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# *Recognizing the $P_4$–structure of claw–free graphs and a larger graph class*

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The $P_4$–structure of a graph $G$ is a hypergraph $\mathcal{H}$ on the same vertex set such that four vertices form a hyperedge in $\mathcal{H}$ whenever they induce a $P_4$ in $G$. We present a constructive algorithm which tests in polynomial time whether a given 4–uniform hypergraph is the $P_4$–structure of a claw–free graph and of (banner,chair,dart)–free graphs. The algorithm relies on new structural results for (banner,chair,dart)–free graphs which are based on the concept of $p$–connectedness. As a byproduct, we obtain a polynomial time criterion for perfectness for a large class of graphs properly containing claw–free graphs.

**Keywords:** Claw-free graphs, reconstruction problem, $P_4$-structure, $p$-connected graphs, homogeneous set, perfect graphs.

## 1 Introduction

Let $\mathcal{H} = (V, \mathcal{E})$ be a 4–uniform hypergraph with vertices $V$ and hyperedges $\mathcal{E}$. We consider the *reconstruction problem* which asks for a graph $G = (V, E)$ whose $P_4$–structure is equal to $\mathcal{H}$. More precisely, is there a graph $G$ such that four vertices from $G$ induce a $P_4$ (that is, a chordless path on four vertices) if and only if these four vertices induce a hyperedge in $\mathcal{H}$? If the answer is yes, how can we find such a graph $G$? This problem has been settled for several classes of graphs including trees [7, 10], bipartite graphs [2], block graphs [4], line graphs of bipartite graphs [15] and line graphs [1].

In this paper we shall provide a polynomial–time algorithm which solves the reconstruction problem for claw–free graphs (that are the graphs containing no induced copy of a $K_{1,3}$), and for BCD–free graphs (that are the graphs containing no induced copy of a *b*anner, a *c*hair, or a *d*art) shown in Figure 1

Our algorithm relies on new structural properties of BCD–free graphs which are obtained by a thorough study of their $P_4$–structure. The results are based on the concept of $p$–connectedness of graphs, which has proved in the past as an extremely powerful tool for the purpose of graph decomposition and for the structural and algorithmic study of graphs with a simple $P_4$–structure (see e.g. the survey paper [3]).

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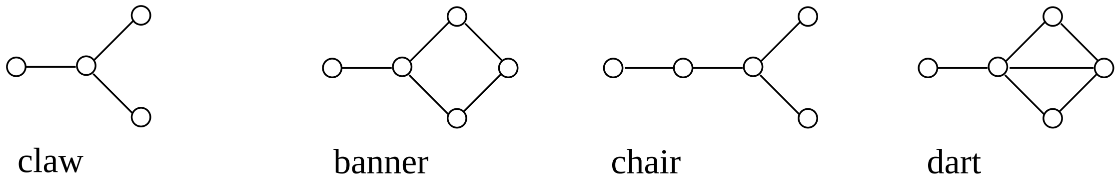



**Fig. 1:** Claw, banner, chair, and dart

The original motivation for the reconstruction problem stems from results concerning the perfectness of graphs. The need for a succinct certificate of perfectness and the observation that odd cycles have unique $P_4$–structure inspired Chvátal to conjecture that a graph is perfect if and only if it has the $P_4$–structure of a perfect graph [8]. This conjecture has been proved by Reed and is known as the Semi–Strong Perfect Graph Theorem [14].

As a byproduct, we will point out a consequence for the class $\mathcal{C}$ of graphs having the $P_4$–structure of a claw–free graph, the most interesting case of BCD–free graphs. First note that graphs belonging to $\mathcal{C}$ need not be claw–free. In fact, $\mathcal{C}$ contains the complements of claw–free graphs (and thus is significantly larger than the class of claw–free graphs). This follows from the observation that a graph and its complement have the same $P_4$–structure. However, $\mathcal{C}$ contains also many graphs such that neither the graph nor its complement are claw–free.

Since claw–free graphs are not perfect in general, graphs belonging to $\mathcal{C}$ need not be perfect. However, the Semi–Strong Perfect Graph Theorem implies that the graph $G \in \mathcal{C}$ is perfect if and only if a claw–free graph $P_4$–isomorphic to $G$ is perfect. Now, since the perfectness of claw–free graphs can be tested efficiently [9], we obtain in this way a criterion for perfectness for graphs in $\mathcal{C}$: Given $G \in \mathcal{C}$, construct the hypergraph $\mathcal{H}$ representing the $P_4$–structure of $G$, and then reconstruct a claw–free graph $H$ having $\mathcal{H}$ as its $P_4$–structure and use the algorithm in [9] for testing perfectness of $H$, and thus for testing perfectness of $G$.

However, a problem remains open: To describe the graphs belonging to $\mathcal{C}$. That is, characterize graphs $P_4$–isomorphic to a claw–free graph. Graphs $P_4$–isomorphic to a tree, a forest, a bipartite graph, a split graph, respectively, are described in [5, 6, 13].

In the next section we recall the notions of $p$–connected graphs and of homogeneous sets which are important tools in our discussion. In Section 3 we study the $p$–connected components of BCD–free graphs, which are of particular interest for the reconstruction problem. In Section 4 we develop the main idea for the reconstruction of $p$–connected BCD–free graphs. In Section 5 we present the algorithm as a whole. The first step consists of the identification of special types of graphs. The second step is an incremental procedure which tries to find some suitable starting graph and successively adds the remaining vertices.

## 2 Basics

We assume familiarity with standard graph–theoretical notions as in [11]. In the following, a $P_k$ always stands for an *induced path* on $k$ vertices and $k-1$ edges. Following [12], a graph $G = (V,E)$ is *p–connected* if for every partition of $V$ into nonempty disjoint sets $V_1, V_2$ there exists a $P_4$ containing vertices from both sets in the partition. Such a $P_4$ is termed as *crossing* between $V_1$ and $V_2$. The *p–connected* *components* of a graph are the maximal induced $p$–connected subgraphs. Note that the $p$–connected components are closed under complementation and are connected subgraphs of both $G$ and $\overline{G}$. Furthermore, it is easy to see that each graph has a unique partition into its $p$–connected components.

A $p$–connected graph is called *separable* if its vertex set $V$ can be partitioned into two nonempty disjoint sets $V_1$ and $V_2$ in such a way that each crossing $P_4$ has its midpoints in $V_1$ and its endpoints in $V_2$. This partition is commonly written as $(V_1, V_2)$ and called the *separation* of $G$. It is obvious that the complement of a separable $p$–connected graph is also separable. The separation $(V_1, V_2)$ of $G$ becomes $(V_2, V_1)$ in $\overline{G}$.

A subset $H$ of $V$ with $1 < |H| < |V|$ is called *homogeneous* if every vertex outside $H$ is either adjacent to all vertices from $H$ or to none of them. A homogeneous set $H$ is *maximal* if no other homogeneous set properly contains $H$. The graph $G^*$ obtained from a $p$–connected graph $G$ by shrinking every maximal homogeneous set to one single vertex is called the *characteristic graph* of $G$. Clearly, $G^*$ is also $p$–connected.

The notions of $p$–connectedness and homogeneous set will be the basic tools for the study of the $P_4$–structure of BCD–free graphs. In [2], this approach has already turned out to be very useful for recognizing the $P_4$–structure of bipartite graphs.

# 3 The structure of BCD–free graphs

We start with an observation about homogeneous sets in $p$–connected BCD–free graphs. It turns out that these sets are of a very simple structure.

**Proposition 3.1** *Every homogeneous set in a $p$–connected BC–free graph is a clique.*

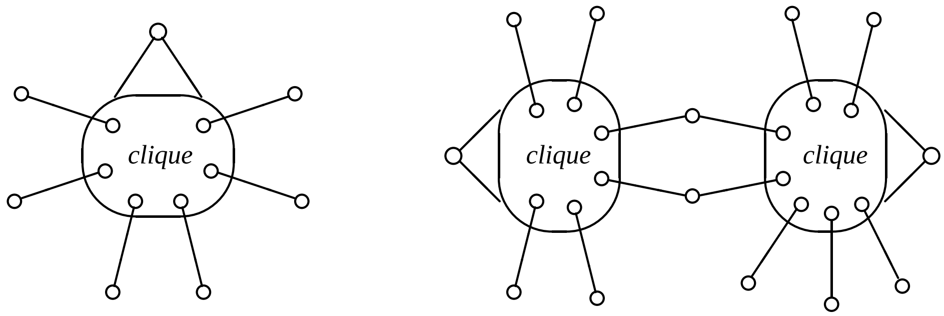
**Proof.** Let $H$ be a homogeneous set in a $p$–connected graph. We consider a $P_4$ which is crossing between $H$ and $V - H$. Clearly, this $P_4$ contains precisely one vertex from $H$. If $H$ contains two nonadjacent vertices, then there is a chair (if the $P_4$ has an end–point in $H$), or a banner (if the $P_4$ has a mid–point in $H$). This proves the claim. □

For further properties of BCD–free graphs, we need the notion of a *spider*¶. This is a graph consisting of a clique of size at least two (the *inner* vertices) and a stable set of equal size (the *outer* vertices) such that every vertex of the clique has precisely one neighbor in the stable set (each such pair of vertices is a *leg*). Furthermore, there is one additional vertex, called the *head* of the spider, which is adjacent precisely to the inner vertices (see Figure 2(a)).

For our purposes it suffices to study the case where the graph contains a stable set with at least three vertices (the recognition of the $P_4$–structure of triangle–free graphs has been solved in [2]; this immediately implies an algorithm for graphs containing no stable set with three vertices). Thereby, the following variants of spiders will play a special role. A *spider with a long leg* is a spider where one leg is replaced by a $P_3$ or a $P_4$. In other words, we subdivide one end–edge by one or two vertices. A *double–spider* consists of two spiders where certain pairs of outer vertices are identified (see Figure 2(b)). Finally, a *3–sun* is a cycle consisting of six vertices and three chords which form a triangle.

**Theorem 3.2** *Let $G$ be a $p$–connected BCD–free graph. If $\alpha(G) \geq 3$ then precisely one of the following statements holds:*

¶ Graphs which are defined in a quite similar way and which are called thin and thick spiders play a crucial role in the theory of $p$–connected graphs, see [3]




(a) *A spider* (b) *A double–spider*

**Fig. 2:** Spiders and Double–Spiders

$(i)$ *G results from a 3–sun or from a headless spider with at least three legs, by replacing the vertices by cliques;*

$(ii)$ *G results from a spider with a long leg or from a double–spider, by replacing the vertices by cliques (the heads of the spiders may be missing);*

$(iii)$ *G contains one of the graphs from Figure 3 as an induced subgraph.*

Because of the long and technical proof, we will divide Theorem 3.2 into two Lemmas 3.3 and 3.4 below, according to whether the graph contains a $P_5$ or not. Theorem 3.2 then follows from these lemmas and Proposition 3.1.

**Lemma 3.3** *Let G be a p–connected BCD–free graph with* $\alpha(G) \geq 3$. *If G is* $P_5$*–free then* $G^*$ *is a headless spider with at least three legs or a* 3*–sun.*

**Proof.** It easily follows from Proposition 3.1 that $\alpha(G^*) \geq 3$ (note that a $P_4$ and a stable set in $G$ contain not more than one vertex from each homogeneous set). Let $S$ denote a maximum stable set and $K$ the remaining vertices in $G^*$. Then $|S| \geq 3$ holds.

We denote by $B$ the bipartite subgraph of $G^*$ containing only the edges between $S$ and $K$. By the maximality of $S$, every vertex $x$ of $K$ has at least one neighbor in $S$. Hence

$$1 \leq d_S(x) \quad \text{for all} \quad x \in K, \tag{1}$$

with $d_S(x)$ denoting the number of neighbors of $x$ in $S$. On the other side, since $G^*$ is connected, every vertex $a$ of $S$ must have at least one neighbor in $K$, i.e.

$$d_K(a) \geq 1 \quad \text{for all} \quad a \in S. \tag{2}$$

*Case 1. B is disconnected.*

First we show that

$$d_S(x) = 1 \quad \text{for all} \quad x \in K. \tag{3}$$

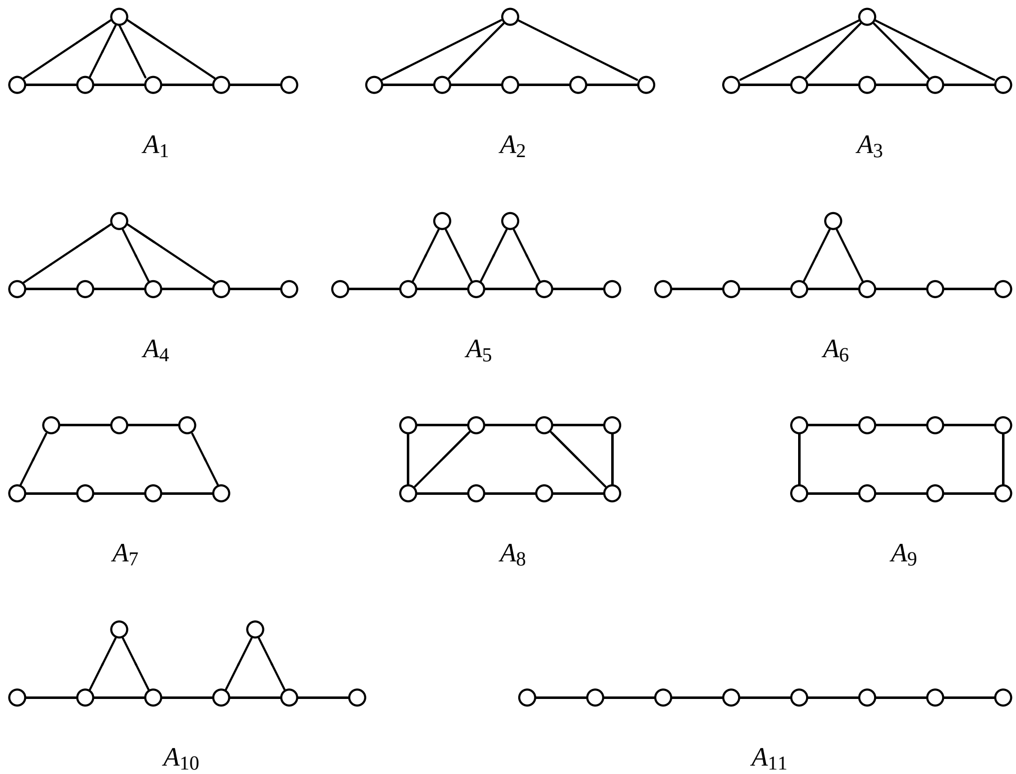



**Fig. 3:** *The starting graphs for the reconstruction procedure*

Otherwise let $d_S(x) \geq 2$ for a vertex $x \in K$ and $a, b$ be two neighbors of $x$ in $S$. We denote by $B_i$ the component of $B$ containing $x$. Let $c \in S$ be a vertex belonging to a different component $B_j$. A shortest path between $x$ and $c$ in $G^*$ consists of at most four vertices, otherwise there would be some $P_5$. On the other side, such a path consists of at least four vertices, otherwise $x$ would be adjacent to a vertex from $B_j$ which implies a chair centered in $x$. Let $xyzc$ be a shortest path. Note that $z$ is nonadjacent to both $a$ and $b$ (because $a, b$ and $c$ belong to different components of $B$), and $y$ must be adjacent to precisely one of $a, b$ (otherwise $G^*$ would have a chair or a dart). We can assume w.l.o.g. that $b$ and $y$ are adjacent. Since $c \in B_j$ and $y \in B_i$, these vertices must be nonadjacent. This provides a contradiction since now $axyzc$ is a $P_5$ in $G^*$.

Hence $B$ consists of at least three components, each containing precisely one vertex from $S$. Consider one of these components, say $B_i$, and let $a$ be the vertex from $B_i \cap S$. We claim that, in $G^*$,

$$\text{every vertex } x \in B_i \cap K \text{ has at least one neighbor outside } B_i. \tag{4}$$

Otherwise let $H$ contain all vertices from $B_i$ without neighbors outside $B_i$. In particular, $a$ belongs to $H$. Since $G^*$ contains no homogeneous sets there must be some vertex outside $H$, say $z$, which distinguishes two vertices from $H$. It is clear that $z$ belongs to $B_i$ and is adjacent to $a$ and nonadjacent to another vertex from $H$, say $x$. Let $y$ be a neighbor of $z$ outside $B_i$ and $c$ the neighbor of $y$ in $S$. Then $xazyc$ is a $P_5$ in $G^*$, which is a contradiction.

Now we show that, in $G^*$,

$$\text{every vertex } x \in B_i \cap K \text{ has a neighbor in } \textit{every} \text{ other component } B_j. \tag{5}$$

Otherwise let $x$ be nonadjacent to $B_j$. By (4) we know that $x$ has at least one neighbor $y$ outside $B_i$, say in $B_k$. Let $a$ and $b$ be the neighbors of $x$ and $y$ in $S$ and let $c \in B_j \cap S$. We consider a shortest path between $b$ and $c$. Clearly, this must be a $P_4$, say $by'zc$ (where $y = y'$ may be possible). Now a shortest path between $a$ and $c$ in the graph induced by $\{a,b,c,x,y,y',z\}$ contains a $P_5$. This is a contradiction.

Finally we prove that

$$d_K(a) = 1 \quad \text{for all } \ a \in S. \tag{6}$$

Otherwise let $H$ contain all neighbors of $a$. Since $H$ is not a homogeneous set, there must be a vertex outside $H$, say $z$, which distinguishes two vertices from $H$. Let $x,x' \in H$ such that $z$ is adjacent to $x$ and nonadjacent to $x'$. Suppose that $x,x' \in B_i$ and $z \in B_j$. By (5) we know that $x'$ must have a neighbor $y$ in $B_k$. Denote by $b$ and $c$ the neighbors of $y$ and $z$ in $S$. Note that $y$ and $z$ must be nonadjacent since otherwise we obtain a chair centered in $y$. Now a shortest path between $b$ and $c$ in the graph induced by $\{a,b,c,x,x',y,z\}$ contains a $P_5$. This is again a contradiction.

Hence every component of $B$ consists of one vertex from $S$ and one vertex from $K$. Now it follows from (5) that $K$ is a clique and therefore $G^*$ is a headless spider with at least three legs.

*Case 2. B is connected.*

First we show that

$$d_S(x) \geq 2 \text{ for all } x \in K. \tag{7}$$

Assume that there is a vertex $x \in K$ with $d_S(x) = 1$. We can assume that the neighbor $a \in S$ of $x$ is not adjacent to all vertices in $K$ (otherwise exchange $a$ and $x$, i.e. set $S := (S - \{a\}) \cup \{x\}$ and $K := (K - \{x\}) \cup \{a\}$; note that $x$ is not adjacent to all vertices from $K$ since otherwise $\{a,x\}$ would be a homogeneous set; if $B$ should now be disconnected then we are in Case 1). We will separate the discussion into two subcases, according to whether there is in $B$ a $P_6$ with $x$ as an end–point. In each case we will get a contradiction.

*Case 2.1. B contains an induced $P_6$ starting at x.*
Let $xaybzc$ be such a $P_6$ in $B$. Then $x,y,z \in K$ and $a,b,c \in S$. Now,

$$y \text{ and } z \text{ are adjacent},$$

otherwise $aybzc$ would be a $P_5$ in $G^*$, and

$$x \text{ cannot be adjacent to both } y \text{ and } z,$$

otherwise $b,c,x,y,z$ would induce a dart. If $x$ is adjacent to $z$ then there is a banner induced by $a,c,x,y,z$. If $x$ is nonadjacent to both $y$ and $z$ then $xayzc$ is a $P_5$. So we have

$$x \text{ is adjacent to } y \text{ and nonadjacent to } z.$$

Now, as $\{a,x\}$ is not homogeneous, there is a vertex $v$ which distinguishes $a$ and $x$. If $v$ is adjacent to $a$ then $v$ cannot be adjacent to both $b$ and $c$. Otherwise there is a chair. Therefore, $v$ is nonadjacent to $z$ (else $xavzb$ or $xavzc$ would be a $P_5$), and also nonadjacent to $y$ (else $v,a,x,y,z$ would induce a dart). But then $vayzc$ is a $P_5$ (if $v$ is nonadjacent to $c$), or $xavcz$ is a $P_5$ (otherwise). This contradiction shows that $v$ is adjacent to $x$. Again, $v$ cannot be adjacent to both $b$ and $c$ (else there is a chair at $v$). Therefore, $v$ cannot be adjacent to $y$ (else $v$ must be adjacent to $b$ and $z$ because $G$ has no dart; but then $czvxa$ is a $P_5$ in $G^*$), and hence $v$ is nonadjacent to $c$ (else $cvxyb$ would be a $P_5$). Now, $v,x,y,z,c$ induce a banner or a $P_5$. This final contradiction settles Case 2.1.

*Case 2.2. There is no induced $P_6$ in B starting at x.*
Among all neighbors of $a$ in $K$ choose a vertex $y$ with maximum $|N_S(y)|$.

Assume $N_S(y)=S$, and consider two vertices $b,c$ in $S-\{a\}$. Note that $x$ and $y$ are nonadjacent otherwise there is a chair. As $\{a,x\}$ is not a homogeneous set, there is a vertex $v$ which distinguishes $a$ and $x$. Now, $v$ cannot be adjacent to both $b$ and $c$, otherwise $a,b,c,v,x$ would induce a chair. Say, $v$ is nonadjacent to $b$. But then $a,b,x,y,v$ induce a dart (if $v$ is adjacent to $y$), or a chair or banner (otherwise). Thus, $N_S(y)\neq S$.

Let $c$ be a vertex in $S-N_S(y)$, and consider a $P_4=xazc$ in $B$ (recall that there is no longer induced path in $B$ between $x$ and $c$). By the choice of $y$, there exists a vertex $b$ in $S$ adjacent to $y$ but nonadjacent to $z$. We note that, in $G^*$,

$y$ and $z$ are adjacent,

otherwise $byazc$ would be a $P_5$, and

$x$ is adjacent to $y$ if and only if it is adjacent to $z$,

otherwise there is a dart. Now, consider a vertex $v\in K-N(a)$. Then

$v$ cannot be adjacent to both $b$ and $c$,

otherwise $xaybvc$ would be a $P_6$ in $B$ starting at $x$.

*Case 2.2.1. v is nonadjacent to b and adjacent to c*
In this case, $v$ cannot be adjacent to both $y$ and $z$, otherwise $a,b,v,y,z$ would induce a dart. On the other hand, $v$ must be adjacent to $y$ or $z$, otherwise $byzcv$ would be a $P_5$. If $v$ is nonadjacent to $z$ then $b,c,y,z$ and $v$ induce a banner. Thus we have

$v$ is adjacent to $z$ and nonadjacent to $y$.

Moreover,

$v$ is nonadjacent to $x$,

otherwise $x$ must be adjacent to $y$ (else $byaxv$ would be a $P_5$), hence also to $z$. But then $byxvc$ is a $P_5$. Now, since $G^*$ has no homogeneous set, there is a vertex $w$ which distinguishes $v$ and $c$.

Assume first that $w$ is adjacent to $c$. Then

$w$ is nonadjacent to $z$,

otherwise $w$ must be adjacent to $y$ (else $y,z,c,v,w$ would induce a dart), and to $a$ (else $aywcv$ would be a $P_5$), and to $b$ (else $bywcv$ would be a $P_5$). But then $a,b,c,w$ and $v$ induce a chair. Moreover,

$w$ is nonadjacent to $a$,

otherwise $w$ is nonadjacent to $b$ (else $a,b,c,w$ and $v$ would induce a chair). But then $b,c,y,z$ and $w$ induce a $P_5$ or a banner. Therefore,

$$w \text{ is nonadjacent to } y,$$

otherwise $aywcv$ would be a $P_5$: But then $aybwc$ is a $P_5$ (if $w$ is adjacent to $b$, or $byzcw$ is a $P_5$ (otherwise). Thus, we must have

$$w \text{ is nonadjacent to } c \text{ and adjacent to } v.$$

Then

$$w \text{ is nonadjacent to } z,$$

otherwise $w$ must be adjacent to $a$ and $y$ (else $a,c,v,z,w$ or $y,c,v,z,w$ would induce a dart), and to $b$ (else $bywvc$ would be a $P_5$). But then $a,b,c,v$ and $w$ induce a chair. And

$$w \text{ is nonadjacent to } y,$$

otherwise $w$ must be adjacent to $b$ (else $b,y,z,v,w$ induce a banner), and to $a$ (else $bwvza$ would induce a $P_5$). But then $a,b,c,v$ and $w$ induce a chair.

Now, $byzvw$ is a $P_5$ (if $w$ is nonadjacent to $b$), or $ybwvc$ is a $P_5$ (otherwise). This final contradiction settles Case 2.2.1. By symmetry, the case where $v$ is adjacent to $b$ and nonadjacent to $c$ cannot occur.

*Case 2.2.2. $v$ is nonadjacent to both $b$ and $c$*
Let $d$ be a neighbor of $v$ in $S$. If $d$ is adjacent to both $y$ and $z$ then $d,a,y,z,c$ induce a dart. If $d$ is adjacent to precisely one of $y,z$ then $d,y,z,b,c$ induce a chair. Thus,

$$d \text{ is nonadjacent to } y \text{ and } z.$$

Consider a $P_4 = xawd$ in $B$. We have seen that $w \in K - \{y,z\}$. Moreover,

$$w \text{ is nonadjacent to } b \text{ and } c,$$

otherwise $xaybwd$ or $xazcwd$ is a $P_6$ in $B$ starting at $x$, and

$$w \text{ is adjacent to } y \text{ and } z,$$

otherwise $byawd$ or $czawd$ would be a $P_5$ in $G^*$. Now we are in the Case 2.2.1 by replacing $c$ by $d$ and $z$ by $w$. Case 2.2.2 is settled, and (7) is completely proved.

Now we show that

$$\text{no two vertices from } K \text{ have the same neighborhood in } S. \tag{8}$$

Let $H$ denote the set of all vertices from $K$ which have the same neighbors in $S$. If $|H| \geq 2$ then, since $H$ is not a homogeneous set, there must exist a vertex $z$ outside $H$ which distinguishes two vertices $x$ and $y$ from $H$, say $z$ is adjacent to $x$ and nonadjacent to $y$. If $z$ is nonadjacent to two vertices in $N_S(x) = N_S(y)$ then there is a banner or a dart. Thus, $z$ is nonadjacent to at most one vertex in $N_S(x)$. If $z$ is nonadjacent to the vertex $a \in N_S(x)$ then $|N_S(x)| = |N_S(y)| = 2$, otherwise two vertices in $N_S(x) - \{a\}$ together with $x,z,a$ would induce a dart. By (7), $z$ is adjacent to a further vertex $c \in S - N_S(x)$. But then $aybzc$ is a $P_5$ where $b$ is the vertex in $N_S(x) - \{a\}$. Thus, $z$ must be adjacent to all vertices in $N_S(x)$. Since $z \notin H$, there must be a vertex in $S - N_S(x)$ adjacent to $z$. This vertex together with $z, x$ and two vertices in $N_S(x)$ induce a dart. This contradiction proves (8).

Next we prove that

$$\text{every two vertices from } K \text{ have a common neighbor in } S. \tag{9}$$

Assume the contrary and let $x, y \in K$ with disjoint neighborhoods in $S$. Consider $a, b \in N_S(x)$, $c, d \in N_S(y)$. Since $G^*$ contains no chair, $x$ and $y$ are nonadjacent. Since $G^*$ contains no $P_5$, a shortest path between $x$ and $y$ consists of three or four vertices. Assume first that $xzy$ is a shortest path. Then $z$ must be adjacent to exactly one vertex from $\{a, b\}$, say $b$ (otherwise there is a chair or a dart), and to exactly one vertex from $\{c, d\}$, say $c$ (by the same reason). But now $axzyd$ is a $P_5$. Hence a shortest path is of the form $xzz'y$. If $z$ or $z'$ belongs to $S$, say $z$, then $z'$ must be adjacent to $a$ or $b$ (else there is a chair). But then there is a banner. Thus, $z$ and $z'$ both belong to $K$. Now, if $a$ is adjacent to $z'$ then $z'$ must be adjacent to $c$ and $d$ (otherwise $xaz'yc$ or $xaz'yd$ would be a $P_5$). But then $x, a, z', c$ and $d$ induce a chair. Hence $a, b$ are nonadjacent to $z'$, and so, $z$ must be adjacent to both $a$ and $b$ (otherwise $axzz'y$ or $bxzz'y$ would be a $P_5$). But then $a, b, x, z$ and $z'$ induce a dart. This contradiction proves (9).

Finally, we claim that

$$d_S(x) = 2 \text{ for all } x \in K. \tag{10}$$

Assume $d_S(x) \geq 3$ for some $x \in K$. As $G^*$ is $p$–connected, there is some vertex $y \in K - \{x\}$. Moreover, by (8) and (9), $N_S(x) \cap N_S(y) \neq \emptyset$, and at least one of $N_S(x) - N_S(y)$, $N_S(y) - N_S(x)$ is nonempty. Thus, if $|N_S(x) \cap N_S(y)| \geq 2$ then there is a banner (if $x$ and $y$ are nonadjacent) or a dart (otherwise). If $|N_S(x) \cap N_S(y)| = 1$ then both $N_S(x) - N_S(y)$ and $N_S(y) - N_S(x)$ are nonempty where the first one contains, by assumption, at least two vertices. But then there is a chair. This proves (10).

Now let $a, b, c \in S$ and $x, y \in K$ such that $x$ is adjacent to $a, b$ and $y$ is adjacent to $b, c$. It is clear that $x$ and $y$ must be adjacent, otherwise there is a $P_5$. Note that there must be further vertices in $G^*$, otherwise $G^*$ is not $p$–connected. It follows from (8) and (9) that there are either vertices $z_i$, $1 \leq i \leq k$, each being adjacent to $b$ and some vertex $d_i \in S$, or precisely one vertex $z$ which is adjacent to $a$ and $c$. Now both $\{x, y, z_1, \ldots, z_k\}$ and $\{x, y, z\}$ must induce a clique, otherwise there is again a $P_5$. In the first case $G^*$ is not $p$–connected since there is no $P_4$ containing $b$. In the second case we obtain a 3–sun. □

**Lemma 3.4** *Let $G$ be a $p$–connected BCD–free graph. If $G$ contains an induced $P_5$ then precisely one of the following statements holds.*

*(i) $G^*$ is a spider with a long leg, or a double–spider;*

*(ii) $G$ contains one of the graphs from Figure 3.*

We will make use of the following observation in proving Lemma 3.4.

**Observation 3.5** *Let $m \geq 5$ be a fixed integer. Let $H$ be a BCD–free graph having no homogeneous set. Let $P = v_1 v_2 \cdots v_m$ be an arbitrary induced path in $H$. Then precisely one of the following statements holds.*

*(i) $H$ contains one of the graphs from $A_1, A_2, A_3, A_4, A_7, A_8, A_9$ and $A_{11}$ from Figure 3;*

*(ii) For any vertex $v$ outside $P$ and any $2 \le k \le m-1$, $\{v, v_k\}$ is not a homogeneous set in the graph $H[P+v]$ induced by $P \cup \{v\}$. Moreover, if $H$ has no $P_{m+1}$ then $\{v, v_k\}$ is not a homogeneous set in $H[P+v]$ for all $k$.*

**Proof.** (of Observation 3.5) Assume that (i) is false. Suppose there is some $k$ such that $\{v, v_k\}$ is a homogeneous set in $H[P+v]$. Then, since $H$ has no homogeneous set, there is a vertex $w$ outside $H[P+v]$ which distinguishes $v$ and $v_k$. We may assume that $w$ is adjacent to $v_k$ and nonadjacent to $v$.

First, consider the case $k = 1$ , i.e. $v$ is adjacent to $v_2$ and nonadjacent to all $v_j, j \ge 3$. Then $v$ must be adjacent to $v_1$ otherwise $v, v_1, v_2, v_3, v_4$ would induce a chair. Now, $w$ cannot be adjacent to $v_3$ otherwise there is a $P_{m+1}$ starting at $v$ (if $w$ is nonadjacent to any $v_j$, $j \ge 4$), or a chair (if $w$ is adjacent to a $v_j$ for some $j \ge 5$), or an $A_1$ or $A_4$ induced by $v_1, \ldots, v_5$ and $w$ (if $w$ is adjacent to $v_4$ and nonadjacent to $v_5$). Therefore, $w$ is nonadjacent to $v_2$ otherwise there is a dart, and so $w$ is also nonadjacent to $v_4$ otherwise $v, v_1, \ldots, v_4$ and $w$ would induce an $A_2$. Since $H$ does not contain a $P_{m+1}$, there is a smallest $j$, $5 \le j \le m$, such that $w$ is adjacent to $v_j$. Then $j = 5$ otherwise there is an $A_7$ or $A_9$ or $A_{11}$. Now, if $m \ge 6$ then there is a chair (if $w$ is nonadjacent to $v_6$) or an $A_8$ (otherwise). Therefore $m = 5$. But then $vv_1wv_5v_4v_3$ is a $P_{m+1}$. This final contradiction shows that $\{v, v_1\}$ cannot be a homogeneous set in $H[P+v]$. By symmetry, the case $k = m$ is also settled.

Consider the case $k = 2$. Note that $v$ and $v_2$ are adjacent otherwise there is a banner. As $H$ has no dart, $w$ must be adjacent to at least one of $v_1$ and $v_3$. Assume first that $wv_1$ and $wv_3$ both are edges of $H$. Then $w$ must be adjacent to $v_4$ otherwise there is a banner, and nonadjacent to $v_5$ by the same reason. But then $v_1, \ldots, v_5$ and $w$ induce an $A_1$. Next, assume that $w$ is adjacent to $v_1$ but nonadjacent to $v_3$. Then $w$ is nonadjacent to $v_4$ otherwise there is a chair (if $w$ is nonadjacent to $v_5$) or an $A_3$ (if $w$ is adjacent to $v_5$). But then $w, v_1, v, v_3, v_4$ and $v_2$ induce an $A_1$. Finally, assume that $w$ is nonadjacent to $v_1$ but adjacent to $v_3$. Then $w$ is adjacent to $v_4$ otherwise there is a dart. But then $v, v_2, w, v_4, v_5$ and $v_3$ (if $w$ is adjacent to $v_5$), or $v_1, \ldots, v_5$ and $w$ (otherwise) induce an $A_1$. The case $k = 2$, and by symmetry, also the case $k = m-1$ is settled. Now, consider the case $3 \le k \le m-2$. Again, $v$ and $v_k$ are adjacent otherwise there is a banner. $w$ must be adjacent to at least one of $v_{k-1}$ and $v_{k+1}$ otherwise there is a dart, say $wv_{k-1} \in E(H)$. Then $w$ must be adjacent to $v_{k-2}$ otherwise there is a dart. By the same reason, $w$ is nonadjacent to $v_{k+2}$, hence $w$ is also nonadjacent to $v_{k+1}$ otherwise there is a banner. But then $w, v_{k-1}, v, v_{k+1}, v_{k+2}$ and $v_k$ induce an $A_1$. This contradiction completes the proof of the Observation. □

**Proof.** (of Lemma 3.4) Assume (ii) is false. We will prove that (i) holds. Note that the assumption implies that $G^*$ also does not contain a graph from Figure 3 because $G^*$ is (isomorphic to) an induced subgraph of $G$.

Let $P = v_1v_2 \cdots v_m$ be a longest induced path in $G^*$. By the assumption, $5 \le m \le 7$. If $G^* = P$ then $G^*$ is a spider with a long leg (if $m \ne 7$), or a double–spider (if $m = 7$), and we are done. Suppose $G^* \ne P$. Then, since $G^*$ is connected, there exists a vertex in $G^* - P$ adjacent to a vertex in $P$. We first note that, for every vertex $v$ outside $P$,

$$\text{if } N_P(v) \ne \emptyset \text{ then } 2 \le |N_P(v)| \le 3. \tag{11}$$

Otherwise $G^*$ would have a $P_{m+1}$ or a chair (if $|N_P(v)| = 1$), or an $A_1$, $A_3$,$A_4$, $A_5$ (if $|N_P(v)| = 4$), or a dart or a chair (if $|N_P(v)| \ge 5$). Next, assume $|N_P(v)| = 2$. Then $N_P(v) \ne \{v_1, v_2\}$ and $N_P(v) \ne \{v_{m-1}, v_m\}$. Otherwise $\{v, v_1\}$, respectively, $\{v, v_m\}$ would be a homogeneous set in the graph induced by $P \cup \{v\}$, contradicting Observation 3.5. Moreover, if $N_P(v) = \{v_1, v_m\}$ then $m = 5$ otherwise $G^*$ would have an $A_7$ or an $A_9$. Furthermore, if $N_P(v) \ne \{v_1, v_m\}$ then the two vertices from $N_P(v)$ must be adjacent otherwise

$G^*$ would have a banner. If $N_P(v) = \{v_k, v_{k+1}\}$ then $k = 2$ or $k = m-2$, otherwise $G^*$ would have an $A_6$. In summary, we have:

$$\text{If } |N_P(v)| = 2 \text{ then } N_P(v) = \{v_2, v_3\} \text{ or } \{v_{m-2}, v_{m-1}\}, \text{ or } m = 5 \text{ and } N_P(v) = \{v_1, v_5\}. \tag{12}$$

We now consider vertices $v$ with $|N_P(v)| = 3$. First, $N_P(v)$ cannot form a subpath $v_{k-1}v_kv_{k+1}$ of $P$ otherwise $\{v, v_k\}$ would be a homogeneous set in the graph induced by $P \cup \{v\}$, contradicting Observation 3.5. Thus, we have:

$$\begin{aligned}&\text{If } |N_P(v)| = 3 \text{ then } m \neq 5 \text{ and } N_P(v) = \{v_1, v_5, v_6\}, \text{ or } m = 6 \text{ and } N_P(v) = \{v_1, v_2, v_6\},\\ &\text{or } m = 7 \text{ and } N_P(v) = \{v_2, v_3, v_7\}.\end{aligned} \tag{13}$$

Otherwise $G^*$ would have an $A_2$, $A_4$ or a banner. For convenience, we will use the following notion: For $I \subseteq \{1, \ldots, m\}$ let $M_I$ denote the set of all vertices $v$ in $G^*$ such that $N_P(v) = \{v_i \mid i \in I\}$. We also write $M_{126}$ instead of $M_{\{1,2,6\}}$, etc. With this notion, we have:

$$M_{23}, M_{m-2;m-1}, M_{126}, M_{156} \text{ and } M_{237} \text{ induce cliques.} \tag{14}$$

Otherwise $G^*$ would have a dart. We now discuss three subcases according to the possiblities of $m$.

*Case 1.* $m = 5$

That is, $G$ has no $P_6$. Assume that there is a vertex $v$ with $N_P(v) = \{v_1, v_5\}$. Then $P \cup \{v\}$ induces a $C_6$ $C$ and by applying Observation 3.5 for the $P_5$'s on $C$ it follows that $G^* = C$. Thus, $G^*$ is a double–spider. Hence we may assume that $M_{15} = \emptyset$. Then, (11), (12) and (13) yields:

$$\text{If } N_P(v) \neq \emptyset \text{ then } N_P(v) = \{v_2, v_3\}, \text{ or } N_P(v) = \{v_3, v_4\}.$$

Since $G^*$ has no $A_5$ and no $P_6$,

$$\text{if } M_{23} \neq \emptyset \text{ then } M_{34} = \emptyset, \text{ and vice versa.}$$

By symmetry, we may assume that $M_{23} \neq \emptyset$. Now, we are going to show that $G^*$ is a spider with a long leg. By (14),

$$\text{at most one vertex in } M_{23} \text{ has no neighbor in } G^* - (P \cup M_{23}),$$

otherwise these vertices would form a homogeneous set in $G^*$. Moreover,

$$\text{every vertex in } M_{23} \text{ has at most one neighbor in } G^* - (P \cup M_{23}),$$

for, if $x \in M_{23}$ has two neighbors $x', x''$ in $G^* - (P \cup M_{23})$ then $\{x, x''\}$ would be a homogeneous set of the graph induced by the $P_5 = x'xv_3v_4v_5$ and $x''$, contradicting Observation 3.5. Furthermore,

$$\text{no two vertices in } M_{23} \text{ have a common neighbor in } G^* - (P \cup M_{23}),$$

For, if $x_1, x_2 \in M_{23}$ are adjacent to $y \in G^* - (P \cup M_{23})$ then $\{x_1, x_2\}$ would be a homogeneous set of the graph induced by the $P_5 = yx_1v_3v_4v_5$ and $x_2$, contradicting Observation 3.5.

Let $N$ be the set of vertices in $G^* - (P \cup M_{23})$ adjacent to a vertex in $M_{23}$. Since $G^*$ has no $P_6$, the facts above show that $V(G^*) = V(P) \cup M_{23} \cup N$, and no two vertices in $N$ are adjacent. That is, $G^*$ is a spider with a long leg. Case 1 is settled.

*Case 2.* $m = 6$
In this case, $G^*$ is $P_7$–free. By (11), every vertex outside $P$ has exactly 0, 2 or 3 neighbors in $P$, and by (12),

$$\text{if } |N_P(v)| = 2 \text{ then } N_P(v) = \{v_2, v_3\} \text{ or } N_P(v) = \{v_4, v_5\}.$$

By (13),

$$\text{if } |N_P(v)| = 3 \text{ then } N_P(v) = \{v_1, v_5, v_6\} \text{ or } N_P(v) = \{v_1, v_2, v_6\}.$$

Since $G^*$ has no $A_{10}$ and no $A_4$, we have:

$$\text{If } M_{23} \neq \emptyset \text{ then } M_{45} = \emptyset \text{ and vice versa.} \qquad (15)$$

Furthermore,

$$\text{if } M_{126} \neq \emptyset \text{ then } M_{156} = \emptyset \text{ and vice versa.} \qquad (16)$$

For, if there exists $v \in M_{126}$ and $v' \in M_{156}$ then $v$ and $v'$ are adjacent otherwise $G^*$ would have a $A_4$ induced by $v_1, v_2, v_3, v_6, v$ and $v'$. But then $\{v, v_1\}$ is a homogeneous set in the graph induced by the $P_5 = v_5 v' v_1 v_2 v_3$ and $v$, contradicting Observation 3.5. This shows (16). Next we have

$$\text{if } M_{23} \neq \emptyset \text{ then } M_{126} = \emptyset \text{ and vice versa.} \qquad (17)$$

For, if there exists $v \in M_{23}$ and $w \in M_{126}$ then $G^*$ would have a $P_7 = v v_3 v_4 v_5 v_6 w v_1$ (if $v$ and $w$ are nonadjacent) or a dart (otherwise). The facts (15), (16) and (17), and the symmetry allow us to assume that $M_{45} = \emptyset$ and $M_{126} = \emptyset$. Now,

$$\text{no vertex in } M_{156} \text{ has a neighbor in } G^* - (P \cup M_{156}), \qquad (18)$$

otherwise $G^*$ would have a chair. By (14) and (18),

$$|M_{156}| \leq 1,$$

otherwise $M_{156}$ would be a homogeneous set in $G^*$. Furthermore,

$$\text{if } |M_{156}| = 1 \text{ then no vertex in } M_{23} \text{ has a neighbor in } G^* - (P \cup M_{23}),$$

because, if $v \in M_{23}$ has a neighbor $v' \in G^* - (P \cup M_{23})$ then, by (18), $v' \notin M_{156}$ and is nonadjacent to the vertex $w$ of $M_{156}$. But then $G^*$ has a $P_7 = v' v v_3 v_4 v_5 w v_1$. Thus,

$$\text{if } |M_{156}| = 1 \text{ then } |M_{23}| \leq 1,$$

otherwise $M_{23}$ would be a homogeneous set in $G^*$. It follows that if $|M_{156}| = 1$ then $V(G^*) = V(P) \cup M_{23} \cup M_{156}$ and hence $G^*$ is a double–spider.

If $|M_{156}| = 0$ then, as in the case $m = 5$, by using Observation 3.5, one can see that

> every vertex in $M_{23}$ has at most one neighbor in $G^* - (P \cup M_{23})$, and no two vertices in $M_{23}$ have a common neighbor in $G^* - (P \cup M_{23})$.

Since $M_{23}$ induces a clique,

$$\text{at most one vertex in } M_{23} \text{ has no neighbor in } G^* - (P \cup M_{23}),$$

otherwise these vertices would form a homogeneous set in $G^*$.

Let $N$ be the set of all vertices in $G^* - (P \cup M_{23})$ adjacent to a vertex in $M_{23}$. As in the case $m = 5$, one can see that no two vertices in $N$ are adjacent, and $V(G^*) = V(P) \cup M_{23} \cup N$. That is $G^*$ is a spider with a long leg. Case 2 is settled.

*Case 3.* $m = 7$
In this case we have, for every vertex $v$ outside $P$ such that $N_P(v) \neq \emptyset$,

> if $|N_P(v)| = 2$ then $N_P(v) = \{v_2, v_3\}$ or $N_P(v) = \{v_5, v_6\}$, and if $|N_P(v)| = 3$ then $N_P(v) = \{v_1, v_5, v_6\}$ or $N_P(v) = \{v_2, v_3, v_7\}$.

Recall that, by (14), the sets $M_{23}, M_{56}, M_{156}$, and $M_{237}$ induce cliques. We now are going to show that $G^*$ is a double–spider.

First, we have

> $M_{56} \cup M_{156}$ and $M_{23} \cup M_{237}$ induce clique

(else $G^*$ would have a dart), and

> no vertex from $M_{56} \cup M_{156}$ is adjacent to every vertex from $M_{23} \cup M_{237}$

(else $G^*$ would have an $A_{10}$ or a chair). Next,

> no vertex from $M_{156}$ has a neighbor in $G^* - (P \cup M_{56} \cup M_{156})$

(else $G^*$ would have a chair), hence

> $|M_{156}| \leq 1$

(else $M_{156}$ would be a homogeneous set in $G^*$). By symmetry, we also have

> $|M_{237}| \leq 1$, and the vertex of $M_{237}$ (if any) has no neighbor in $G^* - (P \cup M_{23})$.

Now, as in the case $m = 5$, we can see that $G^*$ is a double–spider, where $M_{23} \cup M_{237} \cup \{v_2, v_3\}$ is the set of inner vertices of one spider and $M_{56} \cup M_{156} \cup \{v_5, v_6\}$ is the set of inner vertices of the other spider. □

# 4 The reconstruction technique

In this section assume that $\mathcal{H}$ is connected. We want to find a $p$–connected BCD–free graph $G$ such that the $P_4$–structure of $G$ is equal to $\mathcal{H}$. In the following we study the most interesting case where $G$ has stability number at least three and is none of the special graphs from Theorem 3.2 (i)–(ii).

A vertex $v$ is said to have a *partner* in a $P_4$, say $X$, if $v$ together with three vertices from $X$ induces a $P_4$. By checking all possible adjacencies between a vertex and a $P_4$ in a BC–free graph (see Figure 4) one easily realizes that there are four configurations where a vertex has precisely one partner (namely in cases (iv), (v), (vii) and (viii)), and one configuration where it has more than one partner (in case (vi)). We say that $v$ has a partner in a graph if this graph contains a $P_4$ which has a partner for $v$. The following statement will be crucial for the reconstruction of $p$–connected BCD–free graphs, but we formulate it here in a slightly more general way.

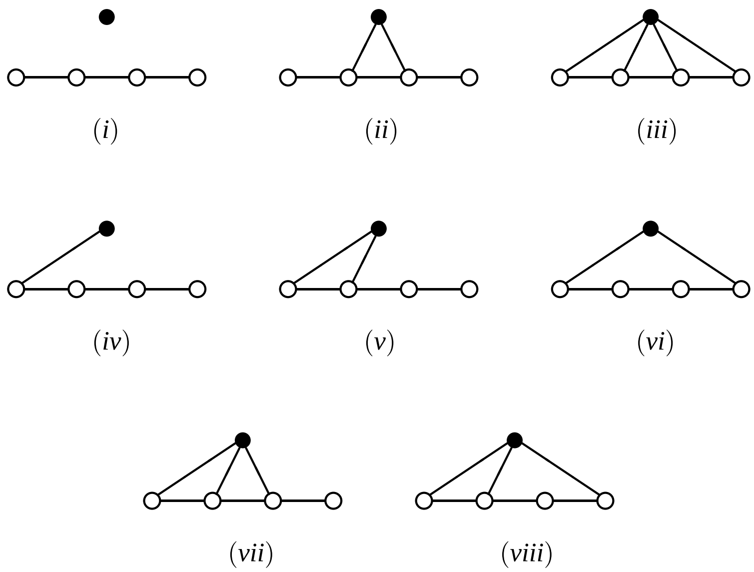



**Fig. 4:** *All adjacencies between a vertex and a $P_4$ in a BC–free graph*

**Lemma 4.1** *Let $G$ be a graph in which every homogeneous set induces a $P_4$–free graph, and let $G'$ be a $p$–connected proper induced subgraph in $G$. If $G'$ is not separable then there exists a vertex $v$ outside $G'$ such that $v$ has a partner in $G'$. In particular, $G' \cup \{v\}$ is $p$–connected.*

**Proof.** Since $G'$ contains a $P_4$ (as it is $p$–connected), it cannot be a homogeneous set of $G$. Hence there exists a vertex $v$ outside $G'$ adjacent to some vertex and nonadjacent to some vertex in $G'$. In particular, the sets $S_1 = N_{G'}(v)$, $S_2 = G' - S_1$ are nonempty. As $G'$ is $p$–connected, there is a $P_4$ crossing $S_1$ and $S_2$. As $G'$ is nonseparable, there is a $P_4$ $P$ crossing $S_1$ and $S_2$ such that if two mid–points of $P$ are in $S_1$ then one of the end–points of $P$ is also in $S_1$. Now, the graph induced by $P$ and $v$ has a $P_4$ containing $v$. That is, $P$ is a partner in $G'$ for $v$. □

Now we show that, in a BC–free graph, the adjacencies of a vertex to a $P_4$ can be determined in a unique way from the partial knowledge of the adjacencies and the $P_4$–structure. Let $X$ be a $P_4$ and $v$ a vertex from outside $X$. We shall say that

- $X$ is of *type 0* if $v$ has no partner in $X$ (see cases (i), (ii) and (iii) of Figure 4);
- $X$ is of *type 1* if $v$ has one partner in $X$ and the three vertices from $X$ which induce a $P_4$ with $v$ induce a $P_3$ (see cases (iv) and (v) of Figure 4);
- $X$ is of *type 2* if $v$ has one partner in $X$ and the three vertices from $X$ which induce a $P_4$ with $v$ induce a $\overline{P_3}$ (see cases (vii) and (viii) of Figure 4);
- $X$ is of *type 3* if $v$ has more than one partner in $X$ (see case (vi) of Figure 4).

More precisely, we prove the following:

**Lemma 4.2** *Let $X$ be a $P_4$ in a BC–free graph and $v$ a vertex outside $X$. If the adjacencies of $v$ with respect to three vertices from $X$ are known, then the adjacency to the fourth vertex from $X$ can be determined from the $P_4$–structure of $X \cup \{v\}$.*

**Proof.** Let $\{w,x,y,z\}$ be the vertex set of $X$ and assume that the adjacencies of $v$ to $x$, $y$ and $z$ are known. We have to find out whether $v$ and $w$ are adjacent or not.

Clearly, the type of $X$ can be determined from the $P_4$–structure of $X \cup \{v\}$. Assume first that $X$ is of type 0. If $v$ has no neighbors in $\{x,y,z\}$ then $v$ must be nonadjacent to $w$ (this corresponds to case (i)). If $v$ has one neighbor in $\{x,y,z\}$ then $v$ must be adjacent to $w$ (this is case (ii)). If $v$ has two neighbors in $\{x,y,z\}$ then $v$ is nonadjacent to $w$ (again case (ii)). Finally, if $v$ has three neighbors in $\{x,y,z\}$ then $v$ is adjacent to $w$ (this is case (iii)).

Assume now that $X$ is of type 1. Then $v$ has at most two neighbors in $\{x,y,z\}$. If $v$ has no neighbor in $\{x,y,z\}$ then $v$ must be adjacent to $w$ (see case (iv)). If $v$ has two neighbors in $\{x,y,z\}$ then $v$ must be nonadjacent to $w$ (see case (v)). If $v$ has one neighbor in $\{x,y,z\}$ then we have to distinguish two cases: If $X$ has edges $wx$, $xy$ and $yz$, i.e. $w$ is an endpoint of $X$, then $v$ is adjacent to $w$ if and only if $v$ is adjacent to $x$. If $X$ has edges $xw$, $wy$ and $yz$, i.e. $w$ is a midpoint of $X$, then $v$ is adjacent to $w$ if and only if $v$ is adjacent to $x$ and $\{v,w,y,z\}$ induces a $P_4$.

If $X$ is of type 2 then $v$ has two or three neighbors in $\{x,y,z\}$. In the first case, $v$ must be adjacent to $w$, in the second case $v$ must be nonadjacent to $w$ (see cases (vii) and (viii)).

Finally, if $X$ is of type 3 then $v$ has one or two neighbors in $\{x,y,z\}$. In the first case, $v$ must be adjacent to $w$, in the second case $v$ must be nonadjacent to $w$ (see case (vi)). □

Note that the previous two lemmas do not hold for arbitrary graphs.

Now we sketch the principle of the reconstruction procedure. First we try to find some *starting graph* $G'$, where $G'$ is a BCD–free graph realization of a subhypergraph $\mathcal{H}'$ of $\mathcal{H}$. This graph should have the property that the adjacencies of all vertices outside $G'$ with respect to $G'$ can be determined in a unique way from the $P_4$–structure. Furthermore, $G'$ should not be separable. The details how to find such a graph are spelled out later. Then we repeatedly extend the starting graph $G'$ by a vertex which has a partner in the current subgraph. If $G'$ is not separable then, by Proposition 3.1 and Lemma 4.1, such a vertex always exists (otherwise there is no realization of $\mathcal{H}$ as a BCD–free graph). Moreover, by Lemma 4.2, the adjacencies of a newly added vertex with respect to all the previously added vertices can be determined in a unique way (otherwise, again, there is no realization of $\mathcal{H}$ as a BCD–free graph).

More precisely, let $v_1,\ldots,v_k$ be a numbering of the vertices from $G-G'$ such that each $v_i$ induces a $P_4$ with some three vertices from $G' \cup \{v_1,\ldots,v_{i-1}\}$. We want to find out the neighbors of $v_j$ in $G' \cup \{v_1,\ldots,v_{j-1}\}$. For that purpose we first determine the neighbors of $v_j$ in $G'$ (which is possible by the above assumption). Now assume inductively that all neighbors of $v_j$ in $G' \cup \{v_1,\ldots,v_{i-1}\}$ are already known (with $1 \le i < j$). We consider a $P_4$, say $\{a,b,c,v_i\}$, with three vertices $a,b,c \in G' \cup \{v_1,\ldots,v_{i-1}\}$. By Lemma 4.2, it can be determined from the $P_4$–structure whether $v_j$ is adjacent to $v_i$ or not.

It remains to find some suitable starting graph. The choice of the starting graph is crucial. Not every graph is suitable in the sense that the adjacencies of the vertices outside are uniquely determined by the $P_4$–structure. We consider the collection of graphs depicted in Figure 3. Note that these graphs are $p$–connected, have stability number at least three and none of them is separable. Furthermore, as shown in

Theorem 3.2, at least one of them occurs as an induced subgraph in $G$. We now prove that all these graphs have the desired property.

**Lemma 4.3** *Let $G'$ be any of the graphs from Figure 3 and let $v$ be a vertex from outside $G'$. Then, in a BCD–free graph, the adjacencies of $v$ with respect to $G'$ can be determined in a unique way from the $P_4$–structure of $G' \cup \{v\}$.*

**Proof.** Consider first a $P_5$ in an arbitrary of the graphs $A_i$ from Figure 3. Once the adjacencies of $v$ to such a $P_5$ are known, we can proceed similarly as this has been proposed above. We extend the $P_5$ successively to the whole subgraph $A_i$ by adding a vertex which has a partner in the current subgraph and determine whether $v$ is adjacent to this vertex or not. In this way we obtain all adjacencies of $v$ with respect to $A_i$.

Denote the vertices of the $P_5$ in the natural order by $a,b,c,d,e$. Let further $X$ and $Y$ be the $P_4$s induced by $\{a,b,c,d\}$ resp. $\{b,c,d,e\}$ and $v$ be a vertex from outside. Assume first that $X$ is of type 0. If $Y$ is also of type 0 then $v$ must be nonadjacent to the $P_5$ (note that in a dart–free graph a vertex cannot be adjacent to all vertices of a $P_5$). If $Y$ is of type 1 then $v$ must induce a $P_4$ together with $c,d,e$. In this case $v$ is either adjacent to $b,c$ or to $e$ only. If $Y$ is of type 2 then $v$ must induce a $P_4$ with $b,d,e$. This means that $v$ is either adjacent to $a,b,c,d$ or to $b,c,e$. We can find out the correct alternative by checking whether $\{a,v,d,e\}$ induces a $P_4$ or not. Clearly $Y$ cannot be of type 3.

Assume now that $X$ is of type 1 and $v$ induces a $P_4$ with $a,b,c$ (in the following we omit the symmetric cases which only exchange the role of $X$ and $Y$). If $Y$ is also of type 1 then $v$ is adjacent to $a$ and $e$. If $Y$ is of type 2 then $v$ is adjacent to $c,d,e$. Again $Y$ cannot be of type 3. If $X$ is of type 1 and $v$ induces a $P_4$ with $b,c,d$ then $Y$ can only be of type 1 or 3. If $Y$ is of type 1 then $v$ is either adjacent to $a,b$ or to $d,e$. If $Y$ is of type 3 then $v$ is adjacent to $a,b,e$.

Assume that $X$ is of type 2 and $v$ induces a $P_4$ with $a,b,d$. If $Y$ is of type 2 and $\{b,v,d,e\}$ induces a $P_4$ then $v$ is adjacent to $b,c,d$. On the other side, if $Y$ is of type 2 and $\{b,c,v,e\}$ induces a $P_4$ then $v$ is adjacent to $a,c,d,e$. Assume that $X$ is of type 2 and $v$ induces a $P_4$ with $a,c,d$. If $Y$ is of type 2 and $\{b,v,e,d\}$ forms a $P_4$ then $v$ is adjacent to $a,b,c,e$. If $\{c,b,v,e\}$ forms a $P_4$ then $v$ is adjacent to $a,b,d,e$. Clearly, in both cases $Y$ cannot be of type 3. Finally, the case where both $X$ and $Y$ are of type 3 cannot occur.

We have shown that the neighbors of $v$ with respect to the $P_5$ can be determined from the $P_4$–structure with the exception of three cases, namely where

(i) $v$ is adjacent either to $b,c$ or to $e$

(ii) $v$ is adjacent either to $a$ or to $c,d$

(iii) $v$ is adjacent either to $a,b$ or to $d,e$.

We can find out the correct alternatives by considering also the remaining vertices of the subgraph $A_i$.

Consider the graph $A_1$ and let $f$ be the sixth vertex which is adjacent to $a,b,c,d$. In order to decide (i) note that in both of the two possible cases $v$ must be nonadjacent to $f$. If $\{v,b,f,d\}$ induces a $P_4$ then $v$ is adjacent to $b,c$, otherwise to $e$. For (ii) note that in the first case $v$ must be nonadjacent to $f$, in the second case $v$ must be adjacent to $f$. If $\{v,a,f,d\}$ induces a $P_4$ then $v$ is adjacent to $a$, otherwise to $c,d$. In (iii) we obtain that in the first case $v$ may be adjacent to $f$ or not, in the second case $v$ must be nonadjacent to $f$. If $\{v,a,f,c\}$ induces a $P_4$ then $v$ is adjacent to $a,b$. If this is not the case and if $\{v,f,d,e\}$ forms a $P_4$ then $v$ is adjacent to $a,b,f$, otherwise to $d,e$.

In $A_2$ the sixth vertex $f$ is adjacent to $a,b,e$. For (i) note that in the first case $v$ must be nonadjacent to $f$, in the second case $v$ must be adjacent to $f$. If $\{v,b,f,e\}$ induces a $P_4$ then $v$ is adjacent to $b,c$, otherwise to

$e$. For (ii) verify that in both cases $v$ must be nonadjacent to $f$. If $\{v,a,f,e\}$ induces a $P_4$ then $v$ is adjacent to $a$, otherwise to $c,d$. In (iii) we obtain that in the first case $v$ must be adjacent to $f$, in the second case $v$ may be adjacent to $f$ or not. If $\{c,b,f,v\}$ induces a $P_4$ then $v$ is adjacent to $d,e,f$. If this is not the case and if $\{b,f,e,v\}$ forms a $P_4$ then $v$ is adjacent to $d,e$, otherwise to $a,b$.

In $A_3$ the sixth vertex $f$ is adjacent to $a,b,d,e$. For (i) we see that in both cases $v$ must be nonadjacent to $f$. If $\{v,b,f,d\}$ induces a $P_4$ then $v$ is adjacent to $b,c$, otherwise to $e$. For (ii) again in both cases $v$ must be nonadjacent to $f$. If $\{v,a,f,e\}$ induces a $P_4$ then $v$ is adjacent to $a$, otherwise to $c,d$. In (iii) we obtain that in both cases $v$ must be adjacent to $f$. Now, if $\{v,f,d,c\}$ is a $P_4$ then $v$ is adjacent to $a,b$, otherwise to $d,e$.

Finally, in $A_4$ the sixth vertex $f$ is adjacent to $a,c,d$. In order to decide (i) note that in both of the two possible cases $v$ must be nonadjacent to $f$. Clearly, if $\{v,b,a,f\}$ induces a $P_4$ then $v$ is adjacent to $b,c$, otherwise to $e$. For (ii) note that in both cases $v$ must be adjacent to $f$. If $\{v,f,d,e\}$ induces a $P_4$ then $v$ is adjacent to $a$, otherwise to $c,d$. In (iii) we obtain that in the first case $v$ may be adjacent to $f$ or not, in the second case $v$ must be nonadjacent to $f$. If $\{v,f,d,e\}$ is a $P_4$ then $v$ is adjacent to $a,b,f$. If this is not the case and if $\{v,a,f,c\}$ is a $P_4$ then $v$ is adjacent to $a,b$, otherwise to $d,e$.

For the remaining graphs we have to use a slightly more involved argumentation (recall that we only want to find out the adjacencies of $v$ with respect to a $P_5$). Consider the graph $A_5$ and let $f$ and $g$ be the vertices which are adjacent to $b,c$ resp. $c,d$. For (i) we obtain that, in the first case, $v$ must be adjacent to $f$ whereas the adjacency to $g$ is open. In the second case $v$ is adjacent to at most one of $f$ and $g$. If $\{v,f,c,g\}$ induces a $P_4$ then the second case is the correct one. If $\{v,f,g,d\}$ induces a $P_4$ then we are in the first case. If neither of the two $P_4$s exists then we are in the second case if and only if $\{v,e,d,g\}$ induces a $P_4$. The decision of (ii) follows by a symmetry argument. In (iii), $v$ is adjacent to at most one of $f$ and $g$. If $\{v,g,d,e\}$ induces a $P_4$ then the first case is the correct one. If $\{e,v,g,c\}$ induces a $P_4$ then the second case is correct. If neither of the two $P_4$s exists then we are in the first case if and only if $\{v,b,c,g\}$ induces a $P_4$.

In $A_6$ let $f$ be the vertex which is adjacent to $e$ and $g$ the vertex which is adjacent to $c,d$. In the first case of (i) the vertex $v$ is adjacent to at most one of $f,g$. In the second case $v$ must be adjacent to $f$ and may also be adjacent to $g$. If $\{b,c,g,v\}$ induces a $P_4$ then we are in the second case. Otherwise, if $\{v,g,d,e\}$ does not induce a $P_4$ then we are in the first case. Finally, if $\{b,c,g,v\}$ does not induce a $P_4$ and $\{v,g,d,e\}$ induces a $P_4$ then we are in the first case if and only if $\{b,v,g,d\}$ induces a $P_4$. For (ii) note that in the second case $v$ must be adjacent to $g$. If $\{b,c,v,f\}$ induces a $P_4$ then we are in the second case. If $\{b,a,v,f\}$ induces a $P_4$ then we are in the first case. If both $P_4$s do not exist then the second case is the correct one if and only if $\{v,d,e,f\}$ induces a $P_4$. In (iii), if $\{v,g,d,e\}$ induces a $P_4$ then we are in the first case. Assume now that $\{v,d,g,e\}$ is no $P_4$. If $\{v,b,c,g\}$ does not induce a $P_4$ then we are in the second case. If $\{v,g,d,e\}$ is not a $P_4$ and $\{v,b,c,g\}$ is a $P_4$ then the second case is the correct one if and only if $\{c,g,v,e\}$ induces a $P_4$.

In $A_8$ let $f$ be adjacent to $e$ and $g$ adjacent to $a,b$ and $h$ adjacent to $e,f,g$. For (i) note that in the second case $v$ must be adjacent to $f$ and $h$. If $\{v,g,b,c\}$ is a $P_4$ then we are in the second case. Otherwise, if $\{v,h,g,a\}$ is not a $P_4$ then we are in the first case. Otherwise, if $\{v,h,g,b\}$ is a $P_4$ then we are in the second case. For (ii) note that, in both cases, if $v$ is adjacent to $h$ then it is also adjacent to $g$. If $\{v,a,g,h\}$ induces a $P_4$ then we are in the first case. Otherwise, if $\{v,g,b,c\}$ is not a $P_4$ then we are in the second case. Finally, if $\{a,v,h,e\}$ or $\{v,g,h,e\}$ is a $P_4$ then the first case is the correct one. In the first case of (iii) vertex $v$ must be adjacent to $g$. If $\{v,d,e,f\}$ or $\{v,d,e,h\}$ is a $P_4$ then we are in the first case. Otherwise, if $\{v,g,h,f\}$ or $\{v,g,h,e\}$ is not a $P_4$ then we are in the second case. Otherwise, the second case is true if and only if $\{c,d,v,f\}$ induces a $P_4$.

In $A_{10}$ let $f$ be the vertex adjacent to $e$, $g$ the vertex adjacent to $b,c$ and $h$ the vertex adjacent to $d,e$. In the first case of (i) the vertex $v$ must be adjacent to $g$, in the second case $v$ must be adjacent to $f$. In both cases $v$ has at most one further neighbor. If $\{v,d,e,f\}$ or $\{v,h,e,f\}$ induce a $P_4$ then we are in the first case. Otherwise, if $\{v,c,d,h\}$ does not induce a $P_4$ then we are in the second case. If $\{v,c,d,h\}$ induces a $P_4$ then the second case is correct if and only if $\{d,h,v,f\}$ is a $P_4$. In (ii), if $\{v,g,c,d\}$ is a $P_4$ then we are in the first case. Otherwise, if $\{v,a,b,g\}$ is not a $P_4$ then we are in the second case. If $\{v,g,c,d\}$ is not a $P_4$ and $\{v,a,b,g\}$ is a $P_4$ then the second case is the correct alternative if and only if $\{b,g,v,d\}$ is a $P_4$. Finally, (iii) can be decided by a symmetry argument from (i).

In $A_{11}$ let $f$ be adjacent to $e$ and $g$ adjacent to $f$ and $h$ adjacent to $g$. For (i) note that in the second case $v$ must be adjacent to $f$. If one of $\{d,c,v,f\}$, $\{d,c,v,g\}$ or $\{d,c,v,h\}$ induces a $P_4$ then we are in the first case. If one of $\{d,e,v,g\}$ or $\{d,e,v,h\}$ induces a $P_4$ then we are in the second case. If none of the $P_4$s exists then we are in the second case if and only if $\{v,f,g,h\}$ induces a $P_4$. In (ii), if one of $\{b,a,v,f\}$, $\{b,a,v,g\}$ or $\{b,a,v,h\}$ induces a $P_4$ then we are in the first case. If one of $\{b,c,v,f\}$, $\{b,c,v,g\}$ or $\{b,c,v,h\}$ induces a $P_4$ then we are in the second case. If none of the $P_4$s exists then the second case is correct if and only if $\{v,d,e,f\}$ induces a $P_4$. In (iii), if one of $\{c,b,v,f\}$, $\{c,b,v,g\}$ or $\{c,b,v,h\}$ induces a $P_4$ then we are in the first case. If one of $\{c,d,v,f\}$, $\{c,d,v,g\}$ or $\{c,d,v,h\}$ induces a $P_4$ then we are in the second case. If no one of the $P_4$s exists then the second case is correct if and only if $\{v,e,f,g\}$ induces a $P_4$.

For $A_7$ and $A_9$ case (i) is analogous to case (i) of an $A_1$. The decision of case (ii) follows by a symmetry argument from (i). Case (iii) is again completely analogous to case (iii) of an $A_{11}$. □

# 5 The algorithm

If $\mathcal{H}$ is not connected then we consider the connected components of $\mathcal{H}$ separately and, for each component, we try to find a $p$–connected BCD–free graph with the corresponding $P_4$–structure. If all these BCD–free graphs exist then their disjoint union (or their disjoint sum) is a realization of $\mathcal{H}$. Hence we can assume that $\mathcal{H}$ is connected.

The algorithm for the reconstruction of BCD–free graphs consists of three parts. In the first part, we check whether there is a graph with stability number less than three whose $P_4$–structure is equal to $\mathcal{H}$. This is done using a method described in [2]. Then we consider the types of graphs which appear in Theorem 3.2 (i)–(ii). If there is such a *special* graph with $P_4$–structure equal to $\mathcal{H}$ then we are done (the recognition of the $P_4$–structure of these special graphs is easy and left to the reader; for details see also [1]). Otherwise, we have to apply the technique described in the previous section.

Note that for a 3–sun or a headless spider with at least three legs (and with the allowed replacements of vertices by cliques) a vertex–by–vertex extension in the sense of Lemma 4.1 is not possible, since all the $p$–connected induced subgraphs are separable. Moreover, if $\mathcal{H}$ is the $P_4$–structure of one of these graphs or of one of the graphs from Theorem 3.2 (ii) then the underlying graph is in general not unique (i.e., there are different realizations). If $G$ is a graph from Theorem 3.2 (ii) and $G'$ is a $p$–connected induced subgraph of $G$ (take as a simple example a $P_5$, $P_6$ or $P_7$) then the adjacencies of vertices outside $G'$ with respect to $G'$ may not be uniquely determined. For these reasons the reconstruction technique of the previous section cannot be applied to the graphs from Theorem 3.2 (i) and (ii).

In order to find a starting graph for the reconstruction procedure we examine all subsets $\mathcal{H}' \subseteq \mathcal{H}$ of six, seven and eight vertices and check whether $\mathcal{H}'$ is the $P_4$–structure of one of the graphs $A_1,\ldots,A_{11}$.

Note that, if $\mathcal{H}'$ is the $P_4$–structure of a graph $A_i$, then the realization might not be unique: e.g., if $\mathcal{H}'$ is the $P_4$–structure of an $A_{10}$ then there are two possible realizations since the two midpoints of the $P_6$ are not uniquely determined. Moreover, $\mathcal{H}'$ may be the $P_4$–structure of different graphs $A_i$: e.g. the graphs $A_3$ and $A_4$ have the same $P_4$–structure; the graph $A_5$ has the same $P_4$–structure as a path $P_7$, etc. Hence *all* realizations must be considered as possible starting graphs (and can be found by a "brute force" approach, in a similar way as described in [2]). Note, however, that the number of possible starting graphs remains polynomial in the size of $\mathcal{H}$.

Here is an informal description of the algorithm as a whole.

`Algorithm` Check–$P_4$–structure

Input: A connected 4–uniform hypergraph $\mathcal{H}$
Output: A BCD–free graph $G$ with $P_4$–structure $\mathcal{H}$
or the answer "No" if no such graph exists

1. Check whether $\mathcal{H}$ is the $P_4$–structure of a graph $G$ with $\alpha(G) \leq 2$.
   If such a graph $G$ exists then output $G$ and STOP.
2. Check whether $\mathcal{H}$ is the $P_4$–structure of a special graph $G$.
   If such a graph $G$ exists then output $G$ and STOP.
3. For all subhypergraphs $\mathcal{H}' \subseteq \mathcal{H}$ with $6 \leq |\mathcal{H}'| \leq 8$ do:
   For $i = 1$ to 11 do:
   If $\mathcal{H}'$ is the $P_4$–structure of the graph $A_i$ then:
   Start the reconstruction procedure for each realization of $\mathcal{H}'$.
   If it produces a graph $G$ then check whether $G$ is BCD–free and
   $\mathcal{H}$ is the $P_4$–structure of $G$. If yes then output $G$ and STOP.
4. Output "No".

It is easy to see that the algorithm can be performed in time polynomial in the size of the hypergraph $\mathcal{H}$. Therefore we have shown:

**Theorem 5.1** *The $P_4$–structure of BCD–free graphs can be recognized in polynomial time.* □

**Corollary 5.2** *The $P_4$–structure of claw–free graphs can be recognized in polynomial time.*

**Proof.** Claw–free graphs are BCD–free. In Step 3 of the algorithm, instead of checking $G$ for being BCD–free one has just to check $G$ for being claw–free. □

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