Asymptotics of several-partition Hurwitz numbers

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Abstract. We derive in this paper the asymptotics of several-partition Hurwitz numbers, relying on a theorem of Kazarian for the one-partition case and on an induction carried on by Zvonkine. Essentially, the asymptotics for several partitions is the same as the one-partition asymptotics obtained by concatenating the partitions.

Résumé. Dans cet article, nous donnons l'asymptotique générale des nombres de Hurwitz à plusieurs partitions, s'appuyant sur un théorème de Kazarian pour le cas d'une partition et s'inspirant d'une récurrence menée par Zvonkine. En substance, l'asymptotique pour plusieurs partitions est la même que celle à une partition obtenue en concaténant les partitions.

Keywords: Hurwitz numbers, asymptotics, many partitions, transitive factorisations

1 Introduction

In the end of the XIXth century, Hurwitz computed the number of ways to factorise in the symmetric group $\mathfrak{S}_n$ a permutation of given cyclic type λ into a product of a minimal number of transpositions which generate a transitive subgroup. If one denotes by $h_n^0(\lambda)$ that number divided par $n!$, Hurwitz proved that

$$\frac{h_n^0(\lambda)}{(n+p-2)!} = \frac{1}{|\mathrm{Aut}\lambda|}\left(\prod_{i=1}^{p}\frac{d_i^{d_i}}{d_i!}\right)n^{p-3} \qquad (\text{write } \lambda = (d_1, ..., d_p)\,).$$

A fruitful generalisation of Hurwitz's original question (see [5] and [6]) is to seek such factorisation numbers $h_n^g\left(\overrightarrow{\lambda}\right)$ with prescribed number of transpositions (the minimal case corresponds to $g = 0$), by remplacing the single permutation σ by a product of an arbitrary number of permutations of given types $\overrightarrow{\lambda} = (\lambda_1, ..., \lambda_k)$ (see Section 2.2 for reminders on partitions), and by adding a transitiveness condition – without the latter, such "disconnected" Hurwitz numbers would be given by Frobenius's formula. Section 2.3 recalls the definitions of the numbers $h_n^g\left(\overrightarrow{\lambda}\right)$ and of their corresponding generating fonction $H^g\left(\overrightarrow{\lambda}\right)$. Section 2.2 defines convenient renormalisations $\mathbb{h}_n^g\left(\overrightarrow{\lambda}\right)$ and $\mathbb{H}_n^g\left(\overrightarrow{\lambda}\right)$.

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In genera 0 and 1, some considerations from algebraic geometry (more precisely the ELSV formula, see [3]) yield explicit formulæ for $h_n^0(\lambda)$ and $h_n^1(\lambda)$, whence closed formulæ for series $H^0(\lambda)$ and $H^1(\lambda)$ (see Section 2.4). Moreover, Kazarian [4] used the integration frame of the ELSV formula to give an explicit formula for the series H^g when $g \geq 2$ (see Section 2.5). However, very little is known on the numbers $h_n^g\left(\overrightarrow{\lambda}\right)$ when the number of partitions is strictly greater than 1. In this paper, we will only be concerned with determining the asymptotics of $h_n^g\left(\overrightarrow{\lambda}\right)$ when n grows to ∞.

Zvonkine introduced in [7] an algebra of power series $\mathcal{A} := \mathbb{Q}[Y, Z]$ which has the following properties (see [8] and [9] for details):

1. the asymptotics of the leading coefficient of a series lying in $\mathcal{A}$ is determined by the leading coefficient in Z (see Claim 1 in Section 2.1);
2. all series $H^g(\lambda_1, ..., \lambda_k)$ but $H^1(\emptyset)$ lay in the algebra $\mathcal{A}$ (see [7]).

Zvonkine proved the latter by induction on the number k of partitions, relying when $k = 1$ on already-known formulæ for spherical and toric genera (see Section 2.4) and on Kazarian's formula for higher genera (see Section 2.5). However, Zvonkine did not make explicit the formula he used; since we want to precisely compute the Z-degree of $H^g\left(\overrightarrow{\lambda}\right)$, we will carry out the explicitation of this reduction formula (see Theorem 10 in Section 3.1). Then, we will be able to prove our main result (Theorem 8) by induction (see Section 3.2), hence the sought asymptotics of all numbers $h_n^g\left(\overrightarrow{\lambda}\right)$ (see Corollary 9). Both Theorem 8 and Corollary 9 are stated in Section 2.6.

The constants in the obtained asymptotics involves some rationnal-valued intersection numbers whose generating function, up to some renormalisation, satisfy Painlevé's equation I ($\frac{d^2f}{dq^2} + f(q)^2 = q$, see [5]) and can hence be recursively computed (see last lines of [1]). Section 2.6 recalls such a recursion formula, allowing one to retrieve the map asymptotics constants t_g defined in [2].

For sake of conciseness, we will use throughout the paper the genus-notation

$$g' := g - 1.$$

2 Hurwitz numbers and the algebra $\mathcal{A}$

2.1 The algebra $\mathcal{A} = \mathbb{Q}\left[\sum_{n\geq 1} \frac{n^{n-1}}{n!} q^n, \sum_{n\geq 1} \frac{n^n}{n!} q^n\right]$

Let us define an algebra $\mathcal{A} := \mathbb{Q}[Y, Z]$ where $Y := \sum_{n\geq 1} \frac{n^{n-1}}{n!} q^n$ and $Z := \sum_{n\geq 1} \frac{n^n}{n!} q^n = DY$ with $D : \sum_{n\geq 0} a_n q^n \mapsto \sum_{n\geq 1} n a_n q^n$. Define also a pseudo-inverse $D^{-1} : \sum_{n\geq 0} a_n q^n \mapsto \sum_{n\geq 1} \frac{a_n}{n} q^n$. The combinatorial identity $Y = qe^Y$ allows one to linearise the product $YZ = Z - Y$, whence the description $\mathcal{A} = \mathbb{Q}[Y] + \mathbb{Q}[Z]$. The latter entitles one to assign to every series in $\mathcal{A}^Z := \mathcal{A} \backslash \mathbb{Q}[Y]$ a polynomial in Z (up to the constant coefficient) that completly describes the asymptotics of the corresponding series thanks to the following claim:

Claim 1 (asymptotics in the algebra $\mathcal{A}$). *One has for any positive integers i and k*

$$\begin{array}{r} \textit{leading coefficient in } \frac{Y^i}{i} \sim C_{-1}\frac{e^n}{n}\sqrt{n}^{-1} \\ \textit{leading coefficient in } Z^k \sim C_k\,\frac{e^n}{n}\sqrt{n}^k \end{array} \quad \textit{where} \quad \begin{array}{l} \frac{1}{C_{-1}} = \sqrt{2\pi} \\ \frac{1}{C_k} = \Gamma\left(\frac{k}{2}\right)2^{\frac{k}{2}} \end{array}.$$

Two series S and T in $\mathcal{A}^Z$ have therefore the same asymptotics if and only if their Z-leading terms are equal, which we will denote by a Z-equality $S \overset{Z}{=} T$. For instance, one has $P(Z)\,Q(Y) \overset{Z}{=} P(Z) \iff Q(1) \neq 0$ for any polynomials P and Q and the equality $DP(Z) \overset{Z}{=} Z^3P'(Z)$ if $P \neq 0$.

2.2 Reminders on partitions

Recall that a *partition* of an integer a is any finite non-increasing sequence $\lambda = (d_1 \geq d_2 \geq \cdots \geq d_p)$ of positive integers (the *parts* of λ) summing up to a. Define the *length* $l(\lambda) := p$, the *size* $|\lambda| := a$, the *multiplicity* $m_k(\lambda) := \mathrm{Card}\{i;\ d_i = k\}$ of any integer k, the *ramification* $r(\lambda) := |\lambda| - l(\lambda)$, the *number of symmetries* $|\mathrm{Aut}\lambda| = \prod_{k\geq 1} m_k(\lambda)!$, the *reduction* $\mathring{\lambda} := \lambda \setminus 1^{m_1(\lambda)}$, the ($n$-th) *completion* $\overline{\lambda} := \lambda \sqcup 1^{n-m_1(\lambda)}$ for any integer $n \geq m_1(\lambda)$. A partition λ is called *reduced* if $m_1(\lambda) = 0$. The *concatenation* $\lambda \sqcup \mu$ of two partitions λ and μ is the partition whose parts are those of λ union those of μ. The length, size and ramification are morphisms from the concatenation to the addition and can therefore be extended to a tuple of partitions by concatenating the latter. At last, it will be convenient to use the following notations and renormalisations (see Definition 2 to define $h_n^g\left(\overrightarrow{\lambda}\right)$ and $H^g\left(\overrightarrow{\lambda}\right)$):

$$\boxed{\lambda} := \frac{1}{|\mathrm{Aut}\lambda|}\frac{d_1^{d_1}\cdots d_p^{d_p}}{d_1!\cdots d_p!}, \quad \boxed{\overrightarrow{\lambda}} := \boxed{\lambda_1}\boxed{\lambda_2}\cdots\boxed{\lambda_k}, \quad \boxed{h}_n^g\left(\overrightarrow{\lambda}\right) := \frac{h_n^g\left(\overrightarrow{\lambda}\right)}{\boxed{\overrightarrow{\lambda}}}, \quad \boxed{H}^g\left(\overrightarrow{\lambda}\right) := \frac{H^g\left(\overrightarrow{\lambda}\right)}{\boxed{\overrightarrow{\lambda}}}.$$

Let $n \geq 1$ an integer and σ a permutation in $\mathfrak{S}_n$. Its *support* is the complement $S\sigma = \mathrm{Supp}\sigma$ in $[1,n]$ of all σ-fixed points and its *type* is the partition $\mathrm{type}(\sigma)$ whose parts are the lengths of the cycles of σ (including fixed cycles). For instance, the type of the disjoint product of two permutations is the concatenation of their types and the cardinality of the support of a permutation equals the size of the reduction of its type. Recall that conjugacy classes in $\mathfrak{S}_n$ are indexed by n-sized partitions.

2.3 Constellations and Hurwitz numbers

Let n and k be positive integers. Define a k-*constellation* of degree n to be a k-tuple $\overrightarrow{\sigma} \in \mathfrak{S}_n^k$ such that $\sigma_1\cdots\sigma_k = \mathrm{Id}$ and that the subgroup $\langle\sigma_1,...,\sigma_k\rangle$ acts transitively on $[1,n]$. The *type* of a k-constellation $\overrightarrow{\sigma}$ is the k-tuple of the types of the σ_i. Its *ramification* r is the sum of those of the σ_i's. Its *genus* $g \geq 0$ is defined by the *Riemann-Hurwitz formula* $r = 2n + 2g'$. (Recall that $g' = g - 1$).

Definition 2 (Hurwitz numbers $h_n^g\left(\overrightarrow{\lambda}\right)$ and Hurwitz series $H^g\left(\overrightarrow{\lambda}\right)$). *Let g and n be two non-negative integers and λ_1, ..., λ_k be partitions of non-negative integers.*

Define $T = T_n = T_n^g(\lambda_1,...,\lambda_k) := 2n + 2g' - r$ where $r := \sum r(\lambda_i)$.

Define $h_n^g(\lambda_1,...,\lambda_k)$ by $\frac{1}{n!}$ times the number of pairs (C,F) where C is a constellation $(\overrightarrow{\sigma},\overrightarrow{\tau}) \in \mathfrak{S}_n^k \times \mathfrak{S}_n^T$ of type $\left\{\begin{array}{l} \forall i,\ \mathrm{type}(\sigma_i) = \overline{\lambda_i} \\ \forall j,\ \mathrm{type}(\tau_j) = \overline{2} \end{array}\right.$ and where $F \subset [1,n]^k$ satisfies $\left\{\forall i, \begin{array}{c} F_i \subset \mathrm{Fix}\sigma_i \\ |F_i| = m_1(\lambda_i) \end{array}\right.$.

Define Hurwitz series by the following generating functions:

$$H^g\left(\overrightarrow{\lambda}\right) := \sum_{n\geq 1} \frac{h_n^g\left(\overrightarrow{\lambda}\right)}{T_n!} q^n \qquad H^g\left(\overline{\lambda_1}, ..., \overline{\lambda_k}\right) := \sum_{n\geq 1} \frac{h_n^g\left(\overline{\lambda_1}, ..., \overline{\lambda_k}\right)}{T_n!} q^n.$$

By choosing first the constellation then the fixed parts, one obtains the relation

$$h_n^g\left(\lambda_1, ..., \lambda_k\right) = h_n^g\left(\overline{\lambda_1}, ..., \overline{\lambda_k}\right) \times \prod_{i=1,\ldots,k} \binom{n - \left(|\lambda_i| - m_1\left(\lambda_i\right)\right)}{m_1\left(\lambda_i\right)}.$$

2.4 *Hurwitz series in genera* 0 *and* 1

In spherical or toric genus, one has closed formulæ stemming from the ELSV formula for one-partition Hurwitz numbers. When one sees these relations in the series $H^g(\lambda)$, one obtains the following claim, which is a reformulation of unpublished results already known by Kazarian in [4]. We will need to define $e_k(\lambda) := \sum_{i_1<i_2<\cdots<i_k} d_{i_1} d_{i_2} \cdots d_{i_k}$ for any partition $\lambda = (d_1, ..., d_p)$ and any integer $k \in [0, p]$.

Claim 3 (Hurwitz series in genera 0 and 1). *Set a partition λ of an integer $a \geq 0$ in $p \geq 0$ parts. Then, one has the identities* $\mathbb{H}^0(\lambda) = D^{p-3}\left(Y^{a-1}Z\right)$ *and*

$$24\mathbb{H}^1(\lambda) = D^{p-1}\left(Y^{a-1}Z^2\right) + (a-1)\, D^{p-1}\left(Y^{a-1}Z\right) - \sum_{x=2}^{p} (x-2)!\, e_x(\lambda)\, D^{p-x}\left(Y^{a-x}Z^x\right).$$

Examples.

$$\mathbb{H}^0_{p=2} = \frac{Y^a}{a} \qquad \mathbb{H}^0_{p=1} = \frac{1}{a}\left(\frac{Y^a}{a} - \frac{Y^{a+1}}{a+1}\right) \qquad \mathbb{H}^0_{p=0} = Y - \frac{3}{2}\left(\frac{Y^2}{2}\right) + \frac{1}{2}\left(\frac{Y^3}{3}\right)$$

$$24\mathbb{H}^1(\emptyset) = D^{-1}Z^2 \qquad 24\mathbb{H}^1((1)) = Z^2 \qquad 24\mathbb{H}^1((2)) = Z^2$$

for any $d \geq 0$: $\quad 24\mathbb{H}^1((d+1)) = Y^a Z\,(Z+d) = Z^2 - Y^2 - 2Y^3 - 3Y^4 - \cdots - (d-1)\,Y^d.$

Corollary 4 (asymptotics of one-partition Hurwitz numbers in genera 0 and 1). *For any partition λ and any genus $g \in \{0, 1\}$, one has the following asymptotics*

$$\frac{\mathbb{h}_n^g(\lambda)}{T_n!} \sim c_g e^n n^{\frac{5}{2}g' + p - 1} \text{ where } (c_0, c_1) := \left(\frac{1}{\sqrt{2\pi}}, \frac{1}{48}\right).$$

Proof of Corollary 4. In null genus, one has the relation $\mathbb{H}^0(\lambda) = D^{p-3}\left(Y^{a-1}Z\right) \stackrel{Z}{\equiv} D^{p-3}Z$. Its leading coefficient is therefore equivalent to n^{p-3} times $C_1 \frac{e^n}{n}\sqrt{n}^1$ thanks to Claim 1. In toric genus, the first term $D^{p-1}\left(Y^{a-1}Z^2\right) \stackrel{Z}{\equiv} D^{p-1}Z^2$ has degree $2 + 2(p-1) = 2p$ whereas the following terms $D^{p-x}\left(Y^{a-x}Z^x\right) \stackrel{Z}{\equiv} D^{p-x}Z^x$ for $x \geq 1$ have Z-degrees $x + 2(p-x) < 2p$. One has therefore $24\mathbb{H}^1(\lambda) \stackrel{Z}{\equiv} D^{p-1}Z^2$, whose leading coefficient is equivalent to n^{p-1} times $C_2 \frac{e^n}{n}\sqrt{n}^2$. □

2.5 Kazarian's formulæ, and the asymptotics of one-partition Hurwitz numbers

Considering integration theory on the moduli space of complex curves with some marked points led Kazarian to the following formula (see [9]). The latter involves some rational-valued intersection numbers $\langle \tau_0^u \tau_2^v \rangle$ defined (in [10]) for some integers $u, v \geq 0$. We will simply write $\langle \tau_2^v \rangle$ for $\langle \tau_0^0 \tau_2^v \rangle$.

Theorem 5 (Kazarian's formula). *Let $\mu = (d_1, ..., d_p)$ be a partition of an integer $a \geq 0$ and $g \geq 0$ be a genus. Then, whenever $n + 2g' > 0$, one has $\mathbb{H}^g(\mu) = Y^a (Z+1)^{2g'+p} P(Z)$ where $P(Z)$ is a polynomial of leading term $\frac{\left\langle \tau_0^p \tau_2^{3g'+p} \right\rangle}{(3g'+p)!} Z^{2g'+p}$.*

Corollary 6 (Kazarian's Z-formula). *For any partition λ and genus $g \geq 0$ such that $p + 2g' > 0$, one has the Z-equality $\mathbb{H}^g(\lambda) \overset{Z}{=} \frac{\left\langle \tau_0^p \tau_2^{3g'+p} \right\rangle}{(3g'+p)!} Z^{5g'+2p}$.*

Combining Corollary 6 with Claim 1 immediately yields the asymptotics of all $h_n^g(\lambda)$'s when $g \geq 2$:

$$\frac{\mathbb{h}_n^g(\lambda)}{T_n!} \sim \frac{\left\langle \tau_2^{3g'+p} \tau_0^p \right\rangle}{(3g'+p)!} C_{5g'+2p} \frac{e^n}{n} \sqrt{n}^{5g'+2p} = Cst_g(\lambda) \times e^n n^{\frac{5}{2}g'+p-1}.$$

A little more work (use the string and dilaton equations in [10]) on the numbers $\langle \tau_2^u \tau_0^v \rangle$ can show that the constant $Cst_g(\lambda) := \frac{\left\langle \tau_2^{3g'+p} \tau_0^p \right\rangle}{(3g'+p)!} C_{5g'+2p}$ is actually λ-free, as we already know in genus 0 and 1 thanks to Corollary 4, hence the following.

Theorem 7 (asymptotics of one-partition Hurwitz numbers in any genus). *For any partition λ and any genus $g \geq 0$, one has the asymptotics*

$$\frac{h_n^g(\lambda)}{T_n!} \overset{n\infty}{\approx} c_g e^n n^{\frac{5}{2}g'+p-1} \text{ where } \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} := \begin{pmatrix} \frac{1}{\sqrt{2\pi}} \\ \frac{1}{48} \end{pmatrix} \text{ and } c_{g \geq 2} := \frac{1}{\Gamma\left(\frac{5}{2}g'\right) 2^{\frac{5}{2}g'}} \frac{\left\langle \tau_2^{3g'} \right\rangle}{(3g')!}.$$

2.6 The main theorem and the general asymptotics of Hurwitz numbers

We can now state our main result, proven in section 3.2, which reduces the understanding of the asymptotics of several-partition Hurwitz numbers to that of single-partition Hurwitz numbers. Recall from [7] that all series $DH^g(\lambda_1, ..., \lambda_k)$ lie in the algebra $\mathcal{A}$ (it is a consequence of the conjunction of Claim 3, Theorem 5 and Theorem 10).

Theorem 8. *For any partitions $\lambda_1, ..., \lambda_k$ and any genus $g \geq 0$, one has the following Z-equality in the algebra $\mathcal{A}^Z$:*

$$D^3 \mathbb{H}^g(\lambda_1, ..., \lambda_k) \overset{Z}{=} D^{3+m_1(\lambda_i)+\cdots+m_1(\lambda_k)} \mathbb{H}^g\left(\mathring{\lambda}_1 \sqcup \mathring{\lambda}_2 \sqcup \cdots \sqcup \mathring{\lambda}_k\right).$$

Corollary 9 (general asymptotics of Hurwitz numbers). *For any partitions* $\lambda_1, ..., \lambda_k$ *and any genus* $g \geq 0$*, one has the following asymptotics for some constant* c_g*:*

$$\frac{h_n^g(\lambda_1, ..., \lambda_k)}{T_n!} \overset{n\infty}{\sim} c_g \frac{e^n}{n} n^{\frac{5}{2}g'} n^{l(\lambda_1)+\cdots+l(\lambda_k)} \text{ with } \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} := \begin{pmatrix} \frac{1}{\sqrt{2\pi}} \\ \frac{1}{48} \end{pmatrix} \text{ and } c_{g\geq 2} := \frac{1}{\Gamma\left(\frac{5}{2}g'\right) 2^{\frac{5}{2}g'}} \frac{\left\langle \tau_2^{3g'} \right\rangle}{(3g')!}.$$

Proof of Corollary 9. If one sets $m_1 := \sum m_1(\lambda_i)$ and $p := \sum l(\lambda_i)$, Theorem 8 states the Z-equality $D^3 H^g(\lambda_1, ..., \lambda_k) \overset{Z}{=} D^{3+m_1} H^g\left(\mathring{\lambda}_1 \sqcup \mathring{\lambda}_2 \sqcup \cdots \sqcup \mathring{\lambda}_k\right)$, whence the asymptotics

$$\frac{h_n^g(\lambda_1, ..., \lambda_k)}{T_n!} \sim n^{m_1} \frac{h_n^g\left(\mathring{\lambda}_1 \sqcup \cdots \sqcup \mathring{\lambda}_k\right)}{T_n^g\left(\mathring{\lambda}_1 \sqcup \cdots \sqcup \mathring{\lambda}_k\right)!} \overset{\text{Theorem 7}}{\sim} n^{m_1} c_g e^n n^{\frac{5}{2}g'+(p-m_1)-1} = c_g e^n n^{\frac{5}{2}g'+p-1}. \square$$

Remark. The constants $\left\langle \tau_2^{3g'} \right\rangle$ can recursively be computed thanks to Witten's conjecture (see [10] and [6]): if one sets $\alpha_0 := \frac{1}{12}$ and $\frac{\alpha_k}{5k(5k+2)} := \frac{\left\langle \tau_2^{3k} \right\rangle}{(3k)!}$ for any $k \geq 1$, then one will obtain the recursion

$$\forall k \geq 1,\ \alpha_k = \frac{25k^2 - 1}{12} \alpha_{k-1} + \frac{1}{2} \sum_{a,b\geq 0}^{a+b=k-1} \alpha_p \alpha_q.$$

The latter being very similar to that in [2] which defines the map asymptotics constants t_g, it is then easy to derive the identity $c_g = \sqrt{2}^{g-3} t_g$ for any integer $g \geq 0$.

3 Reduction formulæ

Zvonkine proved in [7] that all series $H^g(\lambda_1, ..., \lambda_k)$ but $H^1(\emptyset)$ lay in the algebra $\mathcal{A}$ by induction on the number k of partitions, the case $k = 1$ being an immediate corollary from Theorem 5. We explicit the (unexplicited) induction formula used by Zvonkine so as to control the leading coefficients in Z of the series $H^g(\lambda_1, ..., \lambda_k)$ and derive their asymptotics.

3.1 The reduction formula

Let us first carry out an analysis of what becomes a constellation after merging its first two permutations. We reproduce mostly what is explained in [7] but retain some more information.

Let $(\sigma, \rho, \sigma_3, \sigma_4, ..., \sigma_k)$ be a constellation and denote $\pi := \sigma\rho$. One gets $k-1$ permutations $\pi, \sigma_3, ..., \sigma_k$ whose product is the identity, but one generally loses the transitivity condition. Denote Ω^1, ..., Ω^N the orbits of our new group $\langle \pi, \sigma_3, ..., \sigma_k \rangle$ and set σ_i^j for the permutation σ_i induced on Ω^j. One thus obtains for any j a constellation $\left(\pi^j, \sigma_3^j, ..., \sigma_k^j\right)$ on the orbit Ω^j.

Notice that one always has $N \leq \left|\mathring{\lambda}\right| + |\mathring{\mu}| + 1$. This is trivial when $S\sigma \cup S\rho$ is empty (since one then has $\sigma = \mathrm{Id}$ and $N = 1$) and let us explain why, when $S\sigma \cup S\rho$ is non-empty, every orbit must intersect it (hence $N \leq |\lambda| + |\mu|$): if the group $\langle \sigma\rho, \sigma_3, ..., \sigma_k \rangle$ stabilised an orbit disjoint from $S\sigma \cup S\rho$, then so would the group $\langle \sigma, \rho, \sigma_3, ..., \sigma_k \rangle$ since σ and ρ acts trivially out of $S\sigma \cup S\rho$, but the latter group is by assumption transitive, so the mentionned orbit must equal all $[1, n]$, consequently intersecting $S\sigma \cup S\rho$.

The genera g^j's satisfy the Riemann-Hurwitz relation $2n^j + 2g^{j\prime} = r\left(\pi^j\right) + \sum_{i=3}^k r\left(\sigma_i^j\right)$. By summing up these relations and recalling that of our first constellation, one gets $\sum g^{j\prime} = g' - \frac{r(\lambda)+r(\mu)-r(\pi)}{2}$.

Furthermore, since $S\pi \subset S\sigma \cup S\rho$, one can consider the type ν of $\pi_{|S\sigma \cup S\rho}$ as a partition of an integer smaller than $|S\sigma \cup S\rho| \leq |\lambda| + |\mu|$. Let us be more precise and set ν^j for the type of the permutation $\pi_{|S\sigma \cup S\rho}$ induced of Ω^j: the ν^j's are all non-empty (unless $S\sigma \cup S\rho = \emptyset$, namely unless $\mathring{\lambda} = \mathring{\mu} = \emptyset$) and their sizes always sum up to that of $|\nu|$. Then $\pi_{|S\sigma \cup S\rho}$ has $m_1\left(\nu^j\right)$ fixed points in Ω^j and the knowledge of these fixed points for all j's allows one to rebuild $S\sigma \cup S\rho$ (add for any j these $m_1\left(\nu^j\right)$ points to the support of π^j).

Finally, when one wants to factorise back $\pi = \sigma\rho$, one has to choose σ and ρ satisfying the three following conditions: σ and ρ have respective types $\overline{\lambda}$ and $\overline{\mu}$; the union $S\sigma \cup S\rho$ equals $S\pi$ union the preceedingly-chosen point; the group $\langle \sigma, \rho, \sigma_3, ..., \sigma_k, \tau_1, ..., \tau_T \rangle$ acts transitively. A conjugation argument shows that the number $f_{\lambda,\mu}^{\overrightarrow{\nu}}$ of such choices depends only on the partitions $\lambda, \mu, \overrightarrow{\nu}$; it is besides not hard to show that the transitiveness condition amounts to a junction condition on the orbits Ω^j by the cycles of σ or ρ (see Definition 11).

By collecting constellations according to the datas above (N, $\overrightarrow{g}$, $\overrightarrow{\nu}$), one can explicit the formula used by Zvonkine in [7] to prove that all series $H^g\left(\overrightarrow{\lambda}\right)$ but $H^1(\emptyset)$ lay in the algebra $\mathcal{A}$. (Consider the above analysis as a sketch of proof). The reduction formula thus obtained relies on a family $\left(f_{\lambda,\mu}^{\overrightarrow{\nu}}\right)$ of non-negative integers that we will define just after stating the reduction formula.

Theorem 10 (reduction formula). *Let $g \geq 0$ be a genus and $\overrightarrow{\lambda} = (\lambda, \mu, \lambda_3, \lambda_4, ..., \lambda_k)$ be k partitions where $k \geq 2$ is an integer. One then has the following formula*

$$H^g\left(\overline{\lambda}, \overline{\mu}, \lambda_3, ..., \lambda_k\right) = \sum_{\overrightarrow{\nu}, \overrightarrow{g}} f_{\lambda,\mu}^{\overrightarrow{\nu}} \sum_{\overrightarrow{\lambda_3}, ..., \overrightarrow{\lambda_k}} \prod_j H^{g^j}\left(\nu^j, \lambda_3^j, ..., \lambda_k^j\right)$$

where one sums over: integers $N \geq 1$ smaller or equal to $\left|\mathring{\lambda}\right| + |\mathring{\mu}| + 1$; the N-tuples $(\overrightarrow{\nu}, \overrightarrow{g})$ such that $2g' = r(\lambda) + r(\mu) - r(\overrightarrow{\nu}) + \sum 2g^{j\prime}$ (all ν^j's being non-empty unless $\mathring{\lambda} = \mathring{\mu} = \emptyset$); for any $i = 3, ..., k$ partitions[(i)] *$\left(\lambda_i^1, ..., \lambda_i^N\right)$ whose concatenation is λ_i.*

Definition 11 (the numbers $f_{\lambda,\mu}^{\overrightarrow{\nu}}$). *Let N be a positive integer and set $N+2$ partitions $\lambda, \mu, \overrightarrow{\nu}$. For any j, consider Ω^j a $\left|\nu^j\right|$-sized set and π^j a ν^j -typed permutation in $\mathfrak{S}_{\Omega^j}$. Define $f_{\lambda,\mu}^{\overrightarrow{\nu}}$ to be the number of factorisations in $\mathfrak{S}_{\bigsqcup \Omega^j}$ of the permutation $\prod \pi^j$ in a product $\sigma\rho$ satisfying the three conditions:*

1. *the types of σ and ρ are respectively $\overline{\lambda}$ and $\overline{\mu}$;*
2. *the supports of σ and ρ cover all $\bigsqcup \Omega^j$, namely $Fix\sigma \cap Fix\rho = \emptyset$;*

(i) when $k = 2$, one sums (not over nothing but) over the empty list

3. *(junction condition) for any $j \neq j'$, there is a finite sequence $j = j_0, ..., j_L = j'$ such that, for any $p = 1, ..., L$, there is a cycle of σ or ρ which intersects both orbits $\Omega^{j_{p-1}}$ and Ω^{j_p}.*

Remarks. The first condition shows that $f^{\overrightarrow{\nu}}_{\lambda,\mu} = f^{\overrightarrow{\nu}}_{\mathring{\lambda},\mathring{\mu}}$ while the second condition yields the implication $f^{\overrightarrow{\nu}}_{\lambda,\mu} > 0 \implies |\overrightarrow{\nu}| \leq \left|\mathring{\lambda}\right| + |\mathring{\mu}|$, which ensures that the sum in Theorem 10 is *finite*.

When ν is made with the only one partition $\lambda \sqcup \mu$, the above inequality implies that λ and μ are reduced and supports $S\sigma$ and $S\rho$ are disjoint. Then, choosing a factorisation amounts to choosing for any $k \geq 2$ which k-lengthed cycles of π will appear in σ. Therefore, one has $f^{\mathring{\lambda}\sqcup\mathring{\mu}}_{\lambda,\mu} = \prod_{k\geq 2} \binom{m_k(\lambda) + m_k(\mu)}{m_k(\lambda)}$, which can be rewritten in a more convient way (for future application) as $\frac{f^{\mathring{\lambda}\sqcup\mathring{\mu}}_{\lambda,\mu}}{m_1(\lambda)!\, m_1(\mu)!} \frac{\boxed{\mathring{\lambda} \sqcup \mathring{\mu}}}{\boxed{\lambda}\boxed{\mu}} = 1$.

3.2 Proof of Theorem 8

We restate Theorem 8: *for any partitions $\lambda_1, ..., \lambda_k$ and any genus $g \geq 0$, one has the following Z-equality in the algebra $\mathcal{A}^Z$ for M large enough:*

$$D^M \boxed{H}^g (\lambda_1, ..., \lambda_k) \stackrel{Z}{=} D^{M+m_1(\lambda_i)+\cdots+m_1(\lambda_k)} \boxed{H}^g \left(\mathring{\lambda}_1 \sqcup \mathring{\lambda}_2 \sqcup \cdots \sqcup \mathring{\lambda}_k\right).$$

For the wondering reader, the exponent M is a trick to get rid of the exceptional cases.

As a first example, the information about the Z-degree in Kazarian's formulæ can be stated without the condition $p + 2g' > 0$ by the simple equality $\deg_Z D^3 H^g(\lambda) = 5g + 2p + 1$.

Let us prove the following generalisation *for any $M \geq 0$: if the series $D^M H^g(\lambda)$ lies in $\mathcal{A}^Z$, then it has degree* $\deg_Z D^M H^g(\lambda) = 2M + 5g' + 2p$. Indeed, setting $S := H^g(\lambda)$, one can write on the one hand $D^3\left(D^M S\right) = 2 \cdot 3 + \deg_Z D^M S$ and on the other hand $D^M\left(D^3 S\right) = 2M + 5g' + 2p + 6$; equalling both members leads to the conclusion. □

Let us now prove *for any $S \in \mathcal{A}$, $S \in \mathcal{A}^Z \iff \forall M \geq 0,\ \deg_Z D^M S \geq 2M$.* The arrow $\implies$ stems from $DP(Z) \stackrel{Z}{=} Z^3 P'(Z)$. Conversely, if S is a polynomial $P(Y)$, then $DS = P'(Y)Z$ has Z-degree ≤ 1 and hence $D^M S = D^{M-1} DS$ has degree $\leq 1 + 2(M-1) < 2M$. □

Finally, let us prove the following corollary of Theorem 8.

Corollary 12 (which series H^g lie in $\mathcal{A}^Z$). *For any **non-empty** partitions $\lambda_1, ..., \lambda_k, \lambda, \mu$:*

1. *$H^g(\lambda_1, ..., \lambda_k)$ always lies in $\mathcal{A}^Z$ when $k \geq 3$.*
2. *$H^g(\lambda, \mu)$ does **not** lie in $\mathcal{A}^Z$ if and only if $g = 0$ and if both λ and μ have one part.*
3. *$H^g(\lambda)$ does **not** lie in $\mathcal{A}^Z$ if and only if $\binom{g}{l(\lambda)} \in \left\{\binom{0}{0}, \binom{0}{1}, \binom{0}{2}, \binom{1}{0}\right\}$.*

Proof. Take the Z-degree in the given Z-equality and use Corollary 6:

$$\deg D^M \boxed{H}^g (\lambda_1, ..., \lambda_k) = 2M + 2\sum m_1(\lambda_i) + \left(5g' + 2\sum l\left(\mathring{\lambda}_i\right)\right) = 2M + 5g' + 2\sum l(\lambda_i).$$

Since all lengths are ≥ 1, the above degree is $\geq 2M$ when $k \geq 3$. When $k = 2$, the above degree is $< 2M$ if and only if $g = 0$ and $l(\lambda_i) = 1$ for $i = 1, 2$. When $k = 1$, one retrieves the already-known exceptional cases of Kazarian's formulæ. $\square$

We now proceed with the proof of Theorem 8, by induction on the number k of partitions. The case $k = 1$ is immediate from Corollary 6. Because of the number of exceptional cases, the case $k = 2$ will be the longest to deal with, the case $k = 3$ much similar and much easier, and greater k's will be almost straightforward. We start with $k \geq 4$ to get used to the idea, then $k = 3$ and finally $k = 2$, the induction hypothesis allowing one to use the corresponding parts of Corollary 12.

To derive the wanted Z-equality from Theorem 10, one has to analyse the contribution in Z of each product $\prod H^{g^j}$; one will eventually prove the following Z-equality:

$$H^g\left(\overline{\lambda}, \overline{\mu}, \lambda_3, ..., \lambda_k\right) \overset{Z}{=} f_{\lambda,\mu}^{\mathring{\lambda} \sqcup \mathring{\mu}} H^g\left(\mathring{\lambda} \sqcup \mathring{\mu}, \lambda_3, ..., \lambda_k\right)$$

(notice it already stands as a plain equality when $\mathring{\lambda} = \mathring{\mu} = \emptyset$, the reason for which we will leave that case aside below). It is then easy to derive the Z-equality of Theorem 8: remove the bars on top of λ and μ by multiplying by the binomials $\binom{D - (|\lambda| - m_1(\lambda))}{m_1(\lambda)}\binom{D - (|\lambda| - m_1(\lambda))}{m_1(\lambda)}$; since D strictly increases $\deg_Z$, one can multiply instead by $\frac{D^{m_1(\lambda)+m_1(\mu)}}{m_1(\lambda)!\ m_1(\mu)!}$ and still get a Z-equality $H^g(\lambda, \mu, \lambda_3, ..., \lambda_k) \overset{Z}{=} \frac{D^{m_1(\lambda)+m_1(\mu)}}{m_1(\lambda)!\ m_1(\mu)!} f_{\lambda,\mu}^{\mathring{\lambda} \sqcup \mathring{\mu}} H^g\left(\mathring{\lambda} \sqcup \mathring{\mu}, \lambda_3, ..., \lambda_k\right)$; to get from H to $\boxed{H}$, divide both sides by $\boxed{\lambda}\,\boxed{\mu}\,\boxed{\lambda_3}\cdots\boxed{\lambda_k}$; to conclude, use the identity $\frac{f_{\lambda,\mu}^{\mathring{\lambda} \sqcup \mathring{\mu}}}{m_1(\lambda)!\ m_1(\mu)!}\frac{\boxed{\mathring{\lambda} \sqcup \mathring{\mu}}}{\boxed{\lambda}\,\boxed{\mu}} = 1$ and the induction hypothesis.

Case $k \geq 4$. One has $H^g\left(\overline{\lambda}, \overline{\mu}, \lambda_3, ..., \lambda_k\right) = \sum_{\overrightarrow{\nu}, \overrightarrow{g}} f_{\lambda,\mu}^{\overrightarrow{\nu}} \sum_{\overrightarrow{\lambda_3}, ..., \overrightarrow{\lambda_k}} \prod_j H^{g^j}\left(\nu^j, \lambda_3^j, ..., \lambda_k^j\right)$ by Theorem 10 where every factor $H^{g^j}\left(\nu^j, \lambda_3^j, ..., \lambda_k^j\right)$ lies in $\mathcal{A}^Z$ by Corollary 12 for $k-1$ partitions (recall all v^j's are non-empty since we left aside the case $\mathring{\lambda} = \mathring{\mu} = \emptyset$). The product $\prod H^{g^j}\left(\nu^j, \lambda_3^j, ..., \lambda_k^j\right)$ has therefore Z-degree

$$\begin{aligned}
&\sum_j 5g^{j\prime} + 2\left(l\left(\nu^j\right) + \sum_{i\geq 3} l\left(\lambda_i^j\right)\right) \\
&\quad = 5g' - 5\frac{r(\lambda) + r(\mu)}{2} + \frac{5}{2}\left(|\overrightarrow{\nu}| - l\left(\overrightarrow{v}\right)\right) + 2l\left(\overrightarrow{\nu}\right) + 2\sum_{i\geq 3} l(\lambda_i) \\
&\quad = 2\sum_{i\geq 3} l(\lambda_i) + 5g' - \frac{5}{2}\left(r(\lambda) + r(\mu)\right) + \frac{5|\nu| - l(\nu)}{2}.
\end{aligned}$$

Everything is constant except $\frac{5|\nu|-l(\nu)}{2}$. Lemma 13 then shows that the above quantity is maximal if and only if $\nu = \mathring{\lambda} \sqcup \mathring{\mu}$; since this implies $N = 1$ and $\overrightarrow{g} = (g)$, one gets the announced Z-reduction formula.

Case $k=3$. We go along the same idea. Fix a genus $g\geq 0$ and three partitions λ,μ,ξ. Let $p:=l(\nu)+l(\xi)$ and p^j defined alike for all j. Theorem 10 then implies for any integer $M\geq 0$

$$D^M H^g\left(\overline{\lambda},\overline{\mu},\xi\right)=\sum_{\overrightarrow{\nu},\overrightarrow{g},\overrightarrow{\xi},\overrightarrow{M}} f^{\overrightarrow{\nu}}_{\lambda,\mu}\binom{M}{\overrightarrow{M}}\prod_j D^{M^j}H^{g^j}\left(\nu^j,\xi^j\right)$$

where the sum over $\overrightarrow{M}$ is taken over the N-tuples of non-negative integers M^j's which sum up to M. By the induction hypothesis for $k=2$, the term $D^M H^g\left(\mathring{\lambda}\sqcup\mathring{\mu},\xi\right)$ lies in $\mathcal{A}^Z$ for M large enough. Fix such an M. We then show that all other terms have Z-degree smaller than the latter.

By Corollary 12 for two partitions, a factor $D^{M^j}H^{g^j}\left(\nu^j,\xi^j\right)$ will belong to $\mathbb{Q}[Y]$ if and only if $\left(g^j,p^j,M^j\right)=(0,2,0)$; multiplying by such an element will decrease(ii) the Z -degree. As for the other factors, the D-trick combined with Corollary 12 for two partitions shows that their Z-degree is $5g^{j\prime}+2p^j+2M^j$. The product $\prod_j D^{M^j}H^{g^j}\left(\nu^j\right)$ has therefore Z-degree $\leq\sum_Z 5g^{j\prime}+2p^j+2M^j$ where the index Z means that $D^{M^j}H^{g^j}\left(\nu^j\right)$ lies in $\mathcal{A}^Z$.

Set $e:=\#\left\{j;\ \left(g^j,p^j,M^j\right)=(0,2,0)\right\}$ for the number of exceptional factors with no Z. The three previous Z-sums can be linked to the same sums without restriction:

$$\sum_Z g^{j\prime}=e+g'-\frac{r(\lambda)+r(\mu)-r(\overrightarrow{\nu})}{2},\ \sum_Z p^j=l(\overrightarrow{\nu})+l(\xi)-2e,\ \sum_Z M^j=M.$$

One can thus derive the majoration

$$\deg_Z\prod_j D^{M^j}H^{g^j}\left(\nu^j,\xi^j\right)\leq 2M+5g'-\frac{5}{2}\left(r(\lambda)+r(\mu)\right)+l(\xi)+\frac{5|\nu|-l(\nu)}{2}+e.$$

Like when $k\geq 4$, everything is constant except $\frac{5|\nu|-l(\nu)}{2}+e$; since there is at least one $M^j\geq 1$ (thanks to the trick of applying D), one has $e\leq N-1\leq 3(N-1)$ and Lemma 13 still holds: the maximal-Z-degreed term $\prod_j D^{M^j}H^{g^j}\left(\nu^j,\xi^j\right)$ in the sum $D^M H^g\left(\overline{\lambda},\overline{\mu},\xi\right)$ is precisely $D^M H^g\left(\mathring{\lambda}\sqcup\mathring{\mu},\xi\right)$.

Case $k=2$. The proof goes as above. Fix $g\geq 0$ any genus and λ,μ two partitions. For any $M\geq 0$, Theorem 10 implies that $D^M H^g\left(\overline{\lambda},\overline{\mu}\right)=\sum_{\overrightarrow{\nu},\overrightarrow{g},\overrightarrow{M}} f^{\overrightarrow{\nu}}_{\lambda,\mu}\binom{M}{\overrightarrow{M}}\prod_j D^{M^j}H^{g^j}\left(\nu^j\right)$. By Corollary 12 for one partition (namely Corollary 6), a factor $D^{M^j}H^{g^j}\left(\nu^j\right)$ will belong to $\mathbb{Q}[Y]$ if and only if $\left(g^j,p^j,M^j\right)\in\{(0,1,1),(0,1,0),(0,2,0)\}$. For the other factors, we have already stated that their degrees were $5g^{j\prime}+2p^j+2M^j$. The product $\prod_j D^{M^j}H^{g^j}\left(\nu^j\right)$ has therefore Z-degree $\leq\sum_Z 5g^{j\prime}+2p^j+2M^j$. After linking the Z-sums to the (no Z)-sums, one obtains the majoration

$$\begin{aligned}\deg_Z\prod_j D^{M^j}H^{g^j}\left(\nu^j\right) &\leq 2M+5g'-\frac{5}{2}\left(r(\lambda)+r(\mu)\right)+\frac{5|\nu|-l(\nu)}{2}\\ &\quad+\#\left\{j;\ \begin{array}{c}g^j=0\\ p^j=1\\ M^j=1\end{array}\right\}+3\#\left\{j;\ \begin{array}{c}g^j=0\\ p^j=1\\ M^j=0\end{array}\right\}+\left\{j;\ \begin{array}{c}g^j=0\\ p^j=2\\ M^j=0\end{array}\right\}.\end{aligned}$$

(ii) strictly if and only if its (Y-)coefficients sum up to zero

The three sets whose cardinalities are involved being mutually disjoint, the corresponding sum is $\leq N$ and one can even remplace N by $N-1$ if there is at least one $M^j \geq 2$, which can be realised by choosing $M \geq 2\left(\left|\mathring{\lambda}\right| + |\mathring{\mu}| + 1\right) \geq 2N$. Therefore, one can still apply Lemma 13 and conclude, which finishes the proof of Theorem 8. □

Lemma 13. *Let λ, μ be two partitions and σ, ρ two permutation in $\mathfrak{S}_\infty$ of type $\left(\overline{\lambda}, \overline{\mu}\right)$. Denote ν the partition of $\sigma\rho$ induced on $S\sigma \cup S\rho$. Cluster the cycles of ν into N orbits such that the junction condition of Definition 11 is satisfied. Then the quantity $\frac{5|\nu| - l(\nu)}{2} + 3(N-1)$ is maximal if and only if σ and ρ have disjoint supports. (And, in that case, one has $N = 1$.)*

Proof. Call a cycle of σ or ρ to be *interlaced* if it encounters another cycle of σ or ρ (and two such cycles will be called *interlaced with* each other). Set c for the number of interlaced cycles and c' for the number of cycles (included fixed cycles) of the product $\sigma\rho$ induced on the interlaced cycles (of σ and ρ).

A crucial remark is the following: for the junction condition to be satisfied, every cycle of ν must lie in the same orbit as an interlaced cycle, whence the inequality $N \leq c'$.

If one sets $k := |S\sigma \cap S\rho|$ for the number of contact points of the supports, one can write

$$\begin{cases} |\nu| = \left|\mathring{\lambda}\right| + |\mathring{\mu}| - k \\ l(\nu) = l\left(\mathring{\lambda}\right) + l(\mathring{\mu}) - c + c' \end{cases}, \quad \text{hence the quantity to be upper-bounded:} \quad Q := \frac{5(-k) - (c' - c)}{2} + 3(N-1).$$

When $S\sigma \cap S\rho = \emptyset$, all variables $c, c', k, N-1$ equal 0 and so does Q. One has therefore to show $Q < 0$, namely $-2Q \geq 1$, for any other ν than $\mathring{\lambda} \sqcup \mathring{\mu}$. By the crucial remark, it suffices to show the same inequality $5k + c' - c - 2(3N-3) \geq 1$ with some N's been replaced by the same number of c''s: so as to kill the c' in the inequality, we remplace one N out of six, which lead us to wonder if the inequality $5(k+1-N) \geq c$ holds. We are going to show by induction on $|S\sigma| + |S\rho|$ the stronger inequality

$$2(k - N + 1) \geq c.$$

When $\sigma = \rho = \mathrm{Id}$, then all three quantitites $c, k, N-1$ equal 0, whence the above inequality.

Suppose now $\left|\mathring{\lambda}\right| + |\mathring{\mu}| > 0$. Because of the assumption $\nu \neq \mathring{\lambda} \sqcup \mathring{\mu}$, one has $k \geq 1$: take one contact point x in $S\sigma \cap S\rho$, set $y := \sigma(x)$ and $\tau := (x, y)$ the transposition exchanging these points. Finally, write $\sigma = \tau\sigma_*$ where $\sigma_* := \tau\sigma$ fixes x and therefore satifies $|\sigma^*| < |\sigma|$. Thus, one obtains the cycle decomposition of $\sigma\rho$ by multiplying that of $\sigma_*\rho$ by the transposition τ on the left (and conversely). Denote by a $*$-subscript the quantities c_*, k_*, N_* associated to the product of σ_* and ρ; notice that N_* is not well-defined and can be chosen arbitrarily as long as the junction condition is satisfied. For such an N_*, one has the induction hypothesis $c_* \leq 2(k_* - N_* + 1)$. What we want is to dispose of the $*$'s.

Since x is fixed by σ_*, it disappears from the contact points, hence $k_* < k$. Besides, σ_* loses at most one interlaced cycle (it can only be the σ-orbit of x) and ρ loses at most two interlaced cycles (those maybe interlaced with τ), hence $c_* \geq c - 3$. But the case $c_* = c - 3$ implies x's σ-orbit to be a transposition interlaced with two ρ-cycles, each of which not being interlaced with another σ-cycle; since σ and ρ play symmetric roles (set $y := \rho(x)$ instead of $\sigma(x)$), one can avoid this case and hence assume $c_* \geq c - 2$.

Let us look at what happens to the cycles of $\sigma\rho$ when composing (on the left) by τ. If a ($\sigma\rho$-)cycle γ is split in two cycles, cluster them in the same orbit as γ's orbit (hence $N_* = N$). If two cycles are joined, either both cycles were in the same orbit (then, do not change the orbits, hence $N_* = N$) or they were in distinct orbits (then, merge these orbits and do not change the others, hence $N_* = N - 1$). Whenever $N_* = N$, one can conclude by writing

$$c \leq c_* + 2 \leq 2\left(k_* - N_* + 1\right) + 2 \leq 2\left((k-1) - N + 1\right) + 2 = 2\left(k - N + 1\right).$$

We can consequently assume $N_* = N - 1$ and hence τ *joining* two $\sigma\rho$-cycles, which goes the same as saying x and y not to lie in the same $\sigma\rho$-orbit. But that implies both σ- and ρ-orbits of x to remain interlaced for σ_* and ρ (if not, iterate $\sigma\rho$ in a not-interlaced orbit to join x and y), hence $c_* = c$ and the induction hypothesis yields

$$c = c_* \leq 2\left(k_* - N_* + 1\right) \leq 2\left((k-1) - (N-1) + 1\right) = 2\left(k - N + 1\right). \square$$

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