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Algebraic and combinatorial structures on Baxter permutations

Samuele Giraudo

Institut Gaspard Monge, Université Paris-Est Marne-la-Vallée, Marne-la-Vallée, France

Abstract. We give a new construction of a Hopf subalgebra of the Hopf algebra of Free quasi-symmetric functions whose bases are indexed by objects belonging to the Baxter combinatorial family (i.e. Baxter permutations, pairs of twin binary trees, etc.). This construction relies on the definition of the Baxter monoid, analog of the plactic monoid and the sylvestrer monoid, and on a Robinson-Schensted-like insertion algorithm. The algebraic properties of this Hopf algebra are studied. This Hopf algebra appeared for the first time in the work of Reading [Lattice congruences, fans and Hopf algebras, Journal of Combinatorial Theory Series A, 110:237–273, 2005].


Keywords: Hopf algebras, Robinson-Schensted algorithm, quotient monoid, Baxter permutations

1 Introduction

In the recent years, many combinatorial Hopf algebras, whose bases are indexed by combinatorial objects, have been intensively studied. For example, the Malvenuto-Reutenauer Hopf algebra \( FQSym \) of Free quasi-symmetric functions \([19, 7]\) has bases indexed by permutations. This Hopf algebra admits several Hopf subalgebras: The Hopf algebra of Free symmetric functions \( FSym \) \([21, 7]\), whose bases are indexed by standard Young tableaux, the Hopf algebra \( Bell \) \([23]\) whose bases are indexed by set partitions, the Loday-Ronco Hopf algebra \( PBT \) \([18][12]\) whose bases are indexed by planar binary trees and the Hopf algebra \( Sym \) of non-commutative symmetric functions \([10]\) whose bases are indexed by integer compositions. An unifying approach to construct all these structures relies on a definition of a congruence on words leading to the definition of monoids on combinatorial objects. Indeed, \( FSym \) is directly obtained from the plactic monoid \([15]\), \( Bell \) from the Bell monoid \([23]\), \( PBT \) from the sylvestrer monoid \([11][12]\), and \( Sym \) from the hypoplactic monoid \([20]\). The richness of these constructions relies on the fact that, in addition to construct Hopf algebras, the definition of such monoids often brings partial orders, combinatorial algorithms and Robinson-Schensted-like algorithms, of independent interest.

In this paper, we propose to enrich this collection of Hopf algebras by providing a construction of a Hopf algebra whose bases are indexed by objects belonging to the Baxter combinatorial family. This combinatorial family admits various representations as Baxter permutations \([4]\), pairs of twin binary trees \([8]\),...
The author would like to thank Florent Hivert and Jean-Christophe Novelli for their advice and help during all stages of the preparation of this paper. The computations of this work have been done with the open-source mathematical software Sage \cite{25}.

2 Preliminaries

2.1 Words

In the sequel, $A := \{a_1 < a_2 < \ldots \}$ is a totally ordered infinite alphabet and $A^*$ is the free monoid spanned by $A$. Let $u \in A^*$. For $S \subseteq A$, we denote by $u_{|S}$ the restriction of $u$ on the alphabet $S$, that is the longest subword of $u$ made of letters of $S$. The evaluation $\text{eval}(u)$ of the word $u$ is the non-negative integer vector such that its $i$-th entry is the number of occurrences of the letter $a_i$ in $u$. Let $\max(u)$ be the maximal letter of $u$. The Schützenberger transformation $\#$ is defined by $u^\# := \max(u)+1−u_{|\#}$. For example, $(a_5a_3a_1a_5a_3a_1)^\# = a_4a_1a_5a_3a_1$. Note that it is an involution if $u$ has an occurrence of $a_1$. Let $v \in A^*$ and $a, b \in A$. The shuffle product $\shuffle$ is defined on $\mathbb{Z}(A)$ recursively by $u\shuffle\epsilon := \epsilon\shuffle u := u$ and $au \shuffle bv := a(u \shuffle bv) + b(au \shuffle v)$.

2.2 Permutations

Denote by $\mathfrak{S}_n$ the set of permutations of size $n$ and $\mathfrak{S} := \cup_{n \geq 0} \mathfrak{S}_n$. We shall call $(i, j)$ a co-inversion of $\sigma \in \mathfrak{S}$ if $i < j$ and $\sigma_i^{-1} > \sigma_j^{-1}$. Let us recall that the (right) permutohedron order is the partial order $\leq_P$ defined on $\mathfrak{S}_n$ where $\sigma$ is covered by $\nu$ if $\sigma = uabv$ and $\nu = ubav$ where $a < b$. Let $\sigma, \nu \in \mathfrak{S}$. The permutation $\sigma / \nu$ is obtained by concatenating $\sigma$ and the letters of $\nu$ incremented by $|\sigma|$: in the same way, the permutation $\sigma \setminus \nu$ is obtained by concatenating the letters of $\nu$ incremented by $|\sigma|$ and $\sigma$; For example, $312 \div 2314 = 3125647$ and $312 \backslash 2314 = 5647312$. The permutation $\sigma$ is connected if $\sigma = \nu / \pi$ implies $\nu = \sigma$ or $\pi = \sigma$. The shifted shuffle product $\shuffle$ of two permutations is defined by $\sigma \shuffle \nu := \ldots$
\(\sigma \sqcup (\nu_1+|\sigma| \ldots \nu_m+|\sigma|)\); For example, \(12 \sqcup 21 = 1243 \sqcup 43 = 1243+4123=4312+4132+1423+1432\). The standardized word \(\text{std}(u)\) of \(u \in A^*\) is the unique permutation \(\sigma^*\) satisfying \(\sigma_i < \sigma_j\) if \(u_i \leq u_j\) for all \(1 \leq i < j \leq |u|\); For example, \(\text{std}(a_3a_1a_4a_2a_5a_7a_4a_2a_3) = 416289735\).

### 2.3 Binary trees

Denote by \(BT_n\) the set of binary trees with \(n\) internal nodes and \(BT := \bigcup_{n \geq 0} BT_n\). We use in the sequel the standard terminology (i.e., child, ancestor, \ldots) about binary trees \([2]\). The only element of \(BT_0\) is the leaf or empty tree, denoted by \(\perp\). Let us recall that the Tamari order \([14]\) is the partial order \(\leq T\) defined on \(BT_n\) where \(T_0 \in BT_n\) is covered by \(T_1 \in BT_n\) if it is possible to transform \(T_0\) into \(T_1\) by performing a right rotation (see Figure 1).

![Figure 1: The right rotation of root y.](image)

Let \(T_0, T_1 \in BT\). The binary tree \(T_0 \triangleright T_1\) is obtained by grafting \(T_0\) from its root on the leftmost leaf of \(T_1\); In the same way, the binary tree \(T_0 \smallsetminus T_1\) is obtained by grafting \(T_1\) from its root on the rightmost leaf of \(T_0\). The canopy (see \([18]\) and \([26]\)) \(\text{cnp}(T)\) of \(T \in BT\) is the word on the alphabet \(\{0, 1\}\) obtained by browsing the leaves of \(T\) from left to right except the first and the last one, writing 0 if the considered leaf is oriented to the right, 1 otherwise (see Figure 2). Note that the orientation of the leaves in a binary tree is determined only by its nodes so that we can omit to draw the leaves in our next graphical representations.

![Figure 2: The canopy of this binary tree is 0100101.](image)

An \(A\)-labeled binary tree \(T\) is a left (resp. right) binary search tree if for any node \(x\) labeled by \(b\), each label \(a\) of a node in the left subtree of \(x\) and each label \(c\) of a node in the right subtree of \(x\), the inequality \(a < b < c\) (resp. \(a \leq b < c\)) holds. A binary tree \(T \in BT_n\) is a decreasing binary tree if it is bijectively labeled on \(\{1, \ldots, n\}\) and, for all node \(y\) of \(T\), if \(x\) is a child of \(y\), then the label of \(x\) is smaller than the label of \(y\). The shape of a labeled binary tree is the unlabeled binary tree obtained by forgetting its labels.

### 2.4 Baxter permutations and pairs of twin binary trees

A permutation \(\sigma\) is a Baxter permutation if for any subword \(u = u_1u_2u_3u_4\) of \(\sigma\) such that the letters \(u_2\) and \(u_3\) are adjacent in \(\sigma\), \(\text{std}(u) \notin \{2413, 3142\}\). In other words, \(\sigma\) is a Baxter permutation if it avoids the generalized permutation patterns \(2 - 41 - 3\) and \(3 - 14 - 2\) (see \([3]\) for an introduction on generalized permutation patterns). For example, \(42173856\) is not a Baxter permutation; On the other hand \(436975128\) is a Baxter permutation. Let us denote by \(\mathcal{S}^B_n\) the set of Baxter permutations of size \(n\) and \(\mathcal{B} := \bigcup_{n \geq 0} \mathcal{S}^B_n\).

A pair of twin binary trees \((T_L, T_R)\) is made of two binary trees \(T_L, T_R \in BT_n\) such that the canopies of \(T_L\) and \(T_R\) are complementary, that is \(\text{cnp}(T_L)_i \neq \text{cnp}(T_R)_i\) for all \(1 \leq i \leq n - 1\). Denote by \(\mathcal{T}BT_n\)
the set of pairs of twin binary trees where each binary tree has $n$ nodes and $TBT := \bigcup_{n \geq 0} TBT_n$. In [8], Dulucq and Guibert have highlighted a bijection between Baxter permutations and pairs of twin binary trees. In the sequel, we shall make use of a very similar bijection.

3 The Baxter monoid

3.1 Definition and first properties

Recall that an equivalence relation $\equiv$ defined on $A^*$ is a congruence if for all $u, u', v, v' \in A^*$, $u \equiv u'$ and $v \equiv v'$ imply $u.v \equiv u'.v'$.

**Definition 3.1** The Baxter monoid is the quotient of the free monoid $A^*$ by the congruence $\equiv_B$ that is the transitive closure of the adjacency relations $\equiv_B$ and $\equiv_B^{-1}$ defined for $u, v \in A^*$ and $a, b, c, d \in A$ by:

\begin{align}
\text{cubad} & \equiv_B \text{cubad} \quad \text{where} \quad a \leq b < c \leq d, \\
\text{budac} & \equiv_B \text{budac} \quad \text{where} \quad a < b \leq c < d.
\end{align}

For $u \in A^*$, denote by $\hat{u}$ the $\equiv_B$-equivalence class of $u$; For example, the $\equiv_B$-equivalence class of $5273641$ is $\{5237641, 52726341, 5726341, 5762341\}$.

An equivalence relation $\equiv$ defined on $A^*$ is compatible with the restriction of alphabet intervals if for all interval $I$ of $A$ and for all $u, v \in A^*$, $u \equiv v$ implies $u|_I \equiv v|_I$.

**Proposition 3.2** The Baxter monoid is compatible with the restriction of alphabet intervals.

**Proof:** We only have to check the property on adjacency relations.

An equivalence relation $\equiv$ defined on $A^*$ is compatible with the destandardization process if for all $u, v \in A^*$, $u \equiv v$ iff $\text{std}(u) \equiv \text{std}(v)$ and $\text{eval}(u) = \text{eval}(v)$.

**Proposition 3.3** The Baxter monoid is compatible with the destandardization process.

An equivalence relation $\equiv$ defined on $A^*$ is compatible with the Schützenberger involution if for all $u, v \in A^*$, $u \equiv v$ implies $u^\# \equiv v^\#$.

**Proposition 3.4** The Baxter monoid is compatible with the Schützenberger involution.

3.2 Connection with the sylvester monoid

The sylvester monoid $[11, 12]$ is the quotient of the free monoid $A^*$ by the congruence $\equiv_S$ that is the transitive closure of the adjacency relation $\equiv_S$ defined for $u \in A^*$ and $a, b, c \in A$ by:

\[ \text{acub} \equiv_S \text{caub} \quad \text{where} \quad a \leq b < c. \]  

In the same way, let us define the $\#$-sylvester monoid by the congruence $\equiv_{S^\#}$ that is the transitive closure of the adjacency relation $\equiv_{S^\#}$ defined for $u \in A^*$ and $a, b, c \in A$ by:

\[ \text{bucac} \equiv_{S^\#} \text{buca} \quad \text{where} \quad a < b \leq c. \]

Note that this adjacency relation is defined by taking the images by the Schützenberger involution of the sylvester adjacency relation. Indeed, for all $u, v \in A^*$, $u \equiv_{S^\#} v$ iff $u^\# \equiv_S v^\#$. The Baxter monoid and the sylvestor monoid are related in the following way:

**Proposition 3.5** Let $u, v \in A^*$. Then, $u \equiv_B v$ iff $u \equiv_S v$ and $u \equiv_{S^\#} v$.

**Proposition 3.5** shows that the $\equiv_B$-equivalence classes are the intersection of $\equiv_S$-equivalence classes and $\equiv_{S^\#}$-equivalence classes.
4 A Robinson-Schensted-like algorithm

We shall describe here an insertion algorithm \( u \mapsto (P(u), Q(u)) \), such that, given a word \( u \in A^* \), it computes its \( P \)-symbol, that is a pair of \( A \)-labeled twin binary trees \( (T_L, T_R) \) where \( T_L \) (resp. \( T_R \)) is a left (resp. right) binary search tree, and its \( Q \)-symbol, a decreasing binary tree.

4.1 Definition of the insertion algorithm

Let \( T \) be an \( A \)-labeled right binary search tree and \( b \) a letter of \( A \). The lower restricted binary tree of \( T \) compared to \( b \), namely \( T \leq b \), is the right binary search tree uniquely made of the nodes \( x \) of \( T \) labeled by a letter \( a \) satisfying \( a \leq b \) and such that for all nodes \( x \) and \( y \) of \( T \leq b \), if \( x \) is ancestor of \( y \) in \( T \leq b \), then \( x \) is ancestor of \( y \) in \( T \). In the same way, we define the higher restricted binary tree of \( T \) compared to \( b \), namely \( T > b \) (see Figure 3).

![Figure 3](image)

**Figure 3:** A right binary search tree \( T, T \leq 2 \) and \( T > 2 \).

Let \( T \) be an \( A \)-labeled right binary search tree and \( a \) a letter of \( A \). The root insertion of \( a \) into \( T \) consists in modifying \( T \) so that the root of \( T \) is a new node labeled by \( a \), its left subtree is \( T \leq a \) and its right subtree is \( T > a \).

Let \( T \) be an \( A \)-labeled left (resp. right) binary search tree and \( a \) a letter of \( A \). The leaf insertion of \( a \) into \( T \) is recursively defined by: If \( T = \perp \), the result is the one-node binary tree labeled by \( a \); Else, if the label \( b \) of the root of \( T \) satisfies \( a < b \) (resp. \( a \leq b \)), make a leaf insertion of \( a \) into the left subtree of \( T \), else, make a leaf insertion of \( a \) into the right subtree of \( T \).

Given a pair of \( A \)-labeled twin binary trees \( (T_L, T_R) \) where \( T_L \) (resp. \( T_R \)) is a left (resp. right) binary search tree, the insertion of the letter \( a \) of \( A \) into \( (T_L, T_R) \) consists in making a leaf insertion of \( a \) into \( T_L \) and a root insertion of \( a \) into \( T_R \).

The \( P \)-symbol \( (T_L, T_R) \) of a word \( u \in A^* \) is computed by iteratively inserting the letters of \( u \), from left to right, into the pair of twin binary trees \( (\perp, \perp) \). The \( Q \)-symbol of \( u \) is the decreasing binary tree labeled on \( \{1, \ldots, |u|\} \), built by recording the dates of creation of each node of \( T_R \) (see Figure 4).

![Figure 4](image)

**Figure 4:** Steps of computation of the \( P \)-symbol and the \( Q \)-symbol of \( u := 5425424 \).
4.2 Validity of the insertion algorithm

**Lemma 4.1** Let $u \in A^*$. Let $T$ be the right binary search tree obtained by root insertions of the letters of $u$, from left to right. Let $T'$ be the right binary search tree obtained by leaf insertions of the letters of $u$, from right to left. Then, $T = T'$.

**Lemma 4.2** Let $\sigma \in \mathfrak{S}$ and $T \in BT_{|\sigma|}$ be the binary search tree obtained by leaf insertions of the letters of $\sigma$, from left to right. Then, for $1 \leq i \leq |\sigma| - 1$, the $i+1$-st leaf of $T$ is right-oriented iff $(i, i+1)$ is a co-inversion of $\sigma$.

If $(T_L, T_R)$ is a pair of labeled twin binary trees, define its *shape*, that is the pair of unlabeled twin binary trees $(T'_L, T'_R)$ where $T'_L$ (resp. $T'_R$) is the shape of $T_L$ (resp. $T_R$).

**Proposition 4.3** For all word $u \in A^*$, the shape of the $\mathbb{P}$-symbol of $u$ is a pair of twin binary trees.

**Proposition 4.4** Let $u, v \in A^*$. Then, $u \equiv_B v$ iff $\mathbb{P}(u) = \mathbb{P}(v)$.

In particular, we have $\mathbb{P}(\sigma) = \mathbb{P}(\nu)$ iff the permutations $\sigma$ and $\nu$ are $\equiv_B$-equivalent. Moreover, each $\equiv_B$-equivalence class of permutations can be encoded by a pair of unlabeled twin binary trees because there is one unique way to bijectively label a binary tree with $n$ nodes on $\{1, \ldots, n\}$ such that it is a binary search tree.

**Remark 4.5** Let $u, v \in A^*$ and $(T_L, T_R) := \mathbb{P}(u)$. We have $u \equiv_B v$ iff the following two assertions are satisfied:

(i) $v$ is a linear extension of $T_L$ seen as a poset in which the smallest element is its root;

(ii) $v$ is a linear extension of $T_R$ seen as a poset in which minimal elements are the nodes with no descents.

5 The Baxter lattice

5.1 Some properties of the $\equiv_B$-equivalence classes of permutations

**Theorem 5.1** For all $n \geq 0$, each equivalence class of $\mathfrak{S}_n / \equiv_B$ contains exactly one Baxter permutation.

**Proposition 5.2** For all $n \geq 0$, each equivalence class of $\mathfrak{S}_n / \equiv_B$ is an interval of the permutohedron.

For all permutation $\sigma$, let us define $\sigma \uparrow$ (resp. $\sigma \downarrow$) the maximal (resp. minimal) permutation of the $\equiv_B$-equivalence class of $\sigma$ for the permutohedron order.

**Proposition 5.3** Let $\sigma, \nu \in \mathfrak{S}_n$ such that $\sigma \leq_P \nu$. Then, $\sigma \uparrow \leq_P \nu \uparrow$ and $\sigma \downarrow \leq_P \nu \downarrow$.

5.2 A lattice structure on the set of pairs of twin binary trees

**Definition 5.4** For all $n \geq 0$, define the order relation $\leq_B$ on the set $TBT_n$ setting $J_0 \leq_B J_1$, where $J_0, J_1 \in TBT_n$, if there exists $\sigma_0, \sigma_1 \in \mathfrak{S}_n$ such that $\mathbb{P}(\sigma_0) = J_0, \mathbb{P}(\sigma_1) = J_1$ and $\sigma_0 \leq_P \sigma_1$.

Propositions 5.2 and 5.3 ensure that this order is well-defined, and in particular that the relation $\leq_B$ is transitive and antisymmetric.

The pair of twin binary trees $(T_L, T_R)$ is covered by $(T'_L, T'_R) \in TBT$ if one of the three following conditions is satisfied:

1. $T'_R = T_R$ and $T'_L$ is obtained from $T_L$ by performing a left rotation into $T_L$ such that $\text{cnp}(T_L) = \text{cnp}(T'_L)$;
2. $T'_L = T_L$ and $T'_R$ is obtained from $T_R$ by performing a right rotation into $T_R$ such that $\text{cnp}(T_R) = \text{cnp}(T'_R)$;

3. $T'_L$ (resp. $T'_R$) is obtained by performing a left (resp. right) rotation into $T_L$ (resp. $T_R$) such that $\text{cnp}(T_L) \neq \text{cnp}(T'_L)$ (resp. $\text{cnp}(T_R) \neq \text{cnp}(T'_R)$).

Moreover, it is possible to compare two pairs of twin binary trees $J_0 := (T_0^L, T_0^R)$ and $J_1 := (T_1^L, T_1^R)$ very easily by computing the Tamari vector (see [14]) of each binary tree. Indeed, we have $J_0 \leq_B J_1$ iff the Tamari vector of $T_0^L$ (resp. $T_0^R$) is greater (resp. smaller) component by component than the Tamari vector of $T_1^L$ (resp. $T_1^R$).

Propositions 5.2 and 5.3 implies that that $\equiv_B$ is also a lattice congruence [6, 22]. Thus, since the permutohedron is a lattice,

**Proposition 5.5** For all $n \geq 0$, the poset $(\text{TBT}_n, \leq_B)$ is a lattice.

### 6 The Baxter Hopf Algebra

In the sequel, all the algebraic structures have a field of characteristic zero $\mathbb{K}$ as ground field.

#### 6.1 The Hopf algebra $\text{FQSym}$

Recall that the family $\{F_{\sigma}\}_{\sigma \in S}$ form the fundamental basis of $\text{FQSym}$ [7]. Its product and its coproduct are defined by:

$$F_{\sigma} \cdot F_\nu := \sum_{\pi \in \sigma \boxplus \nu} F_\pi,$$

$$\Delta (F_\sigma) := \sum_{0 \leq i \leq |\sigma|} F_{\text{std}(\sigma_1 \ldots \sigma_i)} \otimes F_{\text{std}(\sigma_{i+1} \ldots \sigma_{|\sigma|})}. \quad (5)$$

The following theorem due to Hivert and Nzeutchap [13] shows that an equivalence relation on $A^*$ satisfying some properties can be used to define Hopf subalgebras of $\text{FQSym}$:

**Theorem 6.1** Let $\equiv$ be an equivalence relation defined on $A^*$. If $\equiv$ is a congruence, compatible with the restriction of alphabet intervals and compatible with the destandardization process, then, the family $\{P_\sigma\}_{\sigma \in A^*/\equiv}$ defined by:

$$P_\sigma := \sum_{\sigma \in \hat{\sigma}} F_\sigma \quad (6)$$

spans a Hopf subalgebra of $\text{FQSym}$.

#### 6.2 The Hopf algebra Baxter

By definition, $\equiv_B$ is a congruence, and, by Proposition 3.2 and 3.3 $\equiv_B$ checks the conditions of Theorem 6.1. Moreover, by Proposition 4.4, the $\equiv_B$-equivalence classes of permutations can be encoded by pairs of unlabeled twin binary trees. Hence, we have the following theorem:

**Theorem 6.2** The family $\{P_J\}_{J \in \text{TBT}}$ defined by:

$$P_J := \sum_{\sigma \in \Theta} F_\sigma \quad (7)$$

spans a Hopf subalgebra of $\text{FQSym}$, namely the Hopf algebra Baxter.
The Hilbert series of Baxter is $B(z) := 1 + z + 2z^2 + 6z^3 + 22z^4 + 92z^5 + 422z^6 + 2074z^7 + 10754z^8 + 58202z^9 + 326240z^{10} + 1882960z^{11} + \ldots$, the generating series of Baxter permutations (sequence A001181 of [24]).

One has for example,

\[
P_{12} = F_{12}, \quad P_{2143} = F_{2143} + F_{2413}, \quad P_{542163} = F_{542163} + F_{542613} + F_{546213}.
\]

By Theorem 6.1, the product of Baxter is well-defined. We deduce it from the product of \(\text{FQSym}\) and obtain

\[
P_{J_0} \cdot P_{J_1} = \sum_{P(\pi) = J_0, P(\nu) = J_1} P_{P(\pi)}.
\]

For example,

\[
P \cdot P = P + P + P + P.
\]

In the same way, we deduce the coproduct of Baxter from the coproduct of \(\text{FQSym}\) and obtain

\[
\Delta(P) = \sum_{\sigma := \text{std}(u), \nu := \text{std}(v) \in \Sigma^B} P_{\sigma} \otimes P_{\nu}.
\]

For example,

\[
\Delta P = 1 \otimes P + P \otimes P + P \otimes P + P \otimes P + P \otimes P.
\]

Remark 6.3 It is well-known that the Hopf algebra \(\text{PBT}\) \([18,12]\) is a Hopf subalgebra of \(\text{FQSym}\). Besides, we have the following sequence of injective Hopf maps:

\[
\text{PBT} \xrightarrow{\rho} \text{Baxter} \xrightarrow{} \text{FQSym}.
\]

Indeed, by Proposition 3.5 every \(\equiv_S\)-equivalence class is an union of some \(\equiv_B\)-equivalence classes. Denoting by \(\{P_T\}_{T \in B^T}\) the basis of \(\text{PBT}\) defined in accordance with [6] by the sylvester equivalence relation \(\equiv_S\), we have

\[
\rho(P_T) = \sum_{T' \in B^T} P_J.
\]

For example,

\[
\rho \left( P_{\text{<dot>}} \right) = P_{\text{<dot>}} + P + P.
\]
6.3 Multiplicative bases

Define the elementary family $\{E_J\}_{J \in TBT}$ and the homogeneous family $\{H_J\}_{J \in TBT}$ respectively by:

$$E_J := \sum_{J \trianglelefteq J'} P_{J'} \quad \text{and} \quad H_J := \sum_{J' \triangleright J} P_{J'}.$$  \quad (16)

These families are bases of Baxter since they are defined by triangularity.

Let $J_0 := (T_L^0, T_R^0)$ and $J_1 := (T_L^1, T_R^1)$ be two pairs of twin binary trees. Let us define the pair of twin binary trees $J_0 / J_1$ by $J_0 / J_1 := (T_L^0 / T_L^1, T_R^0 / T_R^1)$. In the same way, the pair of twin binary trees $J_0 \setminus J_1$ is defined by $J_0 \setminus J_1 := (T_L^0 \setminus T_L^1, T_R^0 \setminus T_R^1)$.

Using the multiplicative bases of $\mathbf{FQSym}$, we establish the following proposition:

**Proposition 6.4** For all $J_0, J_1 \in TBT$, we have

$$E_{J_0} \cdot E_{J_1} = E_{J_0 / J_1} \quad \text{and} \quad H_{J_0} \cdot H_{J_1} = H_{J_0 \setminus J_1}.$$  \quad (17)

**Lemma 6.5** Let $C$ be an equivalence class of $\mathfrak{S}_n / \equiv_B$. The Baxter permutation belonging to $C$ is connected if all the permutations of $C$ are connected.

Let us say that a pair of twin binary trees $J$ is connected if the unique Baxter permutation $\sigma$ satisfying $P(\sigma) = J$ is connected.

**Proposition 6.6** The Hopf algebra Baxter is free on the elements $E_J$ where $J$ is a connected pair of twin binary trees.

The generating series $B_C(z)$ of connected Baxter permutations is $B_C(z) = 1 - B(z)^{-1}$. First dimensions of algebraic generators of Baxter are 1, 1, 1, 3, 11, 47, 221, 1113, 5903, 32607, 186143, 1092015.

6.4 Bidendriform bialgebra structure

A Hopf algebra $(H, \cdot, \Delta)$ can be fitted into a bidendriform bialgebra structure if $(H^+, \cdot, \Delta_\prec, \Delta_\succ)$ is a dendriform algebra and $(H^+, \Delta_\prec, \Delta_\succ)$ a codendriform coalgebra, where $H^+$ is the augmentation ideal of $H$. The operators $\prec, \succ, \Delta_\prec$ and $\Delta_\succ$ have to fulfil some compatibility relations. In particular, for all $x, y \in H^+$, the product $\cdot$ of $H$ is retrieved by $x \cdot y = x \prec y + x \succ y$ and the coproduct $\Delta$ of $H$ is retrieved by $\Delta(x) = 1 \otimes x + \Delta_\prec(x) \otimes \Delta_\succ(x) + x \otimes 1$.

The Hopf algebra $\mathbf{FQSym}$ admits a bidendriform bialgebra structure. Indeed, for all $\sigma, \nu \in \mathfrak{S}$ set

$$F_\sigma \prec F_\nu := \sum_{\pi \in \sigma \cup \nu, \pi |_{|\sigma|} = \sigma} F_\pi, \quad F_\sigma \succ F_\nu := \sum_{\pi \in \sigma \cup \nu, \pi |_{|\nu|+1} = \nu} F_\pi.$$  \quad (18)

$$\Delta_\prec(F_\sigma) := \sum_{\sigma_1 \leq i \leq |\sigma|-1} F_{\text{std}(\sigma_1 \ldots \sigma_i)} \otimes F_{\text{std}(\sigma_{i+1} \ldots \sigma_{|\sigma|})},$$  \quad (19)

$$\Delta_\succ(F_\sigma) := \sum_{1 \leq i \leq |\sigma|-1} F_{\text{std}(\sigma_1 \ldots \sigma_i)} \otimes F_{\text{std}(\sigma_{i+1} \ldots \sigma_{|\sigma|})},$$  \quad (20)

**Proposition 6.7** If $\equiv$ is an equivalence relation defined on $A^*$ satisfying the conditions of Theorem 6.1 and additionally, for all $u, v \in A^*$, the relation $u \equiv v$ implies $u_{|u|} = v_{|v|}$, then, the family defined in (2) spans a bidendriform subbialgebra of $\mathbf{FQSym}$, and is free as an algebra, cofree as a coalgebra, self-dual, and the Lie algebra of its primitive elements is free.

The equivalence relation $\equiv_B$ satisfies the premises of Proposition 6.7 so that Baxter is free as an algebra, cofree as a coalgebra, self-dual, and the Lie algebra of its primitive elements is free.
6.5 The dual Hopf algebra $\text{Baxter}^*$

Let $\{\mathbf{P}_j^*\}_{j \in TBT}$ be the dual basis of the basis $\{\mathbf{P}_j\}_{j \in TBT}$. The Hopf algebra $\text{Baxter}^*$, dual of $\text{Baxter}$, is a quotient Hopf algebra of $\text{FQSym}^*$. More precisely,

$$\text{Baxter}^* = \text{FQSym}^*/I$$

(21)

where $I$ is the Hopf ideal of $\text{FQSym}^*$ spanned by the relations $\mathbf{F}_{\sigma}^* = \mathbf{F}_\nu^*$ whenever $\sigma \equiv B \nu$.

Let $\phi : \text{FQSym}^* \to \text{Baxter}^*$ be the canonical projection, mapping $\mathbf{F}_\sigma^*$ on $\mathbf{P}_j^*$ whenever $\mathbb{P}(\sigma) = J$. By definition, the product of $\text{Baxter}^*$ is

$$\mathbf{P}_{j_0}^* \cdot \mathbf{P}_{j_1}^* = \phi(\mathbf{F}_{\sigma}^* \cdot \mathbf{F}_\nu^*)$$

(22)

where $\sigma$ and $\nu$ are any permutations such that $\mathbb{P}(\sigma) = J_0$ and $\mathbb{P}(\nu) = J_1$. For example,

$$\mathbf{P}^* \cdot \mathbf{P}^* = \mathbf{P}^* + \mathbf{P}^* + \mathbf{P}^* + \mathbf{P}^* + \mathbf{P}^* + \mathbf{P}^* + \mathbf{P}^* + \mathbf{P}^*$$

(23)

In the same way, the coproduct of $\text{Baxter}^*$ is

$$\Delta(\mathbf{P}_j) = (\phi \otimes \phi)(\Delta(\mathbf{F}_\sigma^*))$$

(24)

where $\sigma$ is any permutation such that $\mathbb{P}(\sigma) = J$. For example,

$$\Delta \mathbf{P}^* = 1 \otimes \mathbf{P}^* + \mathbf{P}^* \otimes \mathbf{P}^* + \mathbf{P}^* \otimes \mathbf{P}^* + \mathbf{P}^* \otimes \mathbf{P}^* + \mathbf{P}^* \otimes \mathbf{P}^* + \mathbf{P}^* \otimes \mathbf{P}^* + \mathbf{P}^* \otimes \mathbf{P}^* + \mathbf{P}^* \otimes \mathbf{P}^*$$

(25)

**Remark 6.8** By Proposition 6.7, the Hopf algebras $\text{Baxter}$ and $\text{Baxter}^*$ are isomorphic. However, denoting by $\theta : \text{Baxter} \to \text{FQSym}$ the injection from $\text{Baxter}$ to $\text{FQSym}$, $\psi : \text{FQSym} \leftrightarrow \text{FQSym}^*$ the isomorphism from $\text{FQSym}$ to $\text{FQSym}^*$ defined by $\psi(\mathbf{F}_\sigma) := \mathbf{F}_{\tau}^*$, and $\phi : \text{FQSym}^* \to \text{Baxter}^*$ the surjection from $\text{FQSym}^*$ to $\text{Baxter}^*$, the map $\phi \circ \psi \circ \theta : \text{Baxter} \to \text{Baxter}^*$ is not an isomorphism. Indeed:

$$\phi \circ \psi \circ \theta \mathbf{P} = \phi \circ \psi (\mathbf{F}_{2143} + \mathbf{F}_{2413}) = \mathbf{P}^* \oplus \mathbf{P}^*$$

(26)

$$\phi \circ \psi \circ \theta \mathbf{P} = \phi \circ \psi (\mathbf{F}_{3142} + \mathbf{F}_{3412}) = \mathbf{P}^* \oplus \mathbf{P}^*$$

(27)

showing that $\phi \circ \psi \circ \theta$ is not injective.

6.6 Primitive and totally primitive elements

6.6.1 Primitive elements

Since the family $\{\mathbf{E}_j\}_{j \in C}$, where $C$ is the set of connected pairs of twin binary trees, are indecomposable elements of $\text{Baxter}$, its dual family $\{\mathbf{E}_j^*\}_{j \in C}$ forms a basis of the Lie algebra $\mathfrak{p}^*$ of the primitive elements of $\text{Baxter}^*$. By Proposition 6.7, the Lie algebra $\mathfrak{p}^*$ is free.
6.6.2 Totally primitive elements

An element $x$ of a bidendriform bialgebra is totally primitive if $\Delta_<(x) = 0 = \Delta_>(x)$.

Following [9], the generating series $B_T(z)$ of the totally primitive elements of Baxter is $B_T(z) = \frac{B(z)-1}{B(z)^2}$. First dimensions of totally primitive elements of Baxter are 0, 1, 0, 1, 4, 19, 96, 511, 2832, 16215, 95374, 573837. Here follows a basis of the totally primitive elements of Baxter of order 1, 3 and 4:

\begin{align*}
t_{1,1} & = P_{\bullet \bullet}, & (28) \\
t_{3,1} & = P_{\bullet \bullet \bullet} - P_{\bullet \bullet \circ \circ}, & (29) \\
t_{4,1} & = P_{\bullet \bullet \bullet \circ} + P_{\circ \circ \circ \circ} + P_{\circ \circ \circ \bullet} + P_{\circ \circ \bullet \bullet}, & (30) \\
& - P_{\circ \circ \circ \circ} - P_{\circ \circ \circ \bullet} - P_{\circ \circ \bullet \circ} - P_{\circ \circ \bullet \bullet}, & (31) \\
t_{4,2} & = P_{\bullet \bullet \bullet \circ} - P_{\circ \circ \bullet \bullet}, & (32) \\
t_{4,3} & = P_{\bullet \bullet \bullet \circ} - P_{\circ \circ \bullet \bullet}, & (33)
\end{align*}

Baxter is free as dendriform algebra on its totally primitive elements.

References


