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Product decomposition for surjective 2-block NCCA

Felipe García-Ramos[†]

University of British Columbia, Vancouver, Canada

In this paper we define products of one-dimensional Number Conserving Cellular Automata (NCCA) and show that surjective NCCA with 2 blocks (i.e radius 1/2) can always be represented as products of shifts and identites. In particular, this shows that surjective 2-block NCCA are injective.

Keywords: Discrete dynamical systems, cellular automata, number conserving cellular automata, conservation laws, characterization of surjective NCCA

1 Introduction

It is known that injective Cellular Automata (CA) are surjective. In general, the converse is not true, and there are many algebraic CA counterexamples. However, there are interesting subclasses where this might be true. For example, if a surjective CA has entropy 0 then it is almost injective (Moothathu (2011)) and it is not known if it is actually injective. The author believes there are some sub-classes of potential preserving CA, including Number Conserving CA (NCCA), where there are no surjective but not injective CA.

The subclass of NCCA, besides providing interesting mathematical structure, is used for discrete models in scientific disciplines where one simulates systems governed by conservation laws of mass or energy. Many papers have been published on traffic models using NCCA (for example see Maerivoet and Moor (2005)).

The Moore-Myhill theorem says that a CA is surjective iff it is injective on homoclinic classes. Actually, it is easy to see that a NCCA is surjective iff it is bijective on homoclinic classes. This suggests that there might be a closer relationship between surjective NCCA and injective NCCA. So far, it is known that surjective NCCA have dense periodic points (Formenti and Grange (2003)). If it turns out that surjective NCCA are injective we would recover this result, since for bijective CA periodic points are dense.

2 Definitions and classical results

Let $\mathcal{A}$ be a finite set, which will sometimes be referred to as the alphabet. We define the full $\mathcal{A}$-shift as the space of bi-sequences $\mathcal{A}^{\mathbb{Z}}$. We will endow this space with the Cantor (product) topology. If $\omega \in \mathcal{A}^{\mathbb{Z}}$, we

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denote $(\omega)_i$ as the ith coordinate of point x. We will use $\sigma_R : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ as the right shift map, i.e. the map that satisfies $(\omega)_i = (\sigma_R(\omega))_{i+1}$ for all $\omega \in \mathcal{A}^{\mathbb{Z}}$ and $i \in \mathbb{Z}$.

Definition 1 *A **cellular automaton** (CA) is a continuous map $\phi(\cdot) : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ that commutes with the shift.*

Theorem 2 (Curtis-Hedlund-Lyndon) *Hedlund (1969) Let $\phi(\cdot) : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$. The map ϕ is a CA iff there exist two non-negative integers L and R (which represent the left and right radius), and a function $\phi[\cdot] : \mathcal{A}^{L+R+1} \to \mathcal{A}$, such that $(\phi(\omega))_i = \phi\left[(\omega)_{i-m}(\omega)_{i-m+1} ... (\omega)_i ... (\omega)_{i+a}\right]$ (note the use of ( ), and [ ] to distinguish between the two functions that are related).*

We say $L + R + 1$ is the neighbourhood size of ϕ.

Definition 3 *In this paper a **2-block CA** ϕ (also known as CA with radius 1/2) is a map with $L = 1$ and $R = 0$.*

For example the right shift is a 2-block CA. The reader will see that all the results are analogous for $L = 0$ and $R = 1$.

We say two points in $\mathcal{A}^{\mathbb{Z}}$ are *equivalent*, if they differ only on finitely many coordinates. The *homoclinic class* of a point ω is the set of points equivalent to ω.

Theorem 4 (Moore-Myhill) *Moore (1963) Myhill (1963) Let ϕ be a CA. Then ϕ is surjective iff ϕ is injective when restricted to homoclinic classes iff ϕ is injective when restricted to one homoclinic class.*

Definition 5 *We say a cellular automaton $\phi : [0...A)^{\mathbb{Z}} \to [0...A)^{\mathbb{Z}}$ is **number conserving**, also denoted as **NCCA**, if for every point ω in the homoclinic class of 0^∞ we have that $\sum_{i\in\mathbb{Z}} (\phi(\omega))_i < \infty$, and*

$$\sum_{i\in\mathbb{Z}} (\phi(\omega))_i = \sum_{i\in\mathbb{Z}} (\omega)_i.$$

The following result is the particular case for 2-block CA of a general result by Hattori and Takesue (1991), which was used by Boccara and Fuks (2002) to characterize NCCA. We provide a proof of this weaker result for completeness.

Proposition 6 *Let $\phi : [0...a]^{\mathbb{Z}} \to [0...a]^{\mathbb{Z}}$ be a 2-block CA. Then ϕ is a NCCA iff*

$$\phi[pq] = q + \phi[p0] - \phi[q0]. \tag{1}$$

Proof: We have that $\phi(0^\infty p 0^\infty) = 0^\infty\ \phi[0p]\ \phi[p0]\ 0^\infty$. This means that

$$p = \phi[0p] + \phi[p0]. \tag{2}$$

Similarly consider the image of the point $\phi(0^\infty pq 0^\infty) = 0^\infty abc 0^\infty$. We have that $a = \phi[0p]$, $b = \phi[pq]$, and $c = \phi[q0]$. Since ϕ is a NCCA we have that

$$p + q = a + b + c = \phi[0p] + \phi[pq] + \phi[q0]. \tag{3}$$

Combining (3) and (2) we get (1).

Conversely suppose ϕ satisfies (1). Let ω be a point in the homoclinic class of 0^∞. This means there exist j and k, such that $(\omega)_i = 0$ for $i > k$ and $i < j$. So we get that

$$\begin{aligned}\sum_{i\in\mathbb{Z}}(\phi(\omega))_i &= \phi\left[0(\omega)_j\right] + \sum_{i=j}^{k-1}\phi\left[(\omega)_i(\omega)_{i+1}\right] + \phi\left[(\omega)_k 0\right] \\ &= (\omega)_j - \phi\left[(\omega)_j 0\right] + \sum_{i=j}^{k-1}\left((\omega)_{i+1} + \phi\left[(\omega)_i 0\right] - \phi\left[(\omega)_{i+1}0\right]\right) + \phi\left[(\omega)_k 0\right] \\ &= \sum_{i=j}^{k}(\omega)_i.\end{aligned}$$

□

This result tells us that a 2-block NCCA is uniquely determined by the values of $\phi\left[x0\right]$ and if $p = q = 0$ we get $\phi\left[00\right] = 0$.

Example 7 *The reader can check that* $\phi : [0,1,2]^{\mathbb{Z}} \to [0,1,2]^{\mathbb{Z}}$, *with* $\phi\left[10\right] = 0$ *and* $\phi\left[20\right] = 1$, *is a well defined (non-surjective) NCCA but there is no 2-block NCCA* $\phi : [0...3)^{\mathbb{Z}} \to [0...3)^{\mathbb{Z}}$ *with* $\phi\left[10\right] = 1$ *and* $\phi\left[20\right] = 0$, *because the image of the point* $0^\infty 120^\infty$ *cannot have sum equal to 3.*

In general a specification $\phi\left[\cdot 0\right]$ is defines a 2-block NCCA $\phi : [0...a]^{\mathbb{Z}} \to [0...a]^{\mathbb{Z}}$ via (1) iff

$$0 \le q + \phi\left[p0\right] - \phi\left[q0\right] \le a \ \ \forall p, q \in [0...a]\,.$$

We will use the following result several times.

Theorem 8 *A NCCA is surjective iff it is bijective on the homoclinic class of* 0^∞.

Proof: Simply apply Theorem 4 and note that the image and preimage of the homoclinic class of 0^∞ under ϕ is in the homoclinic class of 0^∞. □

We would like to characterize NCCA in a more concrete way, at least under some special assumptions. In this paper we will do so for surjective 2-block NCCA.

We will first show how to represent the product of 2-block NCCA's as a 2-block NCCA. We will establish some properties of the product and finally show that all surjective 2-block NCCA can be represented as the product of shift and identity maps.

3 Products of NCCA

For easier notation we define $A := a+1$, $B := b+1$, and $[0...n) := [0...n-1]$. We will use A and B or a and b, depending on which one gives an easier notation.

Let $\phi : [0...A)^{\mathbb{Z}} \to [0...A)^{\mathbb{Z}}$ and $\psi : [0...B)^{\mathbb{Z}} \to [0...B)^{\mathbb{Z}}$ be two 2-block codes. Consider the function $F(p_1, p_2) = p_2 + Bp_1$, where $p_1 \in [0...A)$ and $p_2 \in [0...B)$. This function is a bijection between $[0...A) \times [0...B)$ and $[0...AB)$. Furthermore, it satisfies

$$F(p_1 + p_1', p_2 + p_2') = F(p_1, p_2) + F(p_1', p_2'),$$

for $p_1 + p_1' \in [0, a]$ and $p_2 + p_2' \in [0, b]$. Now, for all $p, q \in [0...AB)$, we can define $\phi \times \psi [pq] = F(\phi [F_1(p)F_1(q)], \psi [F_2(p)F_2(q)])$, where F_1 and F_2 are the coordinates of the inverse, i.e.

$$\phi \times \psi \left[\binom{p_1}{p_2} \binom{q_1}{q_2} \right] = \binom{\phi [p_1 \; q_1]}{\psi [p_2 \; q_2]},$$

where $\binom{\alpha}{\beta} = F(\alpha, \beta)$.

Now we note that if ϕ and ψ are number conserving, then the product $\chi = \phi \times \psi$ is also number conserving since

$$\begin{aligned} \chi \left[\binom{p_1}{p_2} \binom{q_1}{q_2} \right] &= \binom{\phi [p_1 \; q_1]}{\psi [p_2 \; q_2]} \\ &= \binom{q_1 + \phi [p_1 \; 0] - \phi [q_1 \; 0]}{q_2 + \psi [p_2 \; 0] - \psi [q_2 \; 0]} \\ &= \binom{q_1}{q_2} + \binom{\phi [p_1 \; 0]}{\psi [p_2 \; 0]} - \binom{\phi [q_1 \; 0]}{\psi [q_2 \; 0]} \\ &= \binom{q_1}{q_2} + \chi \left[\binom{p_1}{p_2} \binom{0}{0} \right] - \chi \left[\binom{q_1}{q_2} \binom{0}{0} \right], \end{aligned}$$

and therefore χ satisfies equation (1).

Since F is not symmetric in general, $\phi \times \psi$ need not be the same as $\psi \times \phi$.

Even though we can make products of any two 2-block codes, we will mainly be interested in products of shifts and identities (which will be denoted as σ_R and Id respectively).

If $\phi = Id$, and $\chi = \phi \times \psi$ then

$$\chi \left[\binom{p_1}{p_2} \binom{0}{0} \right] = \binom{\phi [p_1 \; 0]}{\psi [p_2 \; 0]} = \binom{0}{\psi [p_2 \; 0]} = \psi(p_2 \; 0). \tag{4}$$

If $\phi = \sigma_R$, then

$$\chi \left[\binom{p_1}{p_2} \binom{0}{0} \right] = \binom{\phi [p_1 \; 0]}{\psi [p_2 \; 0]} = \binom{p_1}{\psi [p_2 \; 0]} = \psi(p_2 \; 0) + p_1(b+1). \tag{5}$$

Example 9 *Let* $\sigma_R : [0...3)^{\mathbb{Z}} \to [0...3)^{\mathbb{Z}}$, $Id : [0...4)^{\mathbb{Z}} \to [0...4)^{\mathbb{Z}}$, *and* $\chi = \sigma_R \times Id : [0...(3 \cdot 4))^{\mathbb{Z}} \to [0...(3 \cdot 4))^{\mathbb{Z}}$. *For every* $x \in [0...12)$ *there exists* $p_1 \in [0...3)$ *and* $p_2 \in [0...4)$ *such that* $x = p_2 + 4p_1$, *where* $p_1 \in [0...3)$ *and* $p_2 \in [0...4)$. *Hence* $\chi [x0] = p_1$. *In a table it looks like this.*

x	0	1	2	3	4	5	6	7	8	9	10	11
$\chi [x0]$	0	0	0	0	1	1	1	1	2	2	2	2

If we take instead $Id \times \sigma_R$ *we get the following table.*

x	0	1	2	3	4	5	6	7	8	9	10	11
$\chi [x0]$	0	1	2	0	1	2	0	1	2	0	1	2

The following lemma describes an important property when one of the factors is a shift or an identity. The proof uses formulas (4) and (5) and is left to the reader.

Lemma 10 *Let $\phi = Id$. We have that $\chi[x0] = x$ iff $x < B$ and $\psi[x0] = x$; and $\chi[x0] = 0$ iff there exists $n \in [0...A)$ such that $x = y + nB$ and $\psi[y0] = 0$.*

Similarly let $\phi = \sigma_R$. We also have that $\chi[x0] = 0$ iff $x < B$ and $\psi[x0] = 0$; and $\chi[x0] = x$ iff there exists $n \in [0...A)$ and $y \in [0...B)$ such that $x = y + nB$ and $\psi[y\ 0] = y$.

Definition 11 *Let ϕ be a 2-block cellular automata. We say ϕ is a* ***shift-identity product cellular automa (SIPCA)*** *if $\phi = \phi_n \times \cdots \times \phi_1$, where $\phi_i = Id$ for all even $i's$, and $\phi_i = \sigma_R$ for all odd (or vice versa).*

Notation 12 *We will denote $f_\phi(x) = \phi[x0]$. Notice from (1) that the function $f_\phi(x) = \phi[x0]$ completely determines ϕ.*

NCCA arise in the context of particle preserving maps. People have shown (Boccara and Fuks (2002), Pivato (2002), Moreira et al. (2004) for 1-d and recently Kari and Taati (2008) for 2-d) that it is equivalent to give a NCCA as a compatible list of *particle displacement representations*. In the case of 2-block NCCA, the particle displacement representations are given by $f_\phi(x)$, which represent how many particles move to the right when you see x particles in a certain position.

The product of two shifts is a shift, and the product of two identities is an identity. Thus all products of right-shifts and identities are SIPCA. In general if our alphabet is $[0...A)$, for every way that we can write $A = A_n \cdot A_{n-1} \cdots A_1$ (with $A_i \in \mathbb{N}$), we have two SIPCA with that alphabet. We can take $\chi = \phi_n \times \cdots \times \phi_1$ with $\phi_i : [0...A_i)^{\mathbb{Z}} \to [0...A_i)^{\mathbb{Z}}$ alternating between shifts and identities with either $\phi_1 = Id$ or $\phi_1 = \sigma_R$.

It's useful to describe the graph of f_χ. Suppose $\phi_1 : [0...A)^{\mathbb{Z}} \to [0...A)^{\mathbb{Z}}$ is any 2-block NCCA and $\phi_2 : [0...3)^{\mathbb{Z}} \to [0...3)^{\mathbb{Z}}$. Figure 1 represents the graph of f_χ when $\phi_2 = Id$, and Figure 2 when $\phi_2 = \sigma_R$. Figure 3 and 4 represent the graph of f_χ, where χ is a SIPCA and $\phi_1 = Id$ and σ_R respectively.

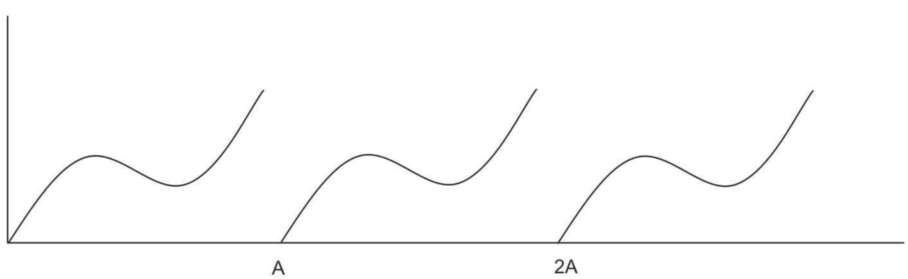

Fig. 1: f_ϕ when $\phi_2 = Id$.

4 Main result

The main goal of this paper is to prove the following result.

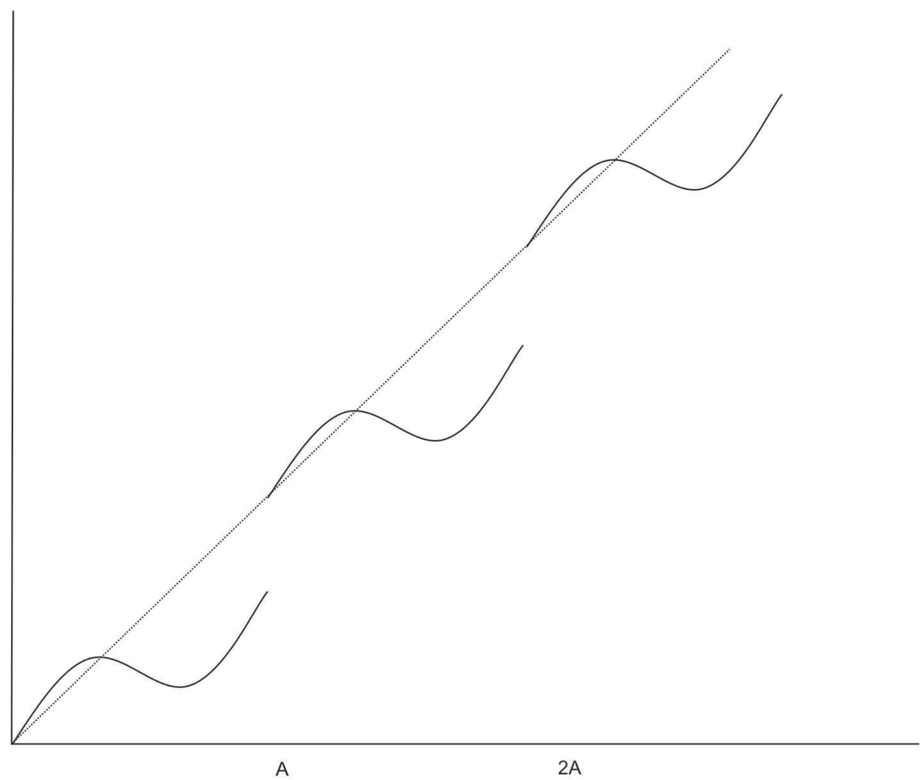

Fig. 2: f_ϕ when $\phi_2 = \sigma_R$.

Theorem 13 *All surjective 2-block NCCA are SIPCA.*

Definition 14 *Let* $\chi : [0...A)^{\mathbb{Z}} \to [0...A)^{\mathbb{Z}}$ *be a SIPCA. We say* $t \leq A$ *is a* ***transition point*** *if* $\chi \mid_{[0...t)^{\mathbb{Z}}}$ *is a SIPCA.*

Example 15 *Let* $Id : [0...A)^{\mathbb{Z}} \to [0...A)^{\mathbb{Z}}$ *and* $\sigma_R : [0...B)^{\mathbb{Z}} \to [0...B)^{\mathbb{Z}}$*. The transition points of* $Id \times \sigma_R$ *are* $[0...B) \cup \{B, 2B, ..., AB\}$.

Example 16 *Let* $Id : [0...A)^{\mathbb{Z}} \to [0...A)^{\mathbb{Z}}$ *and* $\phi : [0...B)^{\mathbb{Z}} \to [0...B)^{\mathbb{Z}}$ *be a SIPCA. The transition points of* $Id \times \phi$ *are the transition points of* ϕ *and* $\{B, 2B, ..., AB\}$.

In general if $\chi = \phi_n \times \cdots \times \phi_1$ is a SIPCA with $\phi_i : [0...A_i)^{\mathbb{Z}} \to [0...A_i)^{\mathbb{Z}}$, then t is a transition point iff $t = \left(\prod_{i=1}^{j-1} A_i \right) B_j$, where $B_j \in [1...A_j)$.

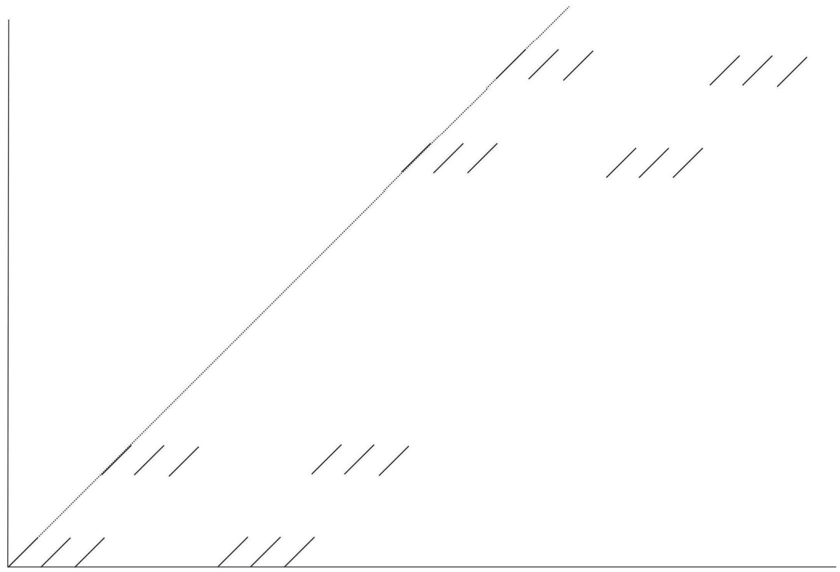

Fig. 3: f_ϕ, where ϕ is a SIPCA and $\phi_1 = Id$.

Lemma 17 *Let* $\chi = \phi_n \times \cdots \times \phi_1 : [0...A)^{\mathbb{Z}} \to [0...A)^{\mathbb{Z}}$ *with* $\phi_i : [0...A_i)^{\mathbb{Z}} \to [0...A_i)^{\mathbb{Z}}$*be a SIPCA and* x *a non-transition point with* $f_\chi(x) = x$ *(or* 0*). If* $t = \left(\prod_{i=1}^{j-1} A_i\right) B_j$ *is the previous transition point then* $f_\chi(t) = t$ *(or* 0*),* $f_\chi(\prod_{i=1}^{j-2} A_i) = 0$ *(or* $\prod_{i=1}^{j-2} A_i$*), and* $x - t < \prod_{i=1}^{j-2} A_i$.

Proof: Suppose x is not a transition point and $f_\chi(x) = x$. Let $t < x$ be the previous transition point, so $t = \left(\prod_{i=1}^{j-1} A_i\right) B_j$ and $\phi_j = \sigma_R$(see Lemma 10). Let $y_1 = x - t < \prod_{i=1}^{j-1} A_i$. We have that $f_\chi(y_1) = y_1$, but since $\chi \mid_{\left[0...\prod_{i=1}^{j-1} A_i\right)^{\mathbb{Z}}} = \phi_{j-1} \times \cdots \times \phi_1$ and $\phi_{j-1} = Id$, we also know that $y_1 < \prod_{i=1}^{j-2} A_i$ (again by Lemma 10).

The other case is analogous. □

We can characterize transition points as follows.

Proposition 18 *Let* $\chi = \phi_n \times \cdots \times \phi_1 : [0...A)^{\mathbb{Z}} \to [0...A)^{\mathbb{Z}}$ *with* $\phi_i : [0...A_i)^{\mathbb{Z}} \to [0...A_i)^{\mathbb{Z}}$*be a SIPCA. Then* $x \in [0...A)^{\mathbb{Z}}$ *is a transition point iff* $\chi \mid_{[0...x)^{\mathbb{Z}}}$ *is a surjective NCCA.*

Proof: If $\chi \mid_{[0...x)^{\mathbb{Z}}}$ is a SIPCA then it is clearly surjective.

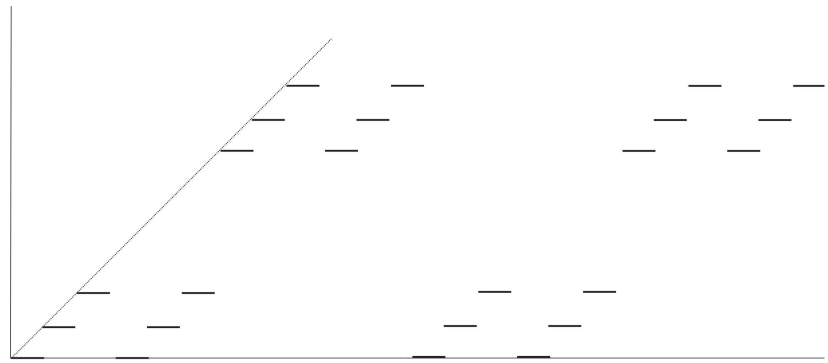

Fig. 4: f_ϕ, where ϕ is a SIPCA and $\phi_1 = \sigma_R$.

For the converse, first we will see that if $\chi \mid_{[0...x)^{\mathbb{Z}}}$ is a surjective NCCA then $f_\chi(x)$ has to be 0 or x. Suppose it's not. Since χ is bijective, there exists only one pair $p, q \leq x$ such that $\chi(0^\infty pq0^\infty) = 0^\infty x 0^\infty$. Note that neither p nor q can be x or 0 because $f_\chi(x)$ is not 0 or x. This means that $p, q < x$, so the image of $\chi \mid_{[0...x)^{\mathbb{Z}}}$ would not be contained in $[0...x)^{\mathbb{Z}}$.

Now suppose x is not a transition point with $f_\chi(x) = x$ and consider the previous transition point $t = \left(\prod_{i=1}^{j-1} A_i\right) B_j < x$.

If we define $m_2 = \prod_{i=1}^{j-2} A_i + t$, then using Lemma 17 we have that $m_2 > x$, $f_\chi(t) = t$, and $f_\chi(\prod_{i=1}^{j-2} A_i) = 0$. Using that $\phi_j = \sigma_R$ and (5) we get $\chi(0^\infty m_2 0^\infty) = 0^\infty \prod_{i=1}^{j-2} A_i \, t \, 0^\infty \in [0...x)^{\mathbb{Z}}$. Since χ is injective, $\chi \mid_{[0...x)^{\mathbb{Z}}}$ cannot be surjective.

The case for $f_\chi(x) = 0$ is similar. $\square$

Lemma 19 *Let* $\phi : [0...A)^{\mathbb{Z}} \to [0...A)^{\mathbb{Z}}$ *and* $\chi : [0...B)^{\mathbb{Z}} \to [0...B)^{\mathbb{Z}}$ *be two 2-block surjective NCCA such that* $A \leq B$ *and there exists* $m \in [0...A-1)$ *such that* $f_\phi(x) = f_\chi(x)$ *for all* $x \in [0...m)$ *. We have the following:*

·a) If $f_\chi(m) = 0$ *or* m*, then* $f_\phi(m) = 0$ *or* m*.*

·b) If $f_\chi(m) \neq 0$ *or* m*, then* $f_\phi(m) = f_\chi(m)$*.*

Proof: We have that

$$\{0^\infty de 0^\infty \mid d + e = m\} \subset \{\phi(0^\infty abc 0^\infty) \mid a + b + c = m \text{ and } a, b, c < m\} \cup \{\phi(0^\infty m 0^\infty)\},$$

and

$$\begin{aligned}\{\chi(0^\infty m0^\infty)\} &= \{0^\infty de0^\infty \mid d+e=m\} - \{\chi(0^\infty abc0^\infty) \mid a+b+c=m \text{ and } a,b,c<m\} \\ &= \{0^\infty de0^\infty \mid d+e=m\} - \{\phi(0^\infty abc0^\infty) \mid a+b+c=m \text{ and } a,b,c<m\} \\ &= \{\phi(0^\infty m0^\infty)\}\,.\end{aligned}$$

a) Theorem 8 says surjective NCCA maps are bijective on the homoclinic class of 0^∞, thus

$$\{0^\infty de0^\infty \mid d+e=m\} - \{\phi(0^\infty abc0^\infty) \mid a+b+c=m \text{ and } a,b,c<m\} = \{0^\infty m0^\infty\}$$

iff $f_\phi(m)=0$ or m.

b) If $f_\chi(m)\neq 0$ or m, then $\{\phi(0^\infty m0^\infty)\} = \{0^\infty de0^\infty\}$, thus $f_\phi(m)=e=f_\chi(m)$. □

To prove Theorem 13, we will need a stronger result, the inductive step shown in the following proposition.

Proposition 20 *Let $\phi : [0...A)^{\mathbb{Z}} \to [0...A)^{\mathbb{Z}}$ be a 2-block surjective NCCA. If there exists a SIPCA $\chi : [0...B)^{\mathbb{Z}} \to [0...B)^{\mathbb{Z}}$ such that $A\leq B$ and there exists $m\in[0...A-1)$ such that $f_\phi(x)=f_\chi(x)$ for all $x\in[0...m)$, then there exists a SIPCA $\chi' : [0...C)^{\mathbb{Z}} \to [0...C)^{\mathbb{Z}}$ such that $A\leq C$, $f_{\chi'}(x)=f_\chi(x)$ for $0\leq x<m$, and $f_{\chi'}(m)=f_\phi(m)$.*

The proof of Proposition 20 is divided in two cases when m is a transition point and when it's not.

Case 1 (m is not a transition point)

This proof is divided into two subcases.

Case 1a) ($f_\chi(m)\neq 0$ or m)

Proof of Case 1a): It's a direct application of Lemma 19. □

Case 1b) ($f_\chi(m)=0$ or m)

Lemma 21 *Let $\chi=\phi_n\times\cdots\times\phi_1 : [0...A)^{\mathbb{Z}} \to [0...A)^{\mathbb{Z}}$ with $\phi_i : [0...A_i)^{\mathbb{Z}} \to [0...A_i)^{\mathbb{Z}}$be a SIPCA. If z is the last zero of f_χ (that is the largest value z in the domain such that $f_\chi(z)=0$), then there are exactly z pairs (x,y) such that $x<y$, $f_\chi(x)=x$, and $f_\chi(y)=0$. Analogously if z is the last point such that $f_\chi(z)=z$, then there exist exactly z pairs (x,y) such that $x<y$, $f_\chi(x)=0$, and $f_\chi(y)=y$.*

Proof: Let $\chi_j=\phi_j\times\cdots\times\phi_1$, where $1\leq j\leq n$. We denote z_j as the last zero of χ_j, and n_j as the number of pairs (x,y) such that $x<y$, $f_{\chi_j}(x)=x$, and $f_{\chi_j}(y)=0$. It is easy to see that $z_1=n_1$. We want to prove that $z_{j+1}-z_j=n_{j+1}-n_j$. Suppose that $\phi_1=Id$. The number of points x such that $f_{\chi_j}(x)=0$ is $\prod\limits_{i\leq j \text{ odd}} A_i$ and the number of points such that $f_{\chi_j}(x)=x$ is $\prod\limits_{i\leq j \text{ even}} A_i$.

If $\phi_{j+1}=\sigma_R$ then we won't have any new zeros of f_χ, so $n_{j+1}-n_j=0$ (see Lemma 10).

If $\phi_{j+1}=Id$ then we have $\left(\prod\limits_{i\leq j \text{ odd}} A_i\right)\cdot(A_{j+1}-1)$ new zeros of f_χ. There are $\prod\limits_{i\leq j \text{ even}} A_i$ points where $f_\chi(x)=x$, hence we have that $n_{j+1}-n_j=\left(\prod\limits_{i\leq j} A_i\right)\cdot(A_{j+1}-1)$.

Now we want to calculate $z_{j+1} - z_j$. If $\phi_{j+1} = \sigma_R$ then we won't have any new zeros of f_χ so $z_{j+1} - z_j = 0$. If $\phi_{j+1} = Id$ then $z_{j+1} - z_j = \left(\prod_{i \le j} A_i\right) \cdot (A_{j+1} - 1)$.

If $\phi_1 = \sigma_R$, we simply interchange odd for even and everything works similarly.

The other part is proved analogously. $\square$

Lemma 22 *Let $\chi = \phi_n \times \cdots \times \phi_1 : [0...A)^{\mathbb{Z}} \to [0...A)^{\mathbb{Z}}$ with $\phi_i : [0...A_i)^{\mathbb{Z}} \to [0...A_i)^{\mathbb{Z}}$ be a SIPCA. For all non-transition points such that $f_\chi(x) = x$, there exist points $p < q < r < x$ such that $x - r = q - p$, $f_\chi(p) = p$, $f_\chi(q) = 0$, and $f_\chi(r) = r$.*

Similarly, for all non-transition points such that $f_\chi(x) = 0$, there exist points $p < q < r < x$ such that $x - r = q - p$, $f(p) = 0$, $f(q) = q$, and $f(r) = 0$.

Proof: We claim that if we have a pair (x, y) such that $x < y$, $f_\chi(x) = x$, and $f_\chi(y) = 0$, then we cannot have a different pair (x', y') such that $y - x = y' - x'$, $f_\chi(x') = x'$, and $f_\chi(y') = 0$. That is because in that case we would have $\chi(0^\infty xy'0^\infty) = 0^\infty\,(x + y')\,0^\infty = 0^\infty\,(x' + y)\,0^\infty = \chi(0^\infty x'y0^\infty)$, which is a contradiction since χ is injective. Thus by Lemma 21 we see that if z is a zero of f_χ then for every $w \le z$ there is a unique pair $x, y \le z$ with $x < y$, $y - x = w$, $f_\chi(x) = x$, and $f_\chi(y) = 0$.

Now suppose x is not a transition point and $f_\chi(x) = x$. Let $t < x$ be the previous transition point, so $t = \left(\prod_{i=1}^{j-1} A_i\right) B_j$, and let $y_1 = x - t$. By Lemma 17 $f_\chi(t) = t$, $f_\chi(\prod_{i=1}^{j-2} A_i) = 0$, and $t \le \prod_{i=1}^{j-2} A_i$. By Lemma 21 we know we have a pair $p, q < x - t$ such that $q - p = x - t$, $f_\chi(p) = p$ and $f_\chi(q) = 0$. $\square$

Proof of Case 1b): Using Lemma 22 if $f_\chi(m) = m$, there exist points $p < q < r < m$ such that $m - r = q - p$, $f_\phi(p) = p$, $f_\phi(q) = 0$, and $f_\phi(r) = r$. Using Lemma 19 we know that $f_\phi(m)$ is either 0 or m. If $f_\phi(m) = 0$, then we have that $\phi(0^\infty pm0^\infty) = 0^\infty(p+m)0^\infty$ and $\phi(0^\infty rq0^\infty) = 0^\infty(r+q)0^\infty$. But since $p + m = r + q$ we have a contradiction. So, $f_\phi(m) = m = f_\chi(m)$. $\square$

Case 2 (**m is a transition point).**

Proof of Case 2: Since m is a transition point we have that $f_\phi(m) = 0$ or m (see Lemma 19). Let $\phi : [0...m)^{\mathbb{Z}} \to [0...m)^{\mathbb{Z}}$, $\chi_1 = \sigma_R \times \phi$, and $\chi_2 = Id \times \phi$ (for σ_R and Id on any alphabet bigger than 1). This means $\chi_1\,[x0] = \chi_2\,[x0] = \phi\,[x0]$ for $x < m$, but $\chi_2\,[m0] = 0$ and $\chi_1\,[m0] = m$. $\square$

Now we can prove Theorem 13.

Proof of Theorem 13: Let $\phi : [0...A)^{\mathbb{Z}} \to [0...A)^{\mathbb{Z}}$ be a surjective 2-block NCCA. By Proposition 20 we can use induction to see there exists a SIPCA $\chi : [0...B)^{\mathbb{Z}} \to [0...B)^{\mathbb{Z}}$ such that $A \le B$ and $f_\phi(x) = f_\chi(x)$ for $0 \le x < A$. If $A = B$ we are done, if $B > A$, we know that A is a transition point of χ. So by Proposition 18 we conclude that ϕ is a SIPCA. $\square$

Corollary 23 *If ϕ is a surjective 2-block NCCA on a prime alphabet then ϕ is a shift or the identity.*

Corollary 24 *All surjective 2-block NCCA are injective.*

5 Further questions

For bigger neighbourhoods not all surjective NCCA are SIPCA.

Example 25 *The CA ϕ on the binary alphabet defined by exchanging the blocks 10100001 and 10010001, is a well-defined bijective NCCA which is not a shift or an identity, but $\phi^2 = Id$. We can construct several similar examples where $\phi^n = \sigma_R^m$ for a certain n and m (where $\sigma_R^0 = Id$).*

We can construct several similar counter-examples where $\phi^n = \sigma_R^m$ for a certain n and m (where $\sigma_R^0 = Id$), but we can ask the following questions.

Question 26 *Are all binary surjective NCCA ψ generalized subshifts? That is, do there exist natural numbers n and m such that $\psi^m = \sigma_R^n$?*

Question 27 *If ϕ is a surjective NCCA do there exist natural numbers n and m and a SIPCA ψ such that $\phi^n = \psi^m$?*

The author is currently investigating these subjects.

NCCA are a particular class of potential conserving CA $\phi : [0...A)^{\mathbb{Z}} \to [0...A)^{\mathbb{Z}}$, when $\mu(x) = x$. The result that proves the density of periodic points for surjective NCCA (Formenti and Grange (2003)) can be easily be extended to unique ground state potentials, that is when $\mu(x) = 0$ for only one state. Theorem 8 holds also for surjective CA that conserves ground state potentials.

Question 28 *Let ϕ be a one-dimensional surjective CA that conserves the potential μ, and $\mu(x) = 0$ for only one state. Is ϕ injective?*

6 addendum

In this appendix we provide a counterexample of a surjective but non injective binary NCCA.

Example 29 *Let $A = [1000100001]$ and $B = [1001000001]$. Notice they do not overlap except at the borders, they have the same length and same weight. Now define the CA as follows.*

Everything stays where it is except where there are two of the previous blocks together, in that case the block in the right changes to A if they are the same and to B if they are different. By together we mean that they overlap in the border. We can identify strings of n blocks, and they will always remain as strings of n blocks. This map is surjective and number conserving, but it is not injective because the image of $(A'B')^\infty$ and $(B'A')^\infty$ is B'^∞. Where $A' = [000100001]$ and $B' = [001000001]$.

Hence powers of this CA are never the the identity nor the power of a shift, but on the homoclinic class of 0^∞ it is a root of the identity so it is a generalized subshift there.

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