

# Bumping algorithm for set-valued shifted tableaux

Takeshi IKEDA<sup>1†</sup>, Hiroshi NARUSE<sup>2</sup> and Yasuhide NUMATA<sup>3,4</sup>

<sup>1</sup>*Department of Applied Mathematics, Okayama University of Science, Ridai-cho, Kita-ku, Okayama, Japan*

<sup>2</sup>*Graduate School of Education, Okayama University, Tsushima-naka, Kita-ku, Okayama, Japan*

<sup>3</sup>*Department of Mathematical Informatics, The University of Tokyo, Hongo, Bunkyo-ku, Tokyo, Japan*

<sup>4</sup>*Japan Science and Technology (JST) CREST*

**Abstract.** We present an insertion algorithm of Robinson–Schensted type that applies to set-valued shifted Young tableaux. Our algorithm is a generalization of both set-valued non-shifted tableaux by Buch and non set-valued shifted tableaux by Worley and Sagan. As an application, we obtain a Pieri rule for a  $K$ -theoretic analogue of the Schur  $Q$ -functions.

**Résumé** Nous présentons un algorithme d’insertion de Robinson–Schensted qui s’applique aux tableaux décalés à valeurs sur des ensembles. Notre algorithme est une généralisation de l’algorithme de Buch pour les tableaux à valeurs sur des ensembles et de l’algorithme de Worley et Sagan pour les tableaux décalés. Comme application, nous obtenons une formule de Pieri pour un analogue en  $K$ -théorie des  $Q$ -fonctions de Schur.

**Keywords:** set-valued shifted tableaux, insertion, Robinson–Schensted, Pieri rule,  $K$ -theory, Schur  $Q$ -functions

## 1 Introduction

This article is an extended abstract of the paper [INN] of the same title. Most details of the proofs are omitted.

In [IN], we introduced a non-homogeneous ( $K$ -theoretic) analogue of Schur  $Q$ -functions. These functions are labeled by strict partitions (or shifted Young diagrams), as are the original  $Q$ -functions. For a strict partition  $\lambda$ , the corresponding  $K$ -theoretic Schur  $Q$ -function  $GQ_\lambda(x)$  can be expressed as a weighted generating function of *shifted set-valued semistandard tableaux* of shape  $\lambda$ , which are the central concern of this article.

The main result of the paper is a Robinson–Schensted type insertion algorithm for the shifted set-valued tableaux (Thm 3.4). Our algorithm is a generalization of both set-valued non-shifted tableaux by Buch [Bu] and non set-valued shifted tableaux by Worley [Wo] and Sagan [Sa]. As an immediate consequence of our algorithm, we have a Pieri rule for  $GQ_\lambda(x)$  (Cor. 3.5).

The original purpose for introducing functions  $GQ_\lambda(x)$  was to apply them to Schubert calculus. In [IN] we introduced function  $GQ_\lambda(x|b)$  (resp.  $GP_\lambda(x|b)$ ) with the *equivariant* parameter  $b = (b_1, b_2, \dots)$ ,

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which represents the structure sheaf of the Schubert variety indexed by  $\lambda$  in the  $K$ -ring of  $T$ -equivariant coherent sheaves on Lagrangian (resp. orthogonal) Grassmannian, where  $T$  is the maximal torus acting on the Grassmannians. Thus our Pieri rule gives an explicit description of  $K$ -theoretic Schubert structure constant for an arbitrary Schubert class times a special (one row type) Schubert class in the  $K$ -ring of Lagrangian Grassmannian.

Recently, a  $K$ -theoretic Littlewood-Richardson rule in terms of the *jeu de taquin* for odd orthogonal Grassmannians of maximal isotropic subspaces has been obtained by Clifford, Thomas and Yong [CTY]. Their method starts from a Pieri rule for the  $K$ -theory by Buch and Ravikumar [BR], which applies to cominuscule Grassmannians. Our approach differs from them substantially. We proceeded independently a different approach of tableaux insertion to result in the same formula as [BR], i.e. the counting of KLG-tableaux. But our method is only applicable to the case of Lagrangian Grassmannians, although there is a set valued tableaux description for  $GP_\lambda(x)$ .

Organization of the paper is as follows. In Section 2, we give the definition of shifted set-valued tableaux, and  $K$ -theoretic Schur  $Q$ -functions  $GQ_\lambda(x)$ . In Section 3, we present our main result, an existence of a Robinson-Schensted type bijection for set-valued shifted tableaux. As a corollary, we have a Pieri rule for  $GQ_\lambda(x)$ . Precise description of the bijection is given by a bumping algorithm which is given in Section 4. In Section 5, we discuss a variant of the bijection, which is analogous to the results by Sagan and Worley. In Section 6, we give an outline of the proof of the main theorem.

## 2 Shifted Young diagrams, set-valued tableaux

### 2.1 Shifted Young diagrams

Let  $\Delta$  denote the set  $\{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq j\}$ . Any element  $\alpha = (i, j)$  is called a *box*. If  $i = j$ , then  $(i, j)$  is called a *diagonal box*. A *shifted Young diagram* is any finite subset  $\lambda$  of  $\Delta$  such that for each  $\alpha = (i, j) \in \lambda$ , any box  $\beta = (i', j') \in \Delta$  satisfying  $i' \leq i$  and  $j' \leq j$  belongs to  $\lambda$ .

We define  $\mathbb{S}$  to be the set of shifted Young diagrams. For  $\lambda \in \mathbb{S}$ , we define  $|\lambda|$  to be the number of boxes in  $\lambda$ . For  $\lambda, \mu \in \mathbb{S}$  such that  $\lambda \subset \mu$ , we define the skew shifted Young diagram  $\mu/\lambda$  to be the set-theoretic difference  $\mu - \lambda$ .

Let  $\alpha = (i, j), \beta = (i', j') \in \Delta$ . We say that  $\alpha$  is *weakly below* (resp. *weakly right of*)  $\beta$  if  $i \geq i'$  (resp.  $j \geq j'$ ). We say that  $\alpha$  is *strictly below* (resp. *strictly right of*)  $\beta$  if  $i > i'$  (resp.  $j > j'$ ). We say that  $\alpha$  is *directly below* (resp. *directly right of*)  $\beta$  if  $i = i' + 1$  and  $j = j'$  (resp.  $i = i'$  and  $j = j' + 1$ ).

We call a skew shifted diagram  $\theta$  a *horizontal strip* (resp. *vertical strip*) if  $\theta$  has no pair of boxes in the same column (resp. row). We call  $\theta$  a *broken border strip* if  $\theta$  contains no  $2 \times 2$  square block.

### 2.2 Tableaux

Define a totally ordered set  $\mathcal{B}$  to be disjoint union of sets  $\mathcal{A} = \{1, 2, \dots\}$  and  $\mathcal{A}' = \{1', 2', \dots\}$  with the following order:

$$1' < 1 < 2' < 2 < \dots$$

We define binary relations  $\leq_r$  and  $\leq_c$  on  $\mathcal{B}$  by

$$x \leq_r y \iff x = y \in \mathcal{A} \text{ or } x < y, \quad x \leq_c y \iff x = y \in \mathcal{A}' \text{ or } x < y.$$

Note that  $x \not\leq_r y$  (resp.  $x \not\leq_c y$ ) is equivalent to  $y \leq_c x$  (resp.  $y \leq_r x$ ) for any  $x, y \in \mathcal{B}$ .

Let  $\mathcal{X}$  denote the set of non-empty finite subsets of  $\mathcal{B}$ . We extend the relations  $\leq_r, \leq_c$  on  $\mathcal{X}$  by  $A \leq_r B \iff \max A \leq_r \min B$  and  $A \leq_c B \iff \max A \leq_c \min B$  for  $A, B \in \mathcal{X}$ .

**Definition 2.1 (Shifted set-valued semistandard tableaux)** Let  $\lambda$  be a shifted Young diagram. A set-valued semistandard tableau of shape  $\lambda$  is a map  $T$  from the set of boxes in  $\lambda$  to  $\mathcal{X}$  satisfying the following “semistandardness”:

1.  $T(\alpha) \leq_r T(\beta)$  if  $\beta \in \lambda$  is directly right of  $\alpha \in \lambda$ .
2.  $T(\alpha) \leq_c T(\beta)$  if  $\beta \in \lambda$  is directly below  $\alpha \in \lambda$ .

**Example 2.2** An example of a set-valued tableau is given by the following:

$$T = \begin{array}{|c|c|c|c|} \hline 1' & 12' & 23 & 34' \\ \hline & 2' & 4' & 6 \\ \hline & & 6 & \\ \hline \end{array}.$$

We denote by  $\mathcal{T}(\lambda)$  the set of all set-valued tableaux of shape  $\lambda$ .

### 2.3 $K$ -theoretic $Q$ -Schur functions

Let  $x = (x_1, x_2, \dots)$  be a sequence of variables. Let  $\lambda \in \mathbb{S}$  and  $T \in \mathcal{T}(\lambda)$ . We define the corresponding monomial  $x^T = \prod_{i=1}^{\infty} x_i^{e_i(T)}$  where  $e_i(T)$  denotes the total number of  $i$  and  $i'$  appearing in  $T$ . The weight of  $T \in \mathcal{T}(\lambda)$  is defined to be  $\beta^{|T|-|\lambda|} x^T$ , where  $\beta$  is a formal parameter and  $|T|$  is the total number of letters in  $T$ . The  $K$ -theoretic  $Q$ -Schur function  $GQ_\lambda(x)$  is defined as the following formal sum of the weights of the elements in  $\mathcal{T}(\lambda)$ :

$$GQ_\lambda(x) = \sum_{T \in \mathcal{T}(\lambda)} \beta^{|T|-|\lambda|} x^T.$$

When  $\beta = 0$  this becomes the Schur  $Q$ -function  $Q_\lambda(x)$ , and when  $\beta = -1$  this represents  $K$ -theory Schubert class corresponding to  $\lambda$  for Lagrangian Grassmannians. See [IN] for other expressions of  $GQ_\lambda(x)$  and geometric background.

## 3 Statements of main results

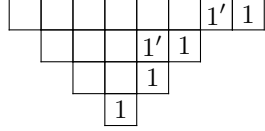
### 3.1 Admissible strips

Let  $\theta = \lambda/\mu$  be a broken border strip. We consider a decomposition  $\theta = C \sqcup C'$ , with  $C, C'$  skew diagrams, i.e. there is a diagram  $\nu$  satisfying  $\mu \subset \nu \subset \lambda$  and  $C = \lambda/\nu$  and  $C' = \nu/\mu$ . Such a decomposition of  $\theta$  is called *admissible* if the following conditions are satisfied:

1. in each of the diagrams  $C$  and  $C'$ , there is no pair of boxes in the same row or column.
2. there is no diagonal box in  $C'$ .

A non-empty broken border strip  $\theta$  is called a *1-admissible strip* if there exists an admissible decomposition of  $\theta$ . For a 1-admissible strip  $\theta$ , we denote by  $\mathcal{C}(\theta)$  the set of all admissible decompositions of  $\theta$ . Later we define the notion of  $m$ -admissible decomposition of a broken border strip.

**Example 3.1** The following is an example of a 1-admissible strip and its 1-admissible decomposition,



where the boxes with entry 1's form  $C$  and 1's form  $C'$ .

The next result shows the role of 1-admissible strip. The detailed construction of the map is given in Section 4. We define the weight of a 1-admissible strip  $\theta$  to be  $\beta^{|\theta|-1}$ .

**Proposition 3.2** *There is a weight preserving bijection:*

$$\phi : \mathcal{T}(\lambda) \times \mathcal{X} \longrightarrow \bigsqcup_{\mu} \mathcal{T}(\mu) \times \mathcal{C}(\mu/\lambda)$$

where  $\mu \in \mathbb{S}$  runs for those  $\mu$  such that  $\mu/\lambda$  is a 1-admissible strip.

### 3.2 Composable admissible strips

Let  $\lambda, \mu, \nu \in \mathbb{S}$  be such that  $\mu \subset \nu \subset \lambda$ . Suppose  $\theta_1 = \nu/\mu, \theta_2 = \lambda/\nu$  are 1-admissible strips. Let  $(C'_i, C_i) \in \mathcal{C}(\theta_i)$  ( $i = 1, 2$ ). We say that  $(C'_1, C_1)$  precedes  $(C'_2, C_2)$  and denote  $(C'_1, C_1) \triangleleft (C'_2, C_2)$ , if the following conditions are satisfied:

1.  $C'_1 \cup C'_2$  is a vertical strip.
2.  $C_1 \cup C_2$  is a horizontal strip.
3. Each box in  $C'_2$  is strictly below any box in  $C'_1$ .
4. Each box in  $C_2$  is strictly right of any box in  $C_1$ .
5. If  $C_1 \neq \emptyset$ , then  $C'_2 = \emptyset$ .

### 3.3 Main results

Let  $\theta = \mu/\lambda$  be a broken border strip, and  $m$  be a positive integer. Suppose there is a nested sequence of shifted diagrams

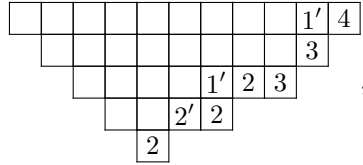
$$\lambda = \nu^{(0)} \subset \nu^{(1)} \subset \nu^{(2)} \subset \dots \subset \nu^{(m)} = \mu \tag{1}$$

such that  $\theta^{(i)} := \nu^{(i)}/\nu^{(i-1)}$  ( $1 \leq i \leq m$ ) are 1-admissible strips. If, moreover, there is a sequence of 1-admissible decompositions  $(C'_i, C_i) \in \mathcal{C}(\theta^{(i)})$  ( $1 \leq i \leq m$ ) such that

$$(C'_i, C_i) \triangleleft (C'_{i+1}, C_{i+1}), \quad (1 \leq i \leq m - 1). \tag{2}$$

then we say  $\theta$  is an  $m$ -admissible strip. For an  $m$ -admissible strip  $\theta$ , let  $\mathcal{C}_m(\theta)$  denote the set of pairs  $(\{\nu^{(i)}\}_{i=1}^m, \{(C'_i, C_i)\}_{i=1}^m)$  satisfying the above conditions, which we call  $m$ -admissible decompositions of  $\theta$ . Note  $\mathcal{C}_1(\theta) = \mathcal{C}(\theta)$  since condition (2) is vacant for  $m = 1$ .

**Example 3.3** The following is a 4-admissible strip



where the boxes with entry  $i$  are  $C_i$ , and  $i'$  are  $C'_i$ .

We denote by  $(m)$  the shifted diagram consisting of one row with  $m$  boxes. We simply denote  $\mathcal{T}(m)$  for  $\mathcal{T}((m))$ . Recall that we define the weight of  $T \in \mathcal{T}(\lambda)$  as  $\beta^{|T|-|\lambda|}x^T$ . Define the weight of  $U \in \mathcal{C}_m(\theta)$  to be  $\beta^{|\theta|-m}$ .

**Theorem 3.4** By algorithm 4.4, we have a weight preserving bijection:

$$\phi_m : \mathcal{T}(\lambda) \times \mathcal{T}(m) \longrightarrow \bigsqcup_{\mu} \mathcal{T}(\mu) \times \mathcal{C}_m(\mu/\lambda), \tag{3}$$

where  $\mu$  runs for shifted diagrams  $\mu$  such that  $\mu \supset \lambda$  and  $\mu/\lambda$  are  $m$ -admissible strips.

As an immediate consequence, we have the following.

**Corollary 3.5 (Pieri rule)** We have

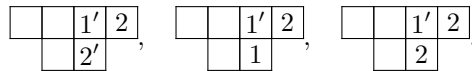
$$GQ_{\lambda}(x) \cdot GQ_m(x) = \sum_{\mu \supset \lambda} \beta^{|\mu|-|\lambda|-m} \#\mathcal{C}_m(\mu/\lambda) \times GQ_{\mu}(x),$$

where  $\mu$  runs for shifted diagrams  $\mu$  such that  $\mu \supset \lambda$  and  $\mu/\lambda$  are  $m$ -admissible strips.

For example we have

$$GQ_{2,1} \cdot GQ_2 = 2GQ_{4,1} + 2GQ_{3,2} + 3\beta GQ_{4,2} + \beta GQ_{5,1} + \beta GQ_{3,2,1} + \beta^2 GQ_{5,2} + \beta^2 GQ_{4,2,1}.$$

In order to give the coefficient of  $GQ_{4,2}$ , we count the elements in  $\mathcal{C}_2(\mu/\lambda)$  with  $\mu = (4, 2)$ ,  $\lambda = (2, 1)$  :



N.B. The elements in  $\mathcal{C}_m(\mu/\lambda)$  are exactly the  $KL\mathcal{G}$ -tableaux of shape  $\mu/\lambda$  with content  $\{1, 2, \dots, m\}$  in [BR].

## 4 Bumping algorithm

The aim of this section is to describe the bijection of Prop 3.2.

The input of our algorithm is a pair  $(T, w)$  with  $T \in \mathcal{T}(\lambda)$  for some  $\lambda \in \mathbb{S}$  and  $w \in \mathcal{X}$ . Basic output is a tableau  $T'$  of some shape  $\mu \in \mathbb{S}$  such that  $\mu \supset \lambda$ . The skew diagram  $\theta = \mu/\lambda$ , the set of “new boxes”, turns out to be a 1-admissible strip. We also have some “recording data” on  $\theta$  which gives an element of  $\mathcal{C}(\theta)$ .

### 4.1 Parts of “L” shape of a tableau

Let  $\lambda \in \mathbb{S}$ . Let  $\ell(\lambda)$  be the number of rows of  $\lambda$ . For  $1 \leq t \leq \lambda_1$  we define a subset of  $\lambda$  by

$$L_t(\lambda) = \{(i, j) \in \lambda \mid i = t \text{ or } j = t\}.$$

For example,  $L_1(\lambda)$  consists of the boxes in the first row. For  $k \geq \ell(\lambda)$ ,  $L_k(\lambda)$  is just the  $k$ -th column. In general, this is a subset of shape “L” including the diagonal box  $(t, t)$ . Let  $T \in \mathcal{T}(\lambda)$ . By restriction we have a map  $L_t(T) : L_t(\lambda) \rightarrow \mathcal{X}$ , which we call the  $t$ -th part of  $T$ .

Our algorithm starts from inserting  $w = w^{(0)} \in \mathcal{X}$  into  $L_1 = L_1(T)$ , the first row of  $T$ , resulting a row  $L'_1$  with possibly a new box at the right end, and a set  $w^{(1)} \in \mathcal{X}$  “bumped out” from the procedure. Then we modify the original tableau  $T = T^{(0)}$  by replacing  $L_1$  with  $L'_1$  to obtain  $T^{(1)}$ . Next we insert  $w^{(1)}$  into the second part of the modified tableau  $T^{(2)}$ . We repeat this procedure until no boxes are bumped out.

### 4.2 Insertion into a part of “L” shape (a rough idea)

We define a procedure to insert some sets  $w \in \mathcal{X}$  into an L part  $X$  of a tableaux.

Here we present a rough idea of constructing the procedure. First, we look at the minimum letters of each boxes in order to decide the box into which a letter in  $w$  to be inserted, in the same manner as the classical bumping procedure (some letters go into empty box at the end). If we might simply insert these letters into  $X$ , some letters in  $w$  may violate the semistandardness, while some letters are not. So we eject some element in  $X$  before inserting  $w$ . Let  $\hat{w}$  be the set of letters in  $w$  which do not conflict any original letters in  $X$ , and let  $\check{w} := w - \hat{w}$  be the complement. If  $\check{w} \neq \emptyset$ , let  $\check{u}$  be the set of elements in  $X$  that conflict some element in  $\check{w}$ . To ensure the semistandardness, we first eject the elements in  $\check{u}$  from the tableau. Furthermore, if a letter in  $\hat{w}$  is inserted into a non-empty box, we eject all the remaining (original) entries of the box. Thus any letter inserted into a non-empty box “does some work” (bumps out at least one letter). This feature is important for constructing the inverse algorithm.

There is a flaw in this idea. For example, we consider a tableau  $T = \begin{bmatrix} 1' \end{bmatrix}$  and  $w = w^{(1)} = \{1'\}$ . According to the naive algorithm above, the resulting tableau is  $T^{(1)} = \begin{bmatrix} 1' \\ 1' \end{bmatrix}$ , and the ejected set is  $w^{(2)} = \{1'\}$ . Since the second part is empty, the final result is  $\begin{bmatrix} 1' \\ 1' \end{bmatrix}$ , which is not semistandard. This is a reason why we need the “unmark” process introduced in the next section. In fact, we should care for the case of inserting elements into the diagonal boxes.

### 4.3 Insertion into a diagonal box

Let  $X \in \mathcal{X}$ , and  $u$  be a subset of  $X$ . We insert  $w \in \mathcal{X}$  into  $X$ , where we consider  $X$  to be a diagonal box.

#### Algorithm 4.1 (Bumping for a diagonal box)

**input**  $X, w, u \in \mathcal{X}$  satisfying  $u \subset X$  and  $\max w \leq_c \min X$ .

**output**  $Y, v$ .

**procedure**

1. If  $X \neq u$ , then let  $Y = (X - u) \cup w$  and  $v = u$ ; and return  $Y, v$ .
2. If  $i' = \max(w) \in \mathcal{A}'$  and  $i \in X$ ,  $i' \notin X$ , then let  $Y = \{i\} \cup (w - \{i'\})$  and  $v = X$ ; and return  $Y, v$ .

3. If  $i' = \max(w) \in \mathcal{A}'$  and  $i' \in X, i \notin X$ , then let  $Y = w$  and  $v = \{i\} \cup (X - \{i'\})$ ; and return  $Y, v$ .
4. If  $i' = \max(w) \in \mathcal{A}'$  and  $i, i' \in X$ , then let  $Y = \{i\} \cup w$  and  $v = X - \{i'\}$ ; and return  $Y, v$ .
5. Otherwise, let  $Y = w$  and  $v = X$ ; and return  $Y, v$ .

For example, if  $u = X = \overline{34}$  and  $w = 13'$ , then we apply (2) to obtain  $Y = \overline{13}$  rather than  $\overline{13'}$ , and  $u = 34$ . Thus letter  $3'$  is unprimed to be 3 in  $u$ . If  $u = X = \overline{3'4}$  and  $w = 13'$ , then we apply (3) to obtain  $Y = \overline{13'}$  and  $u = 34$ , rather than  $u = 3'4$ . In this case, two  $3'$  are involved, and one may think of this process as unpriming “bigger”  $3'$ . Case (4) is a bit strange. If  $u = X = \overline{3'3}$  and  $w = 3'$ , then we have  $Y = \overline{3'3}$  and  $u = 3$ . This case we are unpriming “bigger”  $3'$  also, and let it remain in the box.

#### 4.4 Insertion into a part of “L” shape (definition)

Let  $T$  be a tableau of shape  $\lambda$ , and  $t$  be a positive integer such that  $t \leq \lambda_1$ . Let  $X = L_t(T)$  be the  $t$ -th part of  $T$ . If  $t = 1$ , then  $X$  is a row:  $X = (X_{(1,1)} \leq_r X_{(1,2)} \leq_r \dots \leq_r X_{(1,\lambda_1)})$ . If  $t > \ell(\lambda)$  then  $X$  is a column:  $X = (X_{(1,t)} \leq_c \dots \leq_c X_{(k,t)})$  for some  $k < t$ . We say that  $X$  is a *pure column* in this case (note that  $X$  does not contain diagonal box). If  $1 < t \leq \ell(\lambda)$  then  $X = L_t(T)$  is a sequence of elements in  $\mathcal{X}$ :

$$X = (X_{(1,t)} \leq_c \dots \leq_c X_{(t-1,t)} \leq_c X_{(t,t)} \leq_r X_{(t,t+1)} \leq_r \dots \leq_r X_{(t,t+\lambda_t-1)}).$$

The following algorithm takes as an input a sequence of elements in  $\mathcal{X}$  satisfying

$$X = (X_{-k} \leq_c \dots \leq_c X_{-1} \leq_c X_0 \leq_r X_1 \leq_r \dots \leq_r X_l),$$

for some  $k, l \geq 0$ , and  $w \in \mathcal{X}$ . If  $k = 0$ , we consider  $X$  as a row. Output is a triple  $(Y, Y_+, v)$ , where  $Y$  is a sequence  $Y = (Y_i)_{i=-k}^l$  satisfying the same condition as  $X$ , and  $Y_+, v \in \mathcal{X} \cup \emptyset$ . If  $Y_+ \neq \emptyset$  we will make a new box with entry  $Y_+$  at the right end of  $Y$ .

#### Algorithm 4.2 (Bumping rule for an L part)

**input**  $X = (X_i)_{i=-k}^l$ : tableau of  $L$  shape, i.e.

$$X = (X_{-k} \leq_c \dots \leq_c X_{-1} \leq_c X_0 \leq_r X_1 \leq_r \dots \leq_r X_l),$$

and  $w \in \mathcal{X}$ .

**output**  $Y$  tableau of  $L$  shape of the same length of  $X$ , and  $Y_+, v \in \mathcal{X} \cup \emptyset$ .

#### procedure

1. Define the subsets  $w_{-k}, \dots, w_{l+1}$  of  $w$  by

$$w_t = \begin{cases} \{x \in w \mid x \leq_r \min X_{-k}\} & (t = -k) \\ \{x \in w \mid \min X_{t-1} \leq_c x \leq_r \min X_t\} & (t = -k, \dots, -1) \\ \{x \in w \mid \min X_{-1} \leq_c x \leq_c \min X_0\} & (t = 0) \\ \{x \in w \mid \min X_{t-1} \leq_r x \leq_c \min X_t\} & (t = 1, \dots, l) \\ \{x \in w \mid \min X_l \leq_r x\} & (t = l + 1) \end{cases}$$

2. Decompose  $w_t$  into the subsets  $\check{w}_t$  and  $\hat{w}_t$  defined by

$$\hat{w}_t = \begin{cases} w_t & (t = -k) \\ \{x \in w_t \mid \max X_{t-1} \leq_c x\} & (t = -k + 1, \dots, 0), \\ \{x \in w_t \mid \max X_{t-1} \leq_r x\} & (t = 1, \dots, l + 1) \end{cases}$$

$$\check{w}_t = w_t - \hat{w}_t, \text{ for } t = -k, \dots, l + 1,$$

3. Define  $\check{u}_t$ ,  $\hat{u}_k$ , and  $u_k$  ( $t = -k, \dots, l$ ) by:

$$\check{u}_t = \begin{cases} \emptyset & (\text{if } \check{w}_{t+1} = \emptyset) \\ \{y \in X_t \mid y \not\leq_c \min \check{w}_{t+1}\} & (\text{if } t = -k, \dots, -1 \text{ and } \check{w}_{t+1} \neq \emptyset), \\ \{y \in X_t \mid y \not\leq_r \min \check{w}_{t+1}\} & (\text{if } t = 0, \dots, l \text{ and } \check{w}_{t+1} \neq \emptyset) \end{cases}$$

$$\hat{u}_t = \begin{cases} \emptyset & (\text{if } \hat{w}_t = \emptyset) \\ X_t - \check{u}_t & (\text{if } \hat{w}_t \neq \emptyset) \end{cases}$$

$$u_t = \hat{u}_t \cup \check{u}_t \subset X_t.$$

4. Define  $Y_t = (X_t - u_t) \cup w_t$  and  $v_t = u_t$  for  $t \neq 0$ .

5. Let  $(Y_0, v_0)$  be the pair obtained from the triple  $(X_0, w_0, u_0)$  by Algorithm 4.1 if  $l \geq 0$ .

6. Let  $Y = (Y_{-k}, \dots, Y_l)$ ,  $Y_+ = w_{l+1}$ , and  $v = \bigcup_{t=-k}^l v_t$ ; and return  $Y, Y_+, v$ .

**Example 4.3** Let  $X = (X_{-2}, X_{-1}; X_0; X_1, X_2, X_3)$  be

$$\boxed{13 \mid 4' \mid \mathbf{5} \mid 56 \mid 8 \mid 9}.$$

Let us insert  $w = 25'6'79'9 \in \mathcal{X}$  into  $X$ . Since the minimums in  $X$  is

$$\boxed{1 \mid 4' \mid \mathbf{5} \mid 5 \mid 8 \mid 9},$$

we have  $(w_{-2}, \dots, w_4) = (\emptyset, 2, 5', \emptyset, 6'7, 9', 9)$ . Since the maximums of  $X$  is

$$\boxed{3 \mid 4' \mid \mathbf{5} \mid 6 \mid 8 \mid 9},$$

we have

$t$	-2	-1	0	1	2	3	4
$\hat{w}_t$	$\emptyset$	$\emptyset$	$5'$	$\emptyset$	$7$	$9'$	$9$
$\check{w}_t$	$\emptyset$	$2$	$\emptyset$	$\emptyset$	$6'$	$\emptyset$	$\emptyset$
$\hat{u}_t$	$\emptyset$	$\emptyset$	$5$	$\emptyset$	$8$	$9$	-
$\check{u}_t$	$3$	$\emptyset$	$\emptyset$	$6$	$\emptyset$	$\emptyset$	-

Finally we get

$$Y = \boxed{1 \mid 24' \mid 5 \mid 5 \mid 6'7 \mid 9'}, Y_+ = \{9\}, u = \{3, 5, 6, 8, 9\}.$$

We need to define the bumping algorithm applicable also when  $X = L_t(T)$  is a pure column case, i.e.  $t > \ell(\lambda)$ . However, extension of the algorithm to the column case is straightforward, so we omit detailed description here.



### 4.5 Insertion of $w$ into arbitrary tableau

We define a procedure to insert an element  $w \in \mathcal{X}$  into an arbitrary tableau  $T$ . In the procedure, we insert  $w$  into the first L part of the tableaux. When some letters are bumped out, we insert them into the second L part of the tableaux. Then, while some letters are bumped out, we try to insert them into the next L part of the tableaux until no letters are bumped out.

**Algorithm 4.4**

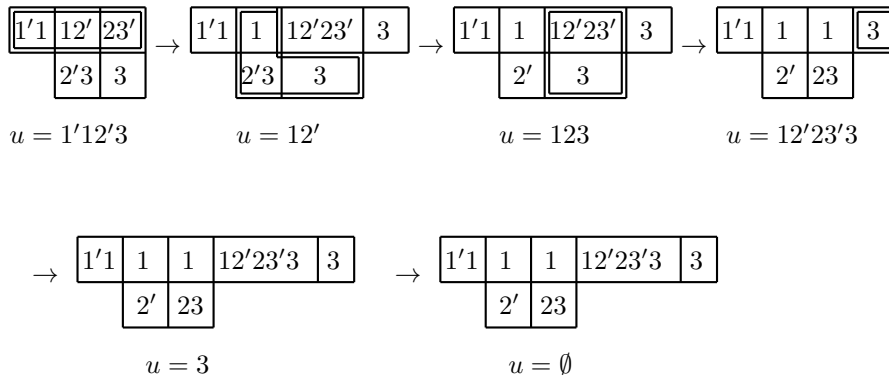
**input**  $T \in \mathcal{T}(\lambda)$  and  $w \in \mathcal{X}$ .

**output**  $U, S', S$ .

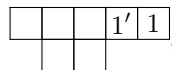
**procedure**

1. Let  $u = w, U = T, S = \emptyset$  and  $S' = \emptyset$ .
2. While  $u \neq \emptyset$ , do the following:
  - (a) Let  $X$  be the  $t$ -th L part of  $U$ ,
  - (b) Let  $(Y, Y_+, u)$  be the triple obtained from  $(X, u)$  by Algorithm 4.2.
  - (c) Let  $U$  be the tableaux obtained from  $U$  by replacing the  $t$ -th L part by  $Y$ .
  - (d) If  $Y_+ \neq \emptyset$ , then do the following:
    - i. Add a new box to the end of  $t$ -th L part of  $U$ , and insert  $Y_+$  into the box.
    - ii. If  $X$  is a *pure column*, then add the new box to  $S$ , else add the new box to  $S'$ .
3. Return  $U, S'$  and  $S$ .

**Example 4.5** Let  $T$  be the leftmost tableau below. We insert  $w = \{1', 1, 2', 3\}$  into  $T$  as follows.



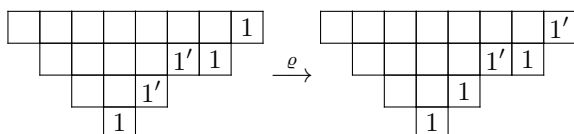
For each step, the relevant part of modification is enclosed. Sets  $S'$  and  $S$  are as follows:



where the box with entry  $1'$  (resp.  $1$ ) is  $S'$  (resp.  $S$ ).

### 4.6 Definition of the map $\phi$

In order to complete the description of the map  $\phi$ , we need one more combinatorial idea. Let  $\theta$  be a 1-admissible strip. We define an involution  $\varrho : \mathcal{C}(\theta) \rightarrow \mathcal{C}(\theta)$ . A box  $\alpha \in \theta$  is said to be *isolated* if  $\alpha$  is not a diagonal box and there is no other box than  $\alpha$  in the row and column where  $\alpha$  presents. For each isolated box, apply its entry the obvious involution  $1 \mapsto 1', 1' \mapsto 1$ , while the non-isolated boxes are untouched. The resulting decomposition of  $\theta$  is obviously admissible. For example, we have



It is obvious that  $\varrho$  is an involution.

**Proposition 4.6** *Let  $\lambda \in \mathbb{S}$ ,  $T \in \mathcal{T}(\lambda)$ , and  $w \in \mathcal{X} = \mathcal{T}(1)$ . We have by Algorithm 4.4 a tableau  $U = (T \leftarrow w) \in \mathcal{T}(\mu)$  for some  $\mu \in \mathbb{S}$  such that  $\mu \supset \lambda$  and a decomposition  $(S', S)$  of  $\theta = \mu/\lambda$ . We have  $(S', S) \in \mathcal{C}(\theta)$ , and therefore  $\theta$  is a 1-admissible strip.*

Let  $T \in \mathcal{T}(\lambda)$  and  $w \in \mathcal{X}$  as in the above proposition. We define  $\phi(T, w)$  to be  $(U, \varrho(S', S)) \in \mathcal{T}(\mu) \times \mathcal{C}(\mu/\lambda)$ .

### 4.7 Proof of Prop. 3.2

To show that  $\phi$  is a bijection, we construct its inverse map. See [INN] for details.

## 5 Robinson–Schensted type correspondence

### 5.1 Quasi-standard tableaux

We will define a notion of “recording” tableaux in our setting. The resulting object is an analogue of a standard tableau, which we will call a *quasi-standard* tableau.

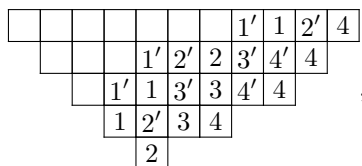
For  $T \in \mathcal{T}(\lambda)$  and  $w \in \mathcal{X}$  we denote by  $T \leftarrow w$  the tableau given in Prop. 3.2. Let  $T \in \mathcal{T}(\lambda)$  and  $(w_1, \dots, w_m) \in \mathcal{X}^m$ . By the consecutive insertions

$$T^{(i)} = (\dots((T \leftarrow w_1) \leftarrow w_2) \dots \leftarrow w_i)$$

we have a tableaux  $T^{(i)} \in \mathcal{T}(\nu^{(i)})$  for some shifted diagram  $\nu^{(i)}$  and an element of  $\mathcal{C}(\nu^{(i)}/\nu^{(i-1)})$  given by Proposition 3.2. Thus we have a nested sequence of shifted diagrams

$$\lambda = \nu^{(0)} \subset \nu^{(1)} \subset \nu^{(2)} \subset \dots \subset \nu^{(m)} = \mu, \tag{4}$$

and also 1-admissible decompositions  $(C'_i, C_i)$  of  $\theta^{(i)} = \nu^{(i)}/\nu^{(i-1)}$ . These objects are expressed as a tableau like



where the boxes filled with  $i$  (resp.  $i'$ ) are  $C_i$  (resp.  $C'_i$ ).

We call such a tableau a *quasi-standard* tableau of degree  $m$ . The precise definition is the following.

**Definition 5.1** A map  $U : \mu/\lambda \rightarrow \mathcal{B}_m := \{1', 1, \dots, m', m\}$  is a *quasi-standard tableau of degree  $m$* , if  $U$  is semistandard in the sense of Def. 2.1 and for any  $1 \leq i \leq m$ ,  $U^{-1}(\{i, i'\})$  is a 1-admissible strip with admissible decomposition given by  $(U^{-1}(i'), U^{-1}(i))$ .

Let  $\mathcal{S}_m(\mu/\lambda)$  denote the set of quasi-standard tableaux of degree  $m$  on  $\mu/\lambda$ .

*Remark.* By the construction,  $\mathcal{S}_1(\mu/\lambda)$  is non-empty if and only if  $\theta = \mu/\lambda$  is an 1-admissible strip. Then we have  $\mathcal{S}_1(\theta) = \mathcal{C}(\theta) = \mathcal{C}_1(\theta)$ . For an  $m$ -admissible strip  $\theta$ , the set  $\mathcal{C}_m(\theta)$  is a subset of  $\mathcal{S}_m(\theta)$ .

### 5.2 Robinson–Schensted correspondence

The following result is an immediate consequence of Prop. 3.2.

**Proposition 5.2** Let  $T \in \mathcal{T}(\lambda)$  and  $(w_1, \dots, w_m) \in \mathcal{X}^m$ . By consecutive insertions

$$T' = (\dots((T \leftarrow w_1) \leftarrow w_2) \dots \leftarrow w_m)$$

we have a tableaux  $T' \in \mathcal{T}(\mu)$  for some shifted diagram  $\mu \supset \lambda$  and the recording tableau  $U$ . Then we have  $U \in \mathcal{S}_m(\mu/\lambda)$ . By this correspondence we have a weight preserving bijection

$$\phi_m : \mathcal{T}(\lambda) \times \mathcal{X}^m \longrightarrow \bigsqcup_{\mu} \mathcal{T}(\mu) \times \mathcal{S}_m(\mu/\lambda), \tag{5}$$

where the sum runs for shifted diagrams  $\mu$  such that  $\mathcal{S}_m(\mu/\lambda) \neq \emptyset$ .

Then we have immediately the following:

**Corollary 5.3** We have

$$GQ_{\lambda}(x) \cdot GQ_1(x)^m = \sum_{\mu} \beta^{|\mu/\lambda| - m} \#\mathcal{S}_m(\mu/\lambda) \times GQ_{\mu}(x),$$

where the sum runs for shifted diagrams  $\mu$  such that  $\mathcal{S}_m(\mu/\lambda) \neq \emptyset$ .

As a special case of  $\lambda = \emptyset$ , we have the following.

**Corollary 5.4 (Robinson–Schensted correspondence)** There is a weight preserving bijection

$$\mathcal{X}^m \longrightarrow \bigsqcup_{\lambda} \mathcal{T}(\lambda) \times \mathcal{S}_m(\lambda).$$

This bijection is a set-valued extension of the results in [Sa] and [Wo].

**Example 5.5** Let  $(w_1, w_2, w_3) = (2'3, 12'2, 134)$ . By the correspondence in Cor. 5.3 we have pair of tableaux

$$\left( \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 34 \\ \hline & & 23' & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2' & 2 & 3' & 3 \\ \hline & & 2 & & \\ \hline \end{array} \right),$$

as a result of bumping process:

$$\emptyset \xrightarrow{w_1} \boxed{23'} \xrightarrow{w_2} \begin{array}{|c|c|c|} \hline 12 & 2 & 2 \\ \hline & & 3' \\ \hline \end{array} \xrightarrow{w_3} \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 34 \\ \hline & & 23' & & \\ \hline \end{array}.$$

## 6 Outline of proof of Thm 3.4

Now we have the bijection  $\phi_m$  in Prop. 5.2. Since a tableau in  $\mathcal{T}(m)$  is a sequence in  $\mathcal{X}$  such that

$$X_1 \leq_r \cdots \leq_r X_m,$$

we can think of  $\mathcal{T}(m)$  as a subset of  $\mathcal{X}^m$ . Thus we only need to determine the image of  $\mathcal{T}(\lambda) \times \mathcal{T}(m)$  under the map  $\phi_m$ . The case  $m = 1$  is obvious since  $\mathcal{T}(1) = \mathcal{X}$ . The case  $m = 2$  is crucial.

**Lemma 6.1** *Let  $T \in \mathcal{T}(\lambda)$  and  $w = (w_1, w_2) \in \mathcal{X}^2$ , and*

$$\phi_2(T, w) = (T', (C'_1, C_1), (C'_2, C_2)).$$

*Then the following are equivalent:*

1.  $w_1 \leq_r w_2$ .
2.  $(C'_1, C_1) \triangleleft (C'_2, C_2)$ .

It is easy to see that the lemma leads to a proof of Thm 3.4. We show this lemma by an argument using “bumping routes”. Details are given in [INN].

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