$$\mathbf{h}_{r-i}\mathbf{e}_i = \sum_w w \; , \tag{14}$$

where the sum is over all words w satisfying $(1')$, $(2')$ with respect to the canonical order, and $\operatorname{asc}(w) \in \{i-1, i\}$.

Proof: $\mathbf{h}_{r-i}$ is the sum over all cyclically decreasing nilCoxeter group elements of length $r-i$ and $\mathbf{e}_i$ is the sum over all cyclically increasing nilCoxeter group elements of length i. Hence

$$\mathbf{h}_{r-i}\mathbf{e}_i = \sum_{\substack{(u,v) \\ u \text{ cycl. dec., } |u| = r-i \\ v \text{ cycl. inc., } |v| = i}} uv.$$

Rearrange each u and v so that they together form a hook with respect to the canonical order associated to the set $\operatorname{supp}(u) \cup \operatorname{supp}(v)$. Either the last letter of u is smaller than the first letter of v, in which case the total ascent is i, or the last letter of u is bigger than the first letter in v, in which case the total number of ascents is $i-1$. This yields a bijection between hook words in the canonical order and pairs appearing in this sum with the number of ascents in $\{i, i-1\}$. In the corner case $i = 0$ (resp. $i = r$) the number of ascents can only be 0 (resp. $r-1$ due to the fact that the words are of length r). □

Example 3.3 *Take $k = 8$, $u = (u_1u_0u_8)(u_5u_4)$ and $v = (u_2u_3)(u_0)$, so that $i = 3$ and $r = 8$. In this case the canonical order is $7 < 8 < 0 < 1 < 2 < 3 < 4 < 5$ and we would write uv as $uv = [(u_5u_4)(u_1u_0u_8)][(u_0)(u_2u_3)]$, giving rise to the word $w = (5410)(8023)$ with $i = 3$ ascents. If on the other hand $u = (u_1u_0)(u_5u_4)$ and $v = (u_2u_3)(u_8u_0)$, so that $i = 4$ and $r = 8$, then we would write $uv = [(u_5u_4)(u_1u_0)][(u_8u_0)(u_2u_3)]$, giving rise to the word $w = (5410)(8023)$ with $i - 1 = 3$ ascents.*

Remark 3.4 *Note that there may be multiplicities in (14) with respect to affine nilCoxeter group elements because there may be several hook words with the same number of ascents that are equivalent to the same affine nilCoxeter element. For example, $(4)(20)$ and $(0)(24)$ are two different hook words with exactly one ascent with respect to the interval $I_{\{0,2,4\}} = \{2 < 4 < 0\}$. Of course, they both correspond to the same affine nilCoxeter element since all letters in the word commute. The element with $u = u_2$ and $v = u_4u_0$ would give rise to the hook word $w = (240)$ with 2 ascents.*

We can use this lemma to get an expression for hook Schur functions.

Corollary 3.5 *For $0 \leq i \leq r \leq k$, the hook Schur function is*

$$\mathbf{s}_{(r-i,1^i)} = \sum_w w \; ,$$

where the sum is over all words w satisfying $(1')$, $(2')$ with respect to the canoncial order, and $\operatorname{asc}(w) = i$.

Proof: From our definition of the noncommutative Schur functions indexed by a hook partition, it follows that

$$\mathbf{s}_{(r-i,1^i)} = \mathbf{h}_{r-i}\mathbf{e}_i - \mathbf{h}_{r-i+1}\mathbf{e}_{i-1} + \cdots + (-1)^i\mathbf{h}_r.$$

Hence by Lemma 3.2 the only words which do not appear in two terms with opposite signs are those that have $\operatorname{asc}(w) = i$, which implies the corollary. □

Example 3.6 *Let $k = 3$. Then for $r = 3$ and $i = 1$ we have*

$$\mathbf{s}_{2,1} = u_1u_0u_1 + u_2u_1u_2 + u_3u_2u_3 + u_0u_3u_0 \\ + u_1u_3u_0 + u_1u_0u_2 + u_2u_0u_1 + u_2u_1u_3 + u_3u_1u_2 + u_3u_2u_0 + u_0u_2u_3 + u_0u_3u_1.$$

We can now write an expression for $\mathbf{p}_r$ by using the definition.

Corollary 3.7 *For $1 \leq r \leq k$,*

$$\mathbf{p}_r = \sum_w (-1)^{\mathrm{asc}(w)} w,$$

where the sum is over all words w satisfying $(1')$ and $(2')$ in the canonical order.

Proof: This follows immediately from the definition

$$\mathbf{p}_r = \sum_{i=0}^{r-1} (-1)^i \mathbf{s}_{(r-i,1^i)} \ .$$

□

In fact, we may restrict our attention to those words in the sum also satisfying $(3')$ because it is possible to show that those not satisfying $(3')$ will cancel.

Lemma 3.8 *For $r \leq k$,*

$$\mathbf{p}_r = \sum_w (-1)^{\mathrm{asc}(w)} w,$$

where the sum is over all words w satisfying $(1')$, $(2')$, and $(3')$.

Proof: Since each canonical interval can be viewed as an interval of the finite nilCoxeter group, the sign-reversing involution described before [4, Theorem 5.1] still holds and there is a sign-reversing involution on the terms which do not satisfy $(3')$. Hence it suffices to sum only over terms which are connected cyclic intervals. □

Example 3.9 *Let $k = 3$. Then*

$$\mathbf{p}_2 = u_1u_0 + u_2u_1 + u_3u_2 + u_0u_3 - (u_1u_2 + u_2u_3 + u_3u_0 + u_0u_1).$$

Theorem 3.1 now follows from the action of words on $\mathbf{s}_\lambda^{(k)}$ given by Equation (12).

4 Outlook

By Corollaries 1.3, 1.4 and 1.5, the Murnaghan-Nakayama rule proved in this paper gives the expansion of the power sum symmetric functions in terms of the k-Schur functions $s_\lambda^{(k)} \in \Lambda_{(k)}$ and the expansion of the dual k-Schur functions $\mathfrak{S}_\lambda^{(k)} \in \Lambda^{(k)}$ in terms of the power sums:

$$p_\nu = \sum_{\lambda \in \mathcal{P}^{(k)}} \chi_{\lambda,\nu}^{(k)} \, s_\lambda^{(k)} \qquad \text{and} \qquad \mathfrak{S}_\nu^{(k)} = \sum_{\lambda \in \mathcal{P}^{(k)}} \frac{1}{z_\lambda} \chi_{\nu,\lambda}^{(k)} \, p_\lambda \ .$$

Unlike in the symmetric function case, where the Schur functions $s_\lambda \in \Lambda$ are self-dual, there should be a dual version of the Murnaghan-Nakayama rule of this paper, namely a combinatorial formula for the coefficients $\tilde{\chi}^{(k)}_{\lambda,\nu}$ in the expansion of the power sum symmetric functions in terms of the dual k-Schur functions

$$p_\nu = \sum_{\lambda \in \mathcal{P}^{(k)}} \tilde{\chi}^{(k)}_{\lambda,\nu}\, \mathfrak{S}^{(k)}_\lambda$$

or, equivalently by the same arguments as in the proof of Corollary 1.5,

$$s^{(k)}_\nu = \sum_{\lambda \in \mathcal{P}^{(k)}} \frac{1}{z_\lambda} \tilde{\chi}^{(k)}_{\nu,\lambda}\, p_\lambda\ .$$

Since the $s^{(k)}_\nu$ are known to be Schur-positive symmetric functions [8], they correspond to representations of the symmetric group under the Frobenius characteristic map. Furthermore, the characters of these representations are given by the $\tilde{\chi}^{(k)}_{\nu,\lambda}$. An explicit description of such representations is an interesting open problem, which has been studied by Li-Chung Chen and Mark Haiman [2]. In the most generality they conjecture a representation theoretical model for the k-Schur functions with a parameter t which keeps track of the degree grading; the $\tilde{\chi}^{(k)}_{\nu,\lambda}$ described above should give the characters of these representations without regard to degree.

Computer evidence suggests that the ribbon condition (2) of Definition 1.1 might be superfluous because it is implied by the other conditions of the definition. This was checked for $k, r \leq 11$ and for all $|\lambda| = n \leq 12$ and $|\mu| = n + r$.

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