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Hierarchical Zonotopal Power Ideals

Matthias Lenz

Abstract. Zonotopal algebra deals with ideals and vector spaces of polynomials that are related to several combinatorial and geometric structures defined by a finite sequence of vectors. Given such a sequence $X$, an integer $k \geq -1$ and an upper set in the lattice of flats of the matroid defined by $X$, we define and study the associated hierarchical zonotopal power ideal. This ideal is generated by powers of linear forms. Its Hilbert series depends only on the matroid structure of $X$. It is related to various other matroid invariants, e. g. the shelling polynomial and the characteristic polynomial.

This work unifies and generalizes results by Ardila-Postnikov on power ideals and by Holtz-Ron and Holtz-Ron-Xu on (hierarchical) zonotopal algebra. We also generalize a result on zonotopal Cox modules due to Sturmfels-Xu.

1 Introduction

Let $X = (x_1, \ldots, x_N) \subseteq \mathbb{R}^r$ be a sequence of vectors that span $\mathbb{R}^r$. For a vector $\eta$, let $m(\eta)$ denote the number of vectors in $X$ that are not perpendicular to $\eta$. A vector $v \in \mathbb{R}^r$ defines a linear polynomial $p_v := \sum_i v_i t_i \in \mathbb{R}[t_1, \ldots, t_r]$. For $Y \subseteq X$, let $p_Y := \prod_{x \in Y} p_x$. Then define

$$\mathcal{P}(X) := \text{span}\{p_Y : X \setminus Y \text{ spans } \mathbb{R}^r\} \quad \text{and} \quad \mathcal{I}(X) := \text{ideal}\{p_{m(\eta)} : \eta \neq 0\} \quad (1.1)$$

The following theorem and several generalizations are well known:

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Theorem 1.1 \((1, 5, 11)\).

\[ \mathcal{P}(X) = \ker \mathcal{I}(X) := \text{span} \left\{ q \in \mathbb{R}[t_1, \ldots, t_r] : f \left( \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r} \right) q = 0 \text{ for all } f \in \mathcal{I}(X) \right\} \]  

\[ (1.2) \]

In addition, \( \mathcal{I}(X) = \mathcal{I}'(X) := \{ p_{\eta}^{m(\eta)} : \text{the vectors in } X \text{ that are perpendicular to } \eta \text{ span a hyperplane} \} \)

We show that a statement as in Theorem 1.1 holds in a far more general setting: we study the kernel of the hierarchical zonotopal power ideal

\[ \mathcal{I}(X, k, J) := \text{ideal} \left\{ p_{\eta}^{m(\eta)+k+\chi_J(\eta)} : \eta \neq 0 \right\} \]  

\[ (1.3) \]

where \( k \geq -1 \) is an integer and \( \chi_J \) is the indicator function of an upper set \( J \) in the lattice of flats of the matroid defined by \( X \). We study those spaces in a slightly more abstract setting, e. g. \( \mathcal{P}(X, k, J) \) is contained in the symmetric algebra over some \( \mathbb{K} \)-vector space, where \( \mathbb{K} \) is a field of characteristic zero.

The choice of a sequence of vectors \( X \) defines a large number of objects in various mathematical fields which are all related to zonotopal algebra \( [11] \). Examples include combinatorics (matroids, matroid and graph polynomials, generalized parking functions and chip firing games if \( X \) is graphic \([15, 18]\)), discrete geometry (hyperplane arrangements, zonotopes and tilings of zonotopes), approximation theory (box splines \([6]\), least map interpolation) and algebraic geometry (Cox rings, fat point ideals \([1, 10, 19]\)).

Central \( \mathcal{P} \)-spaces (in our terminology the kernel of \( \mathcal{I}'(X, 0, \{X\}) \)) were introduced in the literature on approximation theory around 1990 (e. g. \([5]\)). A dual space called \( \mathcal{D}(X) \) appeared almost 30 years ago. See \([11, \text{Section 1.2}] \) for a historic survey.

Recently, Olga Holtz and Amos Ron coined the term zonotopal algebra \( [11] \). They introduced internal \((k = -1)\) and external \((k = +1)\) \( \mathcal{P} \)-spaces and \( \mathcal{D} \)-spaces. Federico Ardila and Alexander Postnikov constructed \( \mathcal{P} \)-spaces for arbitrary integers \( k \geq -1 \) (Combinatorics and geometry of power ideals \([1]\), presented at FPSAC 2009). Olga Holtz, Amos Ron, and Zhiqiang Xu \([12]\) introduced hierarchical zonotopal spaces, i. e. structures that depend on the choice of an upper set \( J \) in addition to \( X \) and \( k \). They studied semi-internal and semi-external spaces (i. e. \( k = -1 \) and \( k = 0 \) and some special upper sets \( J \)). The central case was treated in an algebraic setting in \([8]\). Other related results include \([2, 20]\).

The least map \([7]\) assigns to a finite set \( S \subseteq \mathbb{R}^r \) of cardinality \( m \) an \( m \)-dimensional space of homogeneous polynomials in \( \mathbb{R}[t_1, \ldots, t_r] \). Holtz and Ron \([11]\) showed that in the internal, central and external case, \( \mathcal{P} \)-spaces can be obtained via the least map if \( X \subseteq \mathbb{Z}^r \) is unimodular. In those cases, the \( \mathcal{P} \)-spaces are obtained by choosing the set \( S \) as a certain subset of the set of integral points in the zonotope

\[ Z(X) := \left\{ \sum_{i=1}^{N} \lambda_i x_i : 0 \leq \lambda_i \leq 1 \right\} \]  

\[ (1.4) \]

There is also a discrete theory, where differential operators are replaced by difference operators. \([8]\) and A Tutte polynomial for toric arrangements (Luca Moci) \([16]\), presented at FPSAC 2010). In this theory, the connection with lattice points in zonotope remains valid even if \( X \) is not unimodular.

As an example for the connections between zonotopal algebra and combinatorics, we now explain various relationships between zonotopal spaces and matroid/graph polynomials. They can be deduced from the fact that both, the matroid/graph polynomials \([9]\) and the Hilbert series of the zonotopal spaces \([11]\) are evaluations of the Tutte polynomial.
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The four color theorem is equivalent to the following statement: if \( G \) is a planar graph and \( G^* \) denotes its dual, then

\[
\phi_G(t) = (t - 1)^{N-r} \text{Hilb}(\mathcal{P}(X_G, -1, \{X_G\}), 1/(1 - t))
\]

The Hilbert series is equivalent to the following statement: if \( G \) is a planar graph and \( G^* \) denotes its dual, then

\[
\text{Hilb}(\mathcal{P}(X_{G^*}, -1, \{X_{G^*}\}), -1/3) > 0
\]

Organization of the article: In Section 2, we introduce our notation and review the mathematical background. In Section 3, we define hierarchical zonotopal power ideals. A simple description of their kernels is given by our Main Theorem. In Section 4, we construct bases for the vector spaces \( \mathcal{P}(X, k, J) \) and we deduce several formulas for their Hilbert series. In Section 5, we apply our results to prove a statement about zonotopal Cox modules that were defined by Bernd Sturmfels and Zhiqiang Xu [19]. In Section 6, we give some examples. The full paper is available on the arXiv [14].

2 Preliminaries

Notation: The following notation is used throughout this paper: \( \mathbb{K} \) is a fixed field of characteristic zero. \( V \) denotes a finite-dimensional \( \mathbb{K} \)-vector space of dimension \( r \geq 1 \) and \( U := V^* \) its dual. Our main object of study is a finite sequence \( X = (x_1, \ldots, x_N) \subseteq U \) whose elements span \( U \). We slightly abuse notation by using the symbol \( \subseteq \) for subsequences. For \( Y \subseteq X \), the deletion \( X \setminus Y \) denotes the deletion of a subsequence and not the deletion of a subset, i.e., \( (x_1, x_2) \setminus (x_1) = (x_2) \) even if \( x_1 = x_2 \). The order of the elements in \( X \) is irrelevant except in Section 4.

Matroids and posets: Let \( X = (x_1, \ldots, x_N) \) be a finite sequence whose elements span \( U \). Let \( \mathfrak{M}(X) := \{I \subseteq \{1, \ldots, N\} : \{x_i : i \in I\} \text{ linearly independent}\} \). Then \( \mathfrak{M}(X) \) is a matroid of rank \( r \) on \( N \) elements. \( X \) is called a \( \mathbb{K} \)-representation of the matroid \( \mathfrak{M}(X) \). For more information about matroids, see Oxley's book [17].

We now introduce some additional matroid theoretic concepts. To facilitate notation, we always write \( X \) instead of \( \mathfrak{M}(X) \). The rank of \( Y \subseteq X \) is defined as the cardinality of a maximal independent set contained in \( Y \). It is denoted \( \text{rk}(Y) \). The closure of \( Y \) in \( X \) is defined as \( \text{cl}_X(Y) := \{x \in X : \text{rk}(Y \cup x) = \text{rk}(Y)\} \).

If \( C \subseteq X \) is called a flat if \( C = \text{cl}(C) \). A hyperplane is a flat of rank \( r - 1 \). The set of all hyperplanes in \( X \) is denoted by \( \mathfrak{H} = \mathfrak{H}(X) \).

Given a flat \( C \subseteq X \), we call \( \eta \in V \) a defining normal for \( C \) if \( C = \{x \in X : \eta(x) = 0\} \). Note that for hyperplanes, there is a unique defining normal (up to scaling). The set of bases of the matroid \( X \) (i.e., the subsequences of \( X \) of cardinality \( r \) and rank \( r \)) is denoted \( \mathbb{B}(X) \). If \( x = 0 \), then \( x \) is called a loop. If \( \text{rk}(X \setminus x) = r - 1 \), then \( x \) is called a coloop.

The set of flats of a given matroid \( X \) ordered by inclusion forms a lattice (i.e., a poset with joins and meets) called the lattice of flats \( \mathcal{L}(X) \). An upper set \( J \subseteq \mathcal{L}(X) \) is an upward closed set, i.e., \( C \subseteq C' \).
\[ C \in J \text{ implies } C' \in J. \] We call \( C \in \mathcal{L}(X) \) a maximal missing flat if \( C \not\in J \) and \( C \) is maximal with this property.

The Tutte polynomial \( T_X(x, y) := \sum_{A \subseteq X} (x-1)^{r-rk(A)} (y-1)^{|A|-rk(A)} \) captures a lot of information about the matroid \( \mathfrak{M}(X) \).

**Algebra:** \( \text{Sym}(V) \) denotes the symmetric algebra over \( V \). This is a base-free version of the ring of polynomials over \( V \). The choice of a basis \( B = \{ b_1, \ldots, b_n \} \subseteq V \) yields an isomorphism \( \text{Sym}(V) \cong \mathbb{K}[b_1, \ldots, b_n] \).

A derivation on \( \text{Sym}(V) \) is a \( \mathbb{K} \)-linear map \( D \) satisfying Leibniz’s law, i.e., \( D(fg) = D(f)g + fD(g) \) for \( f, g \in \text{Sym}(V) \). For \( v \in V \), we define the directional derivative in direction \( v \), \( D_v : \text{Sym}(U) \to \text{Sym}(U) \) as the unique derivation which satisfies \( D_v(u) = v(u) \) for all \( u \in U \). For \( \mathbb{K} = \mathbb{R} \) and \( \mathbb{K} = \mathbb{C} \), this definition agrees with the analytic definition of the directional derivative. It follows that \( \text{Sym}(V) \) can be identified with the ring of differential operators on \( U \).

Now we define a pairing \( \langle \cdot, \cdot \rangle : \text{Sym}(U) \times \text{Sym}(V) \to \mathbb{K} \) by \( \langle f, q \rangle := \langle q(D)f \rangle(0) \) where \( q(D)f \) means that \( q \) acts as a differential operator on \( f \).

A graded vector space is a vector space \( V \) that decomposes into a direct sum \( V = \bigoplus_{i \geq 0} V_i \). For a graded vector space, we define its Hilbert series as the formal power series \( \text{Hilb}(V, t) := \sum_{i \geq 0} \dim(V_i)t^i \).

A graded algebra \( V \) has the additional property \( V_iV_j \subseteq V_{i+j} \). We use the symmetric algebra with its natural grading. This grading is characterized by the property that the degree 1 elements are exactly the ones that are contained in \( V \). A \( \mathbb{Z}^n \)-multigraded ring \( R \) is defined similarly: \( R \) decomposes into a direct sum \( R = \bigoplus_{a \in \mathbb{Z}^n} R_a \) and \( R_aR_b \subseteq R_{a+b} \).

A linear map \( f : V \to W \) induces an algebra homomorphism \( \text{Sym}(f) : \text{Sym}(V) \to \text{Sym}(W) \).

**Homogeneous ideals and their kernels:** An ideal \( I \subseteq \text{Sym}(V) \) is called a power ideal \( [1] \) if \( I = \text{ideal}\{ D_\eta^n : \eta \in \mathbb{Z} \} \) for some \( \eta \subseteq V \setminus \{0\} \) and \( e \in \mathbb{Z}_{\geq 0} \). \( D_\eta \) denotes the image of \( \eta \) under the canonical injection \( V \hookrightarrow \text{Sym}(V) \). By definition, power ideals are homogeneous.

**Definition 2.1.** Let \( I \subseteq \text{Sym}(V) \) be a homogeneous ideal. Its kernel (or inverse system) is defined as

\[ \ker I := \{ f \in \text{Sym}(U) : \langle f, q \rangle = 0 \text{ for all } q \in I \} \] (2.1)

It is known that for a homogeneous ideal \( I \subseteq \text{Sym}(V) \) of finite codimension the Hilbert series of \( \ker I \) and \( \text{Sym}(V)/I \) are equal. [10]

**A remark on the notation:** As zonotopal spaces were studied by people from different fields, the notation and level of abstraction used in the literature varies. Authors with a background in spline theory usually work over \( \mathbb{R}^n \) while other authors prefer an abstract setting as we do.

Since the Euclidean setting captures all the important parts of the theory, a reader with no background in abstract algebra may safely make the following substitutions: \( \mathbb{K} = \mathbb{R} \), \( U \cong V \cong \mathbb{R}^n \). \( f \in \mathbb{R}[t_1, \ldots, t_r] = \text{Sym}(V) \) acts on \( \mathbb{R}[t_1, \ldots, t_r] = \text{Sym}(U) \) as the differential operator that is obtained from \( f \) by substituting \( t_i \mapsto \frac{\partial}{\partial t_i} \).

[10] It is sometimes defined slightly differently in the literature: The pairing \( \langle \cdot, \cdot \rangle \) is defined on \( \text{Sym}(V)^* \times \text{Sym}(V) \) and \( \ker I \) is the subset of \( \text{Sym}(V)^* \) that is annihilated by \( I \). In our setting, both definitions agree.
Some authors work in the dual setting and consider a central hyperplane arrangement instead of a finite sequence of vectors \( X \). While hierarchical zonotopal power ideals can be defined in both settings, it is natural for us to work with vectors as we are also interested in the zonotope \( Z \).

### 3 Hierarchical zonotopal power ideals and their kernels

#### Definitions and the Main Theorem:
Recall that \( U = V^* \) denotes an \( r \)-dimensional vector space over a field \( \mathbb{K} \) of characteristic zero and \( X = (x_1, \ldots, x_N) \) denotes a finite sequence whose elements span \( U \).

A vector \( \eta \in V \) defines a flat \( C \subseteq X \). Define \( m_X(C) = m_X(\eta) := |X \setminus C| \). Given an upper set \( J \subseteq \mathcal{L}(X) \), \( \chi : \mathcal{L}(X) \to \{0, 1\} \) denotes its indicator function. The index of \( \chi \) and \( m \) is omitted if it is clear which upper set is meant. We define \( \chi \) for arbitrary sets \( A \subseteq X \) as \( \chi(A) := \chi(\mathrm{cl}(A)) \).

For a given \( x \in U \), we denote the image of \( X \) under the canonical injection \( U \hookrightarrow \mathrm{Sym}(U) \) by \( p_x \). For \( Y \subseteq X \), we define \( p_Y := \prod_{x \in Y} p_x \). For \( \eta \in V \), we write \( D_0 \) for the image of \( \eta \) under the canonical injection \( V \hookrightarrow \mathrm{Sym}(V) \) in order to stress the fact that we mostly think of \( \mathrm{Sym}(V) \) as the algebra generated by the directional derivatives on \( \mathrm{Sym}(U) \).

**Definition 3.1** (Hierarchical zonotopal power ideals and \( P \)-spaces). Let \( \mathbb{K} \) be a field of characteristic zero, \( V \) be a finite-dimensional \( \mathbb{K} \)-vector space of dimension \( r \geq 1 \) and \( U = V^* \). Let \( X = (x_1, \ldots, x_N) \subseteq U \) be a finite sequence whose elements span \( U \). Let \( k \geq -1 \) be an integer and let \( J \subseteq \mathcal{L}(X) \) be a non-empty upper set, where \( \mathcal{L}(X) \) denotes the lattice of flats of the matroid defined by \( X \).

Let \( \chi : \mathcal{L}(X) \to \{0, 1\} \) denote the indicator function of \( J \). Let \( E : \mathcal{L}(X) \to V \) be a function that assigns a defining normal to every flat. Now define

\[
I'(X, k, E) := \text{ideal}\left\{D_{E(C)}^{m(C) + k} + \chi(C) : C \text{ hyperplane or maximal missing flat}\right\} \quad (3.1)
\]

\[
I(X, k, J) := \text{ideal}\left\{D_{\eta}^{m(\eta) + k} + \chi(\eta) : \eta \in V \setminus \{0\}\right\} \subseteq \mathrm{Sym}(V) \quad (3.2)
\]

\[
P(X, k, J) := \text{span} S(X, k, J) \subseteq \mathrm{Sym}(U) \quad (3.3)
\]

where

\[
S(X, k, J) := \begin{cases} 
\{f \partial_Y : Y \subseteq X, 0 \leq \deg f \leq \chi(X \setminus Y) + k - 1\} & \text{for } k \geq 1 \\
\{p_Y : Y \subseteq X, \mathrm{cl}(X \setminus Y) \in J\} & \text{for } k = 0 \\
\{p_Y : |Y \setminus C| < m(C) - 1 + \chi(C) \text{ for all } C \in \mathcal{L}(X) \setminus \{X\}\} & \text{for } k = -1
\end{cases}
\]

Note that the definition of \( S(X, 0, J) \) can be seen as a special case of the definition of \( S(X, k, J) \) for \( k \geq 1 \). For examples, see Section 6, Remark 3.7, and Proposition 3.8.

**Theorem 3.2** (Main Theorem). We use the same terminology as in Definition 3.1. For \( k = -1 \), we assume in addition that \( J \supseteq \mathcal{H} \), i.e. \( J \) contains all hyperplanes in \( X \). Then,

\[
P(X, k, J) = \ker I(X, k, J) \subseteq \ker I'(X, k, J, E) \quad (3.4)
\]

Furthermore, for \( k \in \{-1, 0\} \), \( I'(X, k, J, E) \) is independent of the choice of \( E \) and \( P(X, k, J) = \ker I(X, k, J) = \ker I'(X, k, J, E) \).
Corollary 3.3. In the setting of the Main Theorem, $\mathcal{P}(X, k, \mathcal{L}(X)) = \mathcal{P}(X, k + 1, \{X\})$.

Corollary 3.4. The Hilbert series of $\mathcal{P}(X, k, J)$ depends only on the matroid $\mathfrak{M}(X)$, but not on the representation $X$.

Remark 3.5. One might wonder if similar theorems can be proved for $k \leq -2$. One would of course need to impose extra conditions on $X$ to ensure that the exponents appearing in the definition of the ideals are non-negative. It is easy to prove that $\mathcal{I}$ and $\mathcal{I}'$ are equal in this case. However, we do not know how to construct a generating set for the kernel. A different approach would be required: in general, it is not spanned by a set of polynomials of type $p_Y$ for some $Y \subseteq X$ [I].

Basic results: In this paragraph, we state an important lemma and we give an explicit formula for $\mathcal{P}(X, k, J)$ in two particularly simple cases.

Lemma 3.6. $\mathcal{P}(X, k, J) \subseteq \ker \mathcal{I}(X, k, J) \subseteq \ker \mathcal{I}'(X, k, J)$ holds for all $k \geq -1$ and all $J \subseteq \mathcal{L}(X)$.

Remark 3.7. Suppose that $\dim U = 1$ and that $X$ contains $N'$ non-zero entries. Let $x \in U$ and $y \in V$ be non-zero vectors. Note that $\emptyset$ is the only hyperplane in $X$. Hence, $\mathcal{I}'(X, k, J) = \mathcal{I}(X, k, J) = \text{ideal}\{D_{N'+k+\chi(0)}\}$ and $\mathcal{P}(X, k, J) = \text{span}\{p^k_i : i \in \{0, 1, \ldots, N' - 1 + k + \chi(0)\}\}$.

Proposition 3.8. We use the same terminology as in Definition 3.1. Let $X = (x_1, \ldots, x_r)$ be a basis for $U$. Then $\mathcal{P}(X, k, J) = \ker \mathcal{I}(X, k, J) \subseteq \ker \mathcal{I}'(X, k, J, E)$. Furthermore, for $k \in \{-1, 0\}$, $\ker \mathcal{I}(X, k, J) = \ker \mathcal{I}'(X, k, J, E)$ for arbitrary $E$.

More precisely, writing $p_i := p_{x_i}$ as shorthand notation, we get

$$\mathcal{P}(X, k, J) = \text{span}\left\{\prod_{i \in I} p_i^{a_i + 1} : I \subseteq [r], a_i \in \mathbb{Z}_{\geq 0}, \sum_{i \in I} a_i \leq k + \chi(X \setminus \{x_i : i \in I\}) - 1\right\} \quad (3.5)$$

For $k = 0$, this specializes to $\mathcal{P}(X, 0, J) = \text{span}\{p_Y : X \setminus Y \in J\}$. For $k = -1$, $\mathcal{P}(X, -1, J) = \text{span}\{1\}$ if $J \supseteq \mathcal{H}$ and $\mathcal{P}(X, -1, J) = \{0\}$ otherwise.

For a two-dimensional example, see Example 3.1 and Figure 1.

Deletion and Contraction: In this paragraph, we define deletion and contraction for matroids $X$ and upper sets $J$.

Fix an element $x \in X$. The deletion of $x$ is the matroid defined by the sequence $X \setminus x$. Let $\pi_x : U \to U/x$ denote the projection to the quotient space. The contraction of $x$ is the matroid defined by the sequence $X/x$ which contains the images of the elements of $X \setminus x$ under $\pi_x$.

Let $Y \subseteq X \setminus x$. We write $\bar{Y}$ to denote the subsequence of $X/x$ with the same index set as $Y$ and vice versa.

Let $J \subseteq \mathcal{L}(X)$ be an upper set. Then define

$$J \setminus x := \{C \setminus x : C \in J \text{ and } C = \text{cl}(C \setminus x)\} \subseteq \mathcal{L}(X \setminus x) \quad (3.6)$$

$$J/x := \{(C/x) : x \in C \in J\} \subseteq \mathcal{L}(X/x) \quad (3.7)$$

On the proof of the Main Theorem: The Main Theorem can be proven by induction. Proposition 3.8 is used as the base case. Deletion-contraction gives rise to a short exact sequence from which the result follows. This short exact sequence is recorded in the following proposition:
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**Proposition 3.9.** We use the same terminology as in Definition 3.1. Suppose that \( x \in X \) is neither a loop nor a coloop. For \( k = -1 \), we assume in addition that \( J \supseteq \mathcal{H} \) or \( J = \{X\} \). Then, the following sequence is exact:

\[
0 \to \ker \mathcal{I}(X \setminus x, k, J \setminus x) \xrightarrow{p^U_x} \ker \mathcal{I}(X, k, J) \xrightarrow{\text{Sym}(\pi_x)} \ker \mathcal{I}(X/x, k, J/x) \to 0
\]  

(3.8)

**Remark 3.10.** We do not know if the Main Theorem holds for \( k = -1 \) and \( J \supseteq \mathcal{H} \). In general, our proof does not work in this case and Proposition 3.9 is false. The difficulty of the case \( k = -1 \) was already observed by Holtz and Ron. They conjectured that the Main Theorem holds in the internal case i.e. for \( k = -1 \) and \( J = \{X\} \), but they were unable to prove it [11, Conjecture 6.1]. An incorrect prove of this special case has appeared in the literature [11, Part 3 of Proposition 4.5].

**4 Bases and Hilbert series**

In this section, we show how to select a basis for \( \mathcal{P}(X, k, J) \) from \( S(X, k, J) \) for \( k \geq 0 \) and we give several formulas for the Hilbert series of \( \mathcal{P}(X, k, J) \).

**Bases:** Our construction of a basis depends on the order on \( X \). This order is used to define internal and external activity with respect to a given basis (see [4, Section 6.6.] for a reference). Recall that \( \mathcal{B}(X) \) denotes the set of all bases \( B \subseteq X \). Fix a basis \( B \in \mathcal{B}(X) \). \( b \in B \) is called *internally active* if \( b = \max(X \setminus \text{cl}(B \setminus b)) \), i.e. \( b \) is the maximal element of the unique cocircuit contained in \( (X \setminus B) \cup b \). The set of internally active elements in \( B \) is denoted \( I(B) \). \( x \in X \setminus B \) is called *externally active* if \( x \in \text{cl}\{b \in B : b \leq x\} \), i.e. \( x \) is the maximal element of the unique circuit contained in \( B \cup x \). The set of externally active elements with respect to \( B \) is denoted \( E(B) \).

We generalize [11, Proposition 4.21] to hierarchical spaces:

**Definition 4.1.** We use the same terminology as in Definition 3.1. In addition, let \( k \geq 0 \). Then define

\[
\Gamma(X, k, J) := \left\{(B, I, a_I) : B \in \mathcal{B}(X), I \subseteq I(B), a_I \in \mathbb{Z}_{\geq 0}^I, \sum_{x \in I} a_x \leq k + \chi((B \cup E(B)) \setminus I) - 1 \right\}
\]

\[
\mathcal{B}(X, k, J) := \left\{ p_{X \setminus (B \cup E(B))} \prod_{x \in I} p_{a_x+1}^x : (B, I, a_I) \in \Gamma(X, k, J) \right\} \subseteq \text{Sym}(U)
\]

(4.1)

**Theorem 4.2** (Basis Theorem). We use the same terminology as in Definition 3.1. Let \( k \geq 0 \). Then \( \mathcal{B}(X, k, J) \) is a basis for \( \mathcal{P}(X, k, J) \).

**Remark 4.3.** We do not know if there is a simple method to construct bases for \( \mathcal{P}(X, -1, J) \). This difficulty was already observed in the internal case by Holtz and Ron [11].

**Recursive formulas for the Hilbert series:** The following statement is a direct consequence of Proposition 3.9 and of the Main Theorem:

**Corollary 4.4.** We use the same terminology as in Definition 3.1. Let \( x \in X \) be an element that is not a coloop. For \( k = -1 \), we assume in addition that \( J \supseteq \mathcal{H} \) or \( J = \{X\} \), i.e. \( J \) contains either all or no hyperplanes. Then,
\[
\text{Hilb}(P(X, k, J), t) = \begin{cases} 
\text{Hilb}(P(X \setminus k, J \setminus k), t) & \text{if } x \text{ is a loop} \\
(t \text{ Hilb}(P(X \setminus k, J \setminus k), t) + \text{Hilb}(P(X/k, k, J/k), t)) & \text{otherwise}
\end{cases}
\]

For coloops, the situation is more complicated and requires an additional definition. Fix a coloop \( x \in X \). Then, \( X \setminus x \) is a hyperplane and the following is an upper set:

\[
\text{\( J/x := \{ \hat{C} : x \not\in C \in J \} \cup \{X \setminus x\} \subseteq \mathcal{L}(X/x) \)}
\]

\( J/x \) forgets about the flats containing \( x \), whereas \( J/x \) forgets about the flats not containing \( x \). While the latter is always an upper set in \( \mathcal{L}(X/x) \), some elements of \( \hat{J}/x \) are not closed unless \( X \setminus x \) is a hyperplane.

**Theorem 4.5.** We use the same terminology as in Definition 3.1. Let \( x \in X \) be a coloop and \( k \geq 0 \). Then, 

\[
\text{Hilb}(P(X, k, J), t) = \begin{cases} 
\text{Hilb}(P(X/k, k, J/k), t) \\
+ \sum_{j=0}^{k-1} t^{j+1} \text{Hilb}(P(X/k, k - j, J/x), t) & \text{if } X \setminus x \in J \\
\text{Hilb}(P(X/k, k, J/k), t) + \sum_{j=0}^{k-1} t^{j+1} \text{Hilb}(P(X/k, k - j, J/x), t) & \text{if } X \setminus x \not\in J
\end{cases}
\]

For \( k = -1 \), \( \text{Hilb}(\ker(\mathcal{I}(X, -1, J)), t) = \text{Hilb}(\ker(\mathcal{I}(X/k, k, J/k)), t) \) if \( X \setminus x \in J \) and otherwise \( \text{Hilb}(\ker(\mathcal{I}(X, -1, J)), t) = 0 \). This holds for arbitrary non-empty upper sets \( J \subseteq \mathcal{L}(X) \).

**Combinatorial formulas for \( k \geq 0 \):** Theorem 4.2 provides a method to compute the Hilbert series of a \( P \)-space combinatorially:

**Corollary 4.6.** We use the same terminology as in Definition 3.1. Let \( k \geq 0 \). Then, 

\[
\text{Hilb}(P(X, k, J), t) = \sum_{B \in \mathbb{B}(X)} t^{N_E(B)} \left[ \chi((B \cup E(B)) \setminus I) \sum_{0 \neq I \subseteq I(B)} \sum_{j=0}^{\chi((B \cup E(B)) \setminus I)} t^{|I|+j} \left( \frac{j + |I| - 1}{|I| - 1} \right) \right]
\]

where \( E(B) \) and \( I(B) \) denote the sets of externally resp. internally active elements.

From this, we can deduce a result, which relates the dimension of \( P(X, 0, J) \) and the number of independent sets satisfying a certain property. This was already proven with a different method by Holtz, Ron, and Xu [12].

**Corollary 4.7.** \( \dim P(X, 0, J) = |\{Y \subseteq X : Y \text{ independent, } \text{cl}(Y) \in J\}| \)

Corollary 4.6 gives a formula in terms of the internal and external activity of the bases of \( X \). For \( k = 0 \), there is also a subset expansion formula similar to the one for the Tutte polynomial. In the internal, central and external case, the Hilbert series of the \( P \)-spaces are evaluations of the Tutte polynomial [1]. In particular, \( \text{Hilb}(P(X, 0, \{X\}), t) = t^{N_E(T_X(1, \frac{1}{t}) \text{ and } \text{Hilb}(P(X, 1, \{X\}), t) = t^{N_E(T_X(1 + t, \frac{1}{t}))} \text{.}

The following theorem provides a formula for the semi-external case which “interpolates” between those two formulas:
Theorem 4.8. We use the same terminology as in Definition 3.1
\[ \text{Hilb}(\mathcal{P}(X,0,J), t) = t^{N-r} \sum_{\lambda \in \mathcal{X}} t^{-\text{rk}(A)} (t^{1-1})^{|\lambda| - \text{rk}(A)} \] (4.5)

The case \( k = -1 \): For \( k = -1 \), we do not know if there is such a nice formula as in Corollary 4.6 or Theorem 4.8. However, in a special case, a formula is known.

Fix \( C_0 \in \mathcal{L}(X) \) and set \( J_{C_0} := \{ C \in \mathcal{L}(X) : C \supseteq C_0 \} \). All maximal missing flats in \( J_{C_0} \) are hyperplanes. They have unique defining normals (up to scaling). Holtz, Ron, and Xu [12] showed that \( \ker \mathcal{I}(X,-1,J_{C_0}) = \ker \mathcal{I}(X,-1,J_{C_0}) = \bigcap_{x \in C_0} \mathcal{P}(X \setminus x, 0, \{ X \setminus x \}) \).

Fix an independent set \( K \subseteq X \) that spans \( C_0 \). Choose an order on \( X \) s.t. \( K \) is maximal and define \( \mathcal{B}_-(X,J_{C_0}) = \{ B \in \mathcal{B}(X) : I(B) \cap K = \emptyset \} \). Then, the following theorem holds:

Theorem 4.9 ([12] p. 20). We use the same terminology as in Definition 3.1. In addition, let \( C_0 \in \mathcal{L}(X) \). Then,
\[ \text{Hilb}(\ker \mathcal{I}(X,-1,J_{C_0}), t) = \sum_{B \in \mathcal{B}_-(X,J_{C_0})} t^{N-r-|E(B)|} \] (4.6)

5 Zonotopal Cox Rings

In this section, we briefly describe the zonotopal Cox rings defined by Sturmfels and Xu [19] and we show that our Main Theorem can be used to generalize a result on zonotopal Cox modules due to Ardila and Postnikov [11].

Fix \( m \) vectors \( D_1, \ldots, D_m \in V \) and \( \mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{Z}_{\geq 0}^m \). Sturmfels and Xu [19] introduced the Cox-Nagata ring \( R^G \subseteq \mathbb{K}[s_1, \ldots, s_m, t_1, \ldots, t_m] \). This is the ring of polynomials that are invariant under the action of a certain group \( G \) which depends on the vectors \( D_1, \ldots, D_m \). It is multigraded with a \( \mathbb{Z}^{m+1} \)-grading. For \( r \geq 3 \), \( R^G \) is equal to the Cox ring of the variety \( X_G \) which is gotten from \( \mathbb{P}^{r-1} \) by blowing up the points \( D_1, \ldots, D_m \). Cox rings have received a considerable amount of attention in the recent literature in algebraic geometry. See [13] for a survey.

Cox-Nagata rings are closely related to power ideals: let \( \mathcal{I}_u := \text{ideal}\{ D_1^{u_1+1}, \ldots, D_m^{u_m+1} \} \) and let \( \mathcal{I}^{-1}_{d,u} \) denote the homogeneous component of grade \( d \) of \( \ker \mathcal{I}_u \). Then, \( R^G(d,u) \), the homogeneous component of \( R^G \) of grade \( (d,u) \), is naturally isomorphic to \( \mathcal{I}_{d,u} \) ([19 Proposition 2.1]).

Cox-Nagata rings are an object of great interest but in general, it is quite difficult to understand their structure. However, for some choices of the vectors \( D_1, \ldots, D_m \), we understand a natural subring of the Cox-Nagata ring very well.

Let \( \mathcal{H} = \{ H_1, \ldots, H_m \} \) denote the set of hyperplanes in \( \mathcal{L}(X) \). Let \( \mathcal{H} \in \{ 0, 1 \}^{m \times N} \) denote the non-containment vector-hyperplane matrix, i.e. the 0-1 matrix whose \( (i,j) \) entry is 1 if and only if \( x_j \) is not contained in \( H_i \).

Sturmfels and Xu defined the following structures: the zonotopal Cox ring
\[ Z(X) := \bigoplus_{(d,u) \in \mathbb{Z}_{\geq 0}^{N+1}} R^G(d,u) \] (5.1)
and \( Z(X, \omega) := \bigoplus_{(d, \alpha) \in \mathbb{Z}_{\geq 0}^{m+1}} R_{(d, \alpha) + \omega}^G \), the zonotopal Cox module of shift \( \omega \) for \( \omega \in \mathbb{Z}^n \).

Let \( X(\alpha) \) denote the sequence of \( \sum_i \alpha_i \) vectors in \( U \) that is obtained from \( X \) by replacing each \( x_i \) by \( \alpha_i \) copies of itself and let \( e := (1, \ldots, 1) \in \mathbb{Z}_{\geq 0}^m \). Ardila and Postnikov prove the following isomorphisms [11]

**Proposition 6.3:** \( R_{(d, \alpha + e)}^G \cong \mathcal{P}(X(\alpha), 1, \{X\})_d \), \( R_{(d, \alpha - e)}^G \cong \mathcal{P}(X(\alpha), 0, \{X\})_d \), and \( R_{(d, \alpha - 2e)}^G \cong \mathcal{P}(X(\alpha), -1, \{X\})_d \)

Using the Main Theorem, we can generalize those results. Let \( J_b := \{ C \in \mathcal{L}(X) : b_H = 1 \text{ for all } H \subseteq C \} \). We have the following results on hierarchical zonotopal Cox modules and their Hilbert series:

**Proposition 5.1.** We use the same terminology as in Definition 3.1. For the graded components of the semi-external zonotopal Cox module \( Z(X, \alpha \cdot e + b) \), the following holds:

\[
R_{(d, \alpha - e + b)}^G \cong \mathcal{P}(X(\alpha), 0, J_b)_d \quad \text{for all } d
\]

**Proposition 5.2.** We use the same terminology as in Definition 3.1. Let \( C_0 \in \mathcal{L}(X) \) be a fixed flat and \( J_{C_0} := \{ C \in \mathcal{L}(X) : C \supseteq C_0 \} \) (cf. Subsection 4). If \( b \in \{0, 1\}^n \) satisfies \( b_H = 1 \) if and only if \( H \supseteq C_0 \), then for the graded components of the semi-internal zonotopal Cox module \( Z(X, \alpha - 2e + b) \), the following holds:

\[
R_{(d, \alpha - 2e + b)}^G \cong \ker \mathcal{I}(X(\alpha), -1, J_{C_0})_d \quad \text{for all } d
\]

**Corollary 5.3.** In the setting of Proposition 5.1 the dimension of \( R_{(d, \alpha - e + b)}^G \) equals the coefficient of \( t^d \) in

\[
\text{Hilb}(\mathcal{P}(X(\alpha), 0, J_b), t) = t^{\sum_{A \subseteq X} a_i - r} \sum_{\chi(A) = 1} t^{r - \text{rk}(A)} \sum_{1 \leq \alpha_i \leq a_i} \left( \prod_{k \in A} \binom{a_k}{s_k} \right) \left( \frac{1}{t} - 1 \right)^{\sum_i s_i - \text{rk}(A)}
\]

**Corollary 5.4.** In the setting of Proposition 5.2 the dimension of \( R_{(d, \alpha - 2e + b)}^G \) equals the coefficient of \( t^d \) in

\[
\text{Hilb}(\ker \mathcal{I}(X(\alpha), -1, J_{C_0}), t) = \sum_{B \in \mathcal{B}_{-}(X, J_{C_0})} \sum_{0 \leq s_i \leq a_i - 1} t^{\sum_{a_i \in B} a_i - r - \sum_{s_i \in B} s_i} \]

6 Examples

In this section, we give examples for the structures and constructions appearing in this extended abstract. Here, we do the following identifications: \( \text{Sym}(V) = \text{Sym}(U) = \mathbb{K}[x, y] \) and \( \mathbb{K}[x, y]/x = \mathbb{K}[y] \).

\[
X_1 := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = (x_1, x_2, x_3)
\]

Let \( J_1 := \{ (x_1, x_1), (x_3) \} \). The set of bases is \( \mathcal{B}(X) = \{(x_1 x_2), (x_1 x_3), (x_2 x_3)\} \) and the ideal is \( \mathcal{I}(X_1, 0, J_1) = \text{ideal}\{x^2, xy^2, y^3\} \).
Hierarchical Zonotopal Power Ideals

Now we consider the deletion and contraction of $x_1$. For the upper set $J_1$, we obtain $J_1 \setminus x_1 = \{(x_2, x_3), (x_3)\}$ and $J_1/x_1 = \{(\bar{x}_2, \bar{x}_3), \emptyset\}$.

$$X_1 \setminus x_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = (x_2, x_3) \quad X_1/x_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} = (\bar{x}_2, \bar{x}_3)$$

$$\mathcal{I}(X_1 \setminus x_1, 0, J_1 \setminus x_1) = \text{ideal}\{x, y^2\} \quad \mathcal{I}(X_1/x_1, 0, J_1/x_1) = \text{ideal}\{y^3\}$$

$$\mathcal{P}(X_1 \setminus x_1, 1, J_1 \setminus x_1) = \text{span}\{1, y\} \quad \mathcal{P}(X_1/x_1, 1, J_1/x_1) = \text{span}\{1, y, y^2\}$$

Example 6.1. This is an example for the recursion in Theorem 4.5 and for Proposition 3.8. Let $X_2 := \{x, y, x^2, y^2\}$, and $J_3 := \{x^2, y^3, x^2y^2\}$.

Fig. 1: On the left, we see $\mathcal{P}(X_2, 2, J_3)$ and on the right we see $\mathcal{P}(X_2, 2, J_2)$. For both spaces, the decompositions corresponding to Theorem 4.5 are shown.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (x_1, x_2), J_2 := \{X_2\}, \text{ and } J_3 := \{(x_2, x_1)\}. \text{ This implies } \widehat{J_2/x_2} = \widehat{J_3/x_2} = \{(x_1)\}. \text{ For a graphic description of the } \mathcal{P}\text{-spaces involved in the decomposition, see Figure 1}$$

$$\mathcal{I}(X_2, 2, J_2) = \text{ideal}\{x^3, y^3, x^2y^2\} \quad \mathcal{P}(X_2, 2, J_2) = \text{span}\{1, x, y, x^2, y^2, x^2y, xy^2\}$$

$$\mathcal{I}(X_2, 2, J_3) = \text{ideal}\{x^3, y^4, x^2y^2, xy^3\} \quad \mathcal{P}(X_2, 2, J_3) = \text{span}\{1, x, y, x^2, y^2, x^2y, xy^2, y^3\}$$

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References


