Stokes polyhedra for X-shaped polyminos
Yu. Baryshnikov, L. Hickok, N. Orlow, S. Son

To cite this version:

HAL Id: hal-01197226
https://inria.hal.science/hal-01197226
Submitted on 11 Sep 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Stokes polyhedra for X-shaped polyminos

Yu. Baryshnikov, L. Hickok, N. Orlow, S. Son

Department of Mathematics, University of Illinois, Urbana, IL 61801

Definitions. Consider a pair of interlacing regular convex polygons, each with $2(n + 2)$ vertices, which we will be referring to as red and black ones. One can place these vertices on the unit circle $|z| = 1$ in the complex plane: the vertices of the red polygon at $e^{2k\pi}, k = 0, \ldots, 2n - 1$, of the black polygon at $e^{2k+1}, k = 0, \ldots, 2n - 1$; here $e = \exp(i\pi/(2n + 2))$.

We assign to the vertices of each polygon alternating (within each polygon) signs. Note that all the pairwise intersections of red and black sides are oriented consistently. We declare the corresponding orientation positive.

Definition 1 A collection (perhaps, empty) of non-intersecting chords connecting vertices of opposing sign in one of the polygons will be called a quadrangulation of that polygon. We will color these chords according to which polygon they partition, and orient from $-\pi$ to $\pi$. A maximal quadrangulation partition the polygon into 4-sided polygons which we will be calling the squares.

A pair of quadrangulations of red and black polygons is compatible if any pair of intersecting oriented chords (from different polygons) form a positive frame, just as the intersecting sides of the interlacing polygons do.

All pairs of compatible quadrangulations for $n = 1$ are shown below.

Figure 1: Complete pair of compatible quadrangulations for $n = 1$ are shown. The Stokes polyhedron here is the segment, with two endpoints corresponding to two red quadrangulations compatible with the black quadrangulations in the right third of the display.

Proposition 1 There exists a convex polyhedron $P_{q_v}$ with the set of vertices $P_{q_v}^0$ such that the lattice of incomplete quadrangulations compatible with $q_v$ coincides with the lattice of faces of $P_{q_v}$.

We refer to the polyhedron $P_{q_v}$ as Stokes polyhedron. The family of Stokes polyhedra was introduced in [2], which also noticed that it interpolates between the (combinatorial) d-cubes and Stasheff polyhedra it was noticed that the polyhedra $P_{q_v}$...
Motivation. Our interest to the pairs of compatible quadrangulations stems from the fact that they describe the topology of the stratification of the space of polynomials $\{z^{n+2} + \sum_{i=1}^{n} a_i z^{n+1-i}\} \cong \mathbb{C}^n$ by the union of Stokes and antiStokes sets (see [2][1]).

Specifically, for any collection of chords defining a pair of quadrangulations, $q = (q_1, q_2)$, let $K_q$ be the cone $\mathbb{R}_+^q$ of nonnegative weights on chords in $q$. Inclusions $i : q' \rightarrow q$ induce embeddings of the corresponding cones $K_{q'} \rightarrow K_q$ (by assigning zero weights to the missing chords), and the inductive limit of the corresponding direct system is a polyhedron $\Lambda$, stratified by the relatively open cones $\mathbb{R}_+^q$.

The resulting stratified space is homeomorphic to the pair (Stokes, antiStokes) in $\mathbb{C}^n$.

Polyminos. It is easier to visualize a quadrangulation $q_v$ as a polymino, which is understood here as a collection of $(n + 1)$ unit squares with some sides identified isometrically; the polymino is then the polygon obtained by gluing these sides.

The sides of the squares of the polymino that correspond to the exterior sides of the black polygon will be called exterior sides as well; the sides corresponding to the chords will be called interior. We will call a square of a polymino a leaf if it has three exterior sides, a turning square if it has exactly two adjacent exterior sides and a separating square if it has exactly two opposite exterior sides.

In a snake polymino all non-leaf squares are turning; in a band polymino all non-leaf squares are separating. These two classes of polymino represent two extremes in terms of complexity of the Stokes polyhedra. Indeed, $P_q$, for the band polymino is combinatorially a cube (in $\mathbb{R}^n$). As for the snake polyminos, there are $2^{n-1}$ types (if we fix one end, the the snake polymino is completely characterized whether the turns at the turning squares are left or right).

Perhaps unexpectedly, snake polyminos all have combinatorially equivalent fans, regardless of the type: The Stokes polyhedra for snake polyminos are combinatorially equivalent to the associahedra, or Stasheff polyhedra, see [3].

We remark that over the past 20 years, the associahedra became a staple of algebraic, analytic and combinatorics and combinatorial geometry; a good overview can be found in [3].

Enumeration for $X$-polyminos Our main result is the enumeration of the vertices in the 4-parametric family of $X$-polyminos.

Definition 2 An polymino is called an $X$-polymino, if all but one of its squares are either turning squares or leaf squares. The exceptional square (which is not required to be a turning or leaf one, but can happen to be one) is called central. If, upon removal of the central square the polymino splits into 4 polyminos (which are necessarily snake polyminos) of sizes $n_1, n_2, n_3, n_4$ (in cyclic order around the internal square), we will denote such polymino as $Q_{n}$, where $n = (n_1, n_2, n_3, n_4)$.

Theorem 1 The generating function for the number of vertices of Stokes polyhedra corresponding to $X$-polyminos of size $n$ is

$$T(z) = \frac{(C(z_1) - C(z_2))(C(z_2) - C(z_3))(C(z_3) - C(z_4))(C(z_4) - C(z_1))}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_4)(z_4 - z_1)},$$

where

$$C(z) = \sqrt{4z}.$$
which translates at once into \( T_{1,0,1,n} = 2C_{n+2} + C_{n+1} \). The generating function begins with \( 5 + 12z + 33z^2 + 98z^3 + 306z^4 + 990z^5 + 3289z^6 + 11154z^7 + \ldots \).

References


