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A \textit{q}-analog of Ljunggren’s binomial congruence

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Abstract. We prove a \textit{q}-analog of a classical binomial congruence due to Ljunggren which states that
\[
\begin{pmatrix} ap \\ bp \end{pmatrix} \equiv \begin{pmatrix} a \\ b \end{pmatrix} \pmod{p^3}
\]
modulo \( p^3 \) for primes \( p \geq 5 \). This congruence subsumes and builds on earlier congruences by Babbage, Wolstenholme and Glaisher for which we recall existing \textit{q}-analogs. Our congruence generalizes an earlier result of Clark.

1 Introduction and notation

Recently, \textit{q}-analogs of classical congruences have been studied by several authors including (Cla95), (And99), (SP07), (Pan07), (CP08), (Dil08). Here, we consider the classical congruence
\[
\begin{pmatrix} ap \\ bp \end{pmatrix} \equiv \begin{pmatrix} a \\ b \end{pmatrix} \pmod{p^3}
\]
which holds true for primes \( p \geq 5 \). This also appears as Problem 1.6 (d) in (Sta97). Congruence (1) was proved in 1952 by Ljunggren, see (Gra97), and subsequently generalized by Jacobsthal, see Remark 6.
Let \([n]_q := 1 + q + \ldots + q^{n-1}\), \([n]_q! := [n]_q[n-1]_q \cdots [1]_q\) and
\[
\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}
\]
denote the usual \(q\)-analogs of numbers, factorials and binomial coefficients respectively. Observe that \([n]_1 = n\) so that in the case \(q = 1\) we recover the usual factorials and binomial coefficients as well. Also, recall that the \(q\)-binomial coefficients are polynomials in \(q\) with nonnegative integer coefficients. An introduction to these \(q\)-analogs can be found in \(\text{Sta97}\).

We establish the following \(q\)-analog of (1):

**Theorem 1** For primes \(p \geq 5\) and nonnegative integers \(a, b\),
\[
\binom{ap}{bp}_q \equiv \binom{a}{b}_q \cdot \binom{a}{b+1} \cdot \left(\frac{b+1}{2}\right)^2 \frac{p^2 - 1}{12} (q^p - 1)^2 \mod [p]_q^3.
\]

The congruence (2) and similar ones to follow are to be understood over the ring of polynomials in \(q\) with integer coefficients. We remark that \(p^2 - 1\) is divisible by 12 for all primes \(p \geq 5\).

Observe that (2) is indeed a \(q\)-analog of (1): as \(q \to 1\) we recover (1).

**Example 2** Choosing \(p = 13\), \(a = 2\), and \(b = 1\), we have
\[
\binom{26}{13}_q = 1 + q^{169} - 14(q^{13} - 1)^2 + (1 + q + \ldots + q^{12})^3 f(q)
\]
where \(f(q) = 14 - 41q + 41q^2 - \ldots + q^{132}\) is an irreducible polynomial with integer coefficients. Upon setting \(q = 1\), we obtain \(\binom{26}{13} \equiv 2\) modulo \(13^3\).

Since our treatment very much parallels the classical case, we give a brief history of the congruence (1) in the next section before turning to the proof of Theorem 1.

## 2 A bit of history

A classical result of Wilson states that \((n-1)! + 1\) is divisible by \(n\) if and only if \(n\) is a prime number. “In attempting to discover some analogous expression which should be divisible by \(n^2\), whenever \(n\) is a prime, but not divisible if \(n\) is a composite number”, \(\text{Bab19}\). Babbage is led to the congruence
\[
\binom{2p-1}{p-1}_q \equiv 1 \mod p^2
\]
for primes \(p \geq 3\). In 1862 Wolstenholme, \(\text{Wol62}\), discovered (3) to hold modulo \(p^3\), “for several cases, in testing numerically a result of certain investigations, and after some trouble succeeded in proving it to hold universally” for \(p \geq 5\). To this end, he proves the fractional congruences
\[
\sum_{i=1}^{p-1} \frac{1}{i} \equiv 0 \mod p^2,
\]
\[
\sum_{i=1}^{p-1} \frac{1}{i^2} \equiv 0 \mod p
\]
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for primes \( p \geq 5 \). Using (4) and (5) he then extends Babbage’s congruence (3) to hold modulo \( p^3 \):

\[
\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}
\]

(6)

for all primes \( p \geq 5 \). Note that (6) can be rewritten as \( \binom{2p}{p} \equiv 2 \pmod{p^3} \). The further generalization of (6) to (1), according to (Gra97), was found by Ljunggren in 1952. The case \( b = 1 \) of (1) was obtained by

Glaisher, (Gla00), in 1900.

In fact, Wolstenholme’s congruence (6) is central to the further generalization (1). This is just as true when considering the \( q \)-analogs of these congruences as we will see here in Lemma 5.

A \( q \)-analog of the congruence of Babbage has been found by Clark (Cla95) who proved that

\[
\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^2} \pmod{[p]^2_q}.
\]

(7)

We generalize this congruence to obtain the \( q \)-analog (2) of Ljunggren’s congruence (1). A result similar to (7) has also been given by Andrews in (And99).

Our proof of the \( q \)-analog proceeds very closely to the history just outlined. Besides the \( q \)-analog (7) of Babbage’s congruence (3) we will employ \( q \)-analogs of Wolstenholme’s harmonic congruences (4) and (5) which were recently supplied by Shi and Pan, (SP07):

Theorem 3

For primes \( p \geq 5 \),

\[
\sum_{i=1}^{p-1} \frac{1}{[i]_q} = -\frac{p-1}{2} (q-1) + \frac{p^2-1}{24} (q-1)^2 [p]_q \pmod{[p]^2_q}
\]

(8)

as well as

\[
\sum_{i=1}^{p-1} \frac{1}{[i]^2_q} = -\frac{(p-1)(p-5)}{12} (q-1)^2 \pmod{[p]_q}.
\]

(9)

This generalizes an earlier result (And99) of Andrews.

3 A \( q \)-analog of Ljunggren’s congruence

In the classical case, the typical proof of Ljunggren’s congruence (1) starts with the Chu-Vandermonde identity which has the following well-known \( q \)-analog:

Theorem 4

\[
\binom{m+n}{k}_q = \sum_j \binom{m}{j}_q \binom{n}{k-j}_q q^{j(n-k+j)}.
\]

We are now in a position to prove the \( q \)-analog of (1).

Proof of Theorem 1: As in (Cla95) we start with the identity

\[
\binom{ap}{bp}_q = \sum_{c_1+\ldots+c_a=bp} \binom{p}{c_1}_q \binom{p}{c_2}_q \cdots \binom{p}{c_a}_q q^{\sum_{1 \leq i \leq a} (i-1)c_i - \sum_{1 \leq i < j \leq a} c_i c_j}
\]

(10)
which follows inductively from the \( q \)-analog of the Chu-Vandermonde identity given in Theorem 4. The summands which are not divisible by \( [p]_q^2 \) correspond to the \( c_i \) taking only the values 0 and \( p \). Since each such summand is determined by the indices \( 1 \leq j_1 < j_2 < \ldots < j_b \leq a \) for which \( c_i = p \), the total contribution of these terms is

\[
\sum_{1 \leq j_1 < \ldots < j_b \leq a} q^p \sum_{k=1}^{a} (j_k - 1 - p^2) = \sum_{0 \leq \ell_1 < \ldots < \ell_b \leq a-b} q^p \sum_{k=1}^{a} \ell_k = (a \choose b)_{q^p^2}.
\]

This completes the proof of (7) given in (Cla95).

To obtain (2) we now consider those summands in (10) which are divisible by \( [p]_q^3 \) but not divisible by \( [p]_q^4 \). These correspond to all but two of the \( c_i \) taking values 0 or \( p \). More precisely, such a summand is determined by indices \( 1 \leq j_1 < j_2 < \ldots < j_b < j_{b+1} \leq a \), two subindices \( 1 \leq k < \ell \leq b+1 \), and \( 1 \leq d \leq p-1 \) such that

\[
c_i = \begin{cases} 
  d & \text{for } i = j_k, \\
  p-d & \text{for } i = j_\ell, \\
  p & \text{for } i \in \{j_1, \ldots, j_{b+1}\} \setminus \{j_k, j_\ell\}, \\
  0 & \text{for } i \notin \{j_1, \ldots, j_{b+1}\}.
\end{cases}
\]

For each fixed choice of the \( j_i \) and \( k, \ell \) the contribution of the corresponding summands is

\[
\sum_{d=1}^{p-1} \left( \begin{array}{c} p \\ d \end{array} \right)_q \left( \begin{array}{c} p \\ p-d \end{array} \right)_q q^p \sum_{i \leq \ell, e_i = \ell} c_i - \sum_{1 \leq i < j \leq a} c_i c_j
\]

which, using that \( q^p \equiv 1 \) modulo \( [p]_q \), reduces modulo \( [p]_q^3 \) to

\[
\sum_{d=1}^{p-1} \left( \begin{array}{c} p \\ d \end{array} \right)_q \left( \begin{array}{c} p \\ p-d \end{array} \right)_q q^{d^2} = \left( \begin{array}{c} 2p \\ p \end{array} \right)_q - [2]_{q^p^2}.
\]

We conclude that

\[
\left( \begin{array}{c} ap \\ bp \end{array} \right)_{q^p^2} = \left( \begin{array}{c} a \\ b \end{array} \right)_{q^p^2} + \left( \begin{array}{c} a \\ b+1 \end{array} \right) \left( \begin{array}{c} b+1 \\ 2 \end{array} \right) \left( \begin{array}{c} 2p \\ p \end{array} \right)_q - [2]_{q^p^2} \right) \mod [p]_q^3. \tag{11}
\]

The general result therefore follows from the special case \( a = 2, b = 1 \) which is separately proved next. \( \square \)

4 A \( q \)-analog of Wolstenholme’s congruence

We have thus shown that, as in the classical case, the congruence (2) can be reduced, via (11), to the case \( a = 2, b = 1 \). The next result therefore is a \( q \)-analog of Wolstenholme’s congruence (6).

**Lemma 5** For primes \( p \geq 5 \),

\[
\left( \begin{array}{c} 2p \\ p \end{array} \right)_q \equiv [2]_{q^p^2} - \frac{p^2 - 1}{12} (q^p - 1)^2 \mod [p]_q^3.
\]
**Proof:** Using that 
\[
\binom{2p}{p}_q = \binom{2p}{p}_q \binom{p+1}{1}_q \cdot \binom{p+2}{2}_q \ldots \binom{p+q-1}{q-1}_q
\]
we compute
\[
\binom{2p}{p}_q = \frac{2q^p \prod_{k=1}^{p-1} (p)_q + q^p (p-k)_q}{(p-1)_q!, \cdot} \]
which modulo \([p]_q^3\) reduces to (note that \([p-1]_q\) is relatively prime to \([p]_q^3\))
\[
[2]_q^p \left( \binom{p-1}{p} + q^{(p-3)p} \sum_{1 \leq i < j \leq p-1} \frac{[i]_q}{[i]_q} + \binom{p-3}{3} \sum_{1 \leq i < j \leq p-1} \frac{[i]_q [j]_q}{[i]_q [j]_q} \right).
\]
Combining the results \([8]\) and \([9]\) of Shi and Pan, (SP07), given in Theorem \([3]\) we deduce that for primes \(p \geq 5\),
\[
\sum_{1 \leq i < j \leq p-1} \frac{1}{[i]_q [j]_q} \equiv \frac{(p-1)(p-2)}{6} (q-1)^2 \quad \mod [p]_q.
\]
Together with \([8]\) this allows us to rewrite \([12]\) modulo \([p]_q^3\) as
\[
[2]_q^p \left( \binom{p-1}{p} + q^{(p-3)p} \left( -\frac{p-1}{2} (q^p - 1) + \frac{p^2 - 1}{24} (q^p - 1)^2 \right) + \right.
\]
\[
\left. + q^{(p-3)p} \frac{(p-1)(p-2)}{6} (q^p - 1)^2 \right).
\]
Using the binomial expansion
\[
q^{mp} = ((q^p - 1) + 1)^m = \sum_k \binom{m}{k} (q^p - 1)^k
\]
to reduce the terms \(q^{mp}\) as well as \([2]_q^p = 1 + q^p\) modulo the appropriate power of \([p]_q\) we obtain
\[
\binom{2p}{p}_q \equiv 2 + q^p (q^p - 1) + \frac{(p-1)(5p-1)}{12} (q^p - 1)^2 \quad \mod [p]_q^3.
\]
Since
\[
[2]_q^{p^2} \equiv 2 + q^p (q^p - 1) + \frac{(p-1)p}{2} (q^p - 1)^2 \quad \mod [p]_q^3
\]
the result follows. \(\square\)

**Remark 6** Jacobsthal, see (Gra97), generalized the congruence \([1]\) to hold modulo \(p^{3+r}\) where \(r\) is the \(p\)-adic valuation of
\[
ab(a-b) \left( \begin{array}{c} a \\ b \end{array} \right) = 2a \left( \begin{array}{c} a \\ b+1 \end{array} \right) \left( \frac{b+1}{2} \right).
\]
It would be interesting to see if this generalization has a nice analog in the \(q\)-world.
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