Toward the asymptotic count of bi-modular hidden patterns under probabilistic dynamical sources: a case study

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Consider a countable alphabet $\mathcal{A}$. A multi-modular hidden pattern is an r-tuple $(w_1, \dots, w_r)$, where each w_i is a word over $\mathcal{A}$ called a module. The hidden pattern is said to occur in a text t when the later admits the decomposition $t = v_0 w_1 v_1 \cdots v_{r-1} w_r v_r$, for arbitrary words v_i over $\mathcal{A}$. Flajolet, Szpankowski and Vallée (2006) proved via the method of moments that the number of matches (or occurrences) with a multi-modular hidden pattern in a random text $X_1 \cdots X_n$ is asymptotically Normal, when $(X_n)_{n\geq 1}$ are independent and identically distributed $\mathcal{A}$-valued random variables. Bourdon and Vallée (2002) had conjectured however that asymptotic Normality holds more generally when $(X_n)_{n\geq 1}$ is produced by an expansive dynamical source. Whereas memoryless and Markovian sequences are instances of dynamical sources with finite memory length, general dynamical sources may be non-Markovian i.e. convey an infinite memory length. The technical difficulty to count hidden patterns under sources with memory is the context-free nature of these patterns as well as the lack of logarithm- and exponential-type transformations to rewrite the product of non-commuting transfer operators. In this paper, we address a case study in which we have successfully overpassed the aforementioned difficulties and which may illuminate how to address more general cases via auto-correlation operators. Our main result shows that the number of matches with a bi-modular pattern (w_1, w_2) normalized by the number of matches with the pattern w_1, where w_1 and w_2 are different alphabet characters, is indeed asymptotically Normal when $(X_n)_{n\geq 1}$ is produced by a holomorphic probabilistic dynamical source.

Keywords: hidden pattern, non-Markovian sequence, pattern matching, probabilistic dynamical source

1 Introduction

The number of occurrences of a word or a set of them in a generic text is an important parameter to understand the complexity of pattern searching algorithms [14]. The statistics of patterns in random

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text or sequences also lies at the core of various applied fields, ranging from molecular biology [18], to speech recognition [27] and text mining, among others. Memoryless, Markovian and more recently hidden Markov models have been proposed in the literature to model from genomic sequences to spoken utterances [28]. On the other hand, the patterns considered in the literature have ranged from single words to sets of them described by regular expressions [29] and even certain non-regular languages such as palindromes. Underlying the assessment of patterns in random sequences is the guiding principle that exceptional patterns in a text (i.e. patterns observed much more or less frequently than expected by chance alone) carry useful information, for instance, to identify instructions in a genome that regulate the translation of DNA into proteins [25, 26], or to set up thresholds to warn of potential security breaks in secured databases [30, 31].

In what follows, $\mathcal{A}$ is a given and denumerable (i.e. finite or countable) set. We call $\mathcal{A}$ the alphabet and $\mathcal{A}^*$ denotes the set of all possible words obtained by concatenating a finite number of alphabet characters. For $u, v \in \mathcal{A}^*$, uv denotes the word in $\mathcal{A}^*$ obtained by concatenating v to the right of u.

The string matching problem is defined by the exact matching of a word in a text (i.e. without gaps between consecutive characters of the word). More precisely, a word w is said to occur in a text t when $t = v_0 w v_1$ for arbitrary words v_0, v_1 in $\mathcal{A}^*$. Similarly, the approximate string matching problem is defined by the exact matching of a set of words regarded as close variants of w under a certain metric. (Both of these problems fall in the more general setting of matching a regular pattern i.e. a subset of words $\mathcal{L}$ in $\mathcal{A}^*$ described by a regular expression.) If gaps are allowed between consecutive letters of w, we are led to the so called hidden pattern matching problem. More generally, a multi-modular hidden pattern is an r-tuple $(w_1, \ldots, w_r)$, with w_i is a word in $\mathcal{A}^*$ called a module. The hidden pattern $(w_1, w_2, \ldots, w_r)$ is said to occur in t when the later admits the decomposition $t = v_0 w_1 v_1 w_2 v_2 \ldots v_{t-1} w_t v_t$, where the v_i's are arbitrary words in $\mathcal{A}^*$ called gaps. Variations of this problem allow the modules to be regular languages—the so called generalized patterns—and may also impose restrictions on the gap lengths. Note that hidden patterns with unbounded gaps are non-regular types of patterns.

In [9, 10, 11], Guibas and Odlyzko highlighted the role of autocorrelation of a word and correlation between pairs of words for the string matching problem in the context of memoryless sources. In [17], Régnier and Szpankowsky proved that the number of occurrences of a word in a text admits a Gaussian limit law when the text is more generally produced by a Markovian source. Follow up work in [19], extended this result to the approximate string matching problem but only for memoryless sources. The Gaussian limit law for the hidden pattern matching problem was obtained by the authors of [7] again in the memoryless setting.

Memoryless and Markovian sources are instances of more general models called probabilistic dynamical sources, introduced by Vallée in the context of information theory [23]. In [2], Bourdon and Vallée proved that the Gaussian limit law holds for regular pattern matching problems. In [1], they obtained the mean and the variance of the number of occurrences of generalized patterns under probabilistic dynamical sources—encompassing all the aforementioned pattern matching problems. The asymptotic Normality of hidden patterns for sources with memory was conjectured by the authors of [1], however, it has remained an open problem due to various technical difficulties. On one hand, hidden patterns with unbounded gaps are context-free but non-regular, which rules out Markovian embeddings to tackle the problem [32]. On the other hand, the approach used in [7] and based on the method of moments to settle the Gaussian limit law for memoryless sources appears rather intractable for sources with memory—even Markovian ones.

Motivated by the above observations we tackle the simplest possible case study associated with Bourdon and Vallée's ten-year old conjecture, namely the number of matches with a bi-modular motif of the form

(a, b), with $a, b \in \mathcal{A}$ different alphabet characters, renormalized by the number of matches with a, in a long random text produced by a holomorphic probabilistic dynamical source. The interest of this case study lies in the methodology of the solution—which is non-trivial—and may illuminate how to address more general cases with auto-correlation. In fact, the main contribution of the present paper is the use of analytic and probabilistic ideas to extrapolate spectral information about a transfer operator $\mathbf{K}(z, u)$ defined for all (z, u) in a neighborhood of $(0, 1)$ in the complex plane to identify the dominant singularity of a generating function derived from the transfer operators

$$\prod_{i=m}^{1} \mathbf{K}(z, u^{i/m}), \text{ with } m \geq 1.$$

The chief difficulty here is the non-commutativity of the involved operators to rewrite the above product in terms of logarithm- and exponential-type transformations. We remark that the indices in the above product range from m down to 1 due to the identity in equation (9). Since transfer operators much like transfer matrices [6] do not commute in general, the indices cannot be arranged in an arbitrary order.

Plan of the paper. In § 2 we introduce some notation, the problem and describe our main result. In § 3–3.2, we review the notions of probabilistic dynamical source and generating operators associated with languages, as well as results from the spectral theory of linear operators relevant for the asymptotic analysis of generating functions derived from transfer operators. The general scheme of the proof with all the important results concerning the operators is addressed in § 4. The complete proof is given in Annex A and B.

2 Notation and problem

In the sequel, $\mathcal{S}$ denotes a probabilistic source over the alphabet $\mathcal{A}$ that generates an infinite sequence $(X_n)_{n\geq 1}$ of a $\mathcal{A}$-valued random variables (see § 3 for the precise meaning of this). In addition, $a, b \in \mathcal{A}$ are given and different alphabet characters, and the ordered-pair (a, b) is a bi-modular hidden pattern. For $\omega \in \mathcal{A}^\star$, we associate the costs

$$\begin{aligned} C(\omega) &= \text{number of occurrences of } (a,b) \text{ in } \omega; && (1)\\ |\omega|_a &= \text{number of occurrences of } (a) \text{ in } \omega; && (2)\\ \Omega(\omega) &= C(\omega)/|\omega|_a. && (3) \end{aligned}$$

All throughout the paper, Ω is called the "normalized" number of occurrences of (a, b). Above it is understood that $\Omega(\omega) = 0$ when $|\omega|_a = 0$ and hence $C(\omega) = 0$. Furthermore, the restriction of C and Ω to $\mathcal{A}^n$ are from now on denoted C_n and Ω_n, respectively.

It follows from the analysis in [7] that C_n admits a Gaussian limit law when $\mathcal{S}$ is a memoryless source. We did not succeed in proving the same result for probabilistic dynamical sources, however, as stated next, we can show that Ω_n is asymptotically Gaussian when the source is holomorphic (see Annex A for the definition). Our result involves the dominant eigenvalue $\lambda(z, u)$ of some operator $\mathbf{K}(z, u)$ defined in equation (15). Let $A(z, u)$ be the bivariate function

$$A(z, u) = \exp\left(\int_0^1 \log \lambda(z, u^t)\, dt\right),$$

and consider the function $\sigma(u)$ such that $\sigma(1) = 1$ and $A(\sigma(u), u) = 1$, for all u in a neighborhood of $u = 1$. We now state our main result.

Theorem 2.1 *Let Ω_n be the "normalized" number of occurrences of the hidden pattern (a, b) in a random text of length n produced by an holomorphic dynamical source. The random variable Ω_n follows asymptotically a Gaussian limit law. More precisely, for each $x \in \mathbb{R}$, we have that*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\Omega_n - \mathbb{E}[\Omega_n]}{\sqrt{\mathbb{V}[\Omega_n]}} \le x\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt,$$

where the mean and variance of Ω_n satisfy

$$\begin{aligned} \mathbb{E}[\Omega_n] &\sim -\sigma'(1)\cdot n; \\ \mathbb{V}[\Omega_n] &\sim \big(\sigma'(1)^2 - \sigma'(1) - \sigma''(1)\big)\cdot n. \end{aligned}$$

As we mentioned in § 1, the chief interest in the above result is in the method of proof that we hope to adjust for modules with auto-correlation in a follow up paper. Theorem 2.1 is obtained via an analytic combinatorics approach [6]. A key ingredient is the bivariate generating function $\mathcal{H}(z, u)$ relative to the cost Ω—see equation (11), which we use to prove the pointwise convergence of the Laplace transforms of the renormalized random variable $\Omega_n^\star := (\Omega_n - \mathbb{E}[\Omega_n])/\sqrt{\mathbb{V}[\Omega_n]}$ to that of the Gaussian law.

3 Probabilistic dynamical sources.

In what remains of this manuscript, I denotes the open interval $(0, 1)$. A probabilistic dynamical source $\mathcal{S}$ over the alphabet $\mathcal{A}$ is defined by the following elements:

(a) a probability density function f over I;

(b) a topological partition of I into open intervals $(I_\alpha)_{\alpha\in\mathcal{A}}$; in particular, we may define a transformation $\sigma : \cup_{\alpha\in\mathcal{A}} I_\alpha \to \mathcal{A}$ to satisfy that $\sigma(x) = \alpha$ if and only if $x \in I_\alpha$; and

(c) a transformation $T : I \to I$ such that, for each $\alpha \in \mathcal{A}$, the restriction $T : I_\alpha \to I$ is bijective and twice continuously differentiable.

A dynamical source generates a random text in the following way: first, a random variable Y with probability density function f is drawn and the characters emitted by the source are $X_i = \sigma(T^{i-1}(Y))$, for each $i \ge 1$. A probabilistic dynamical source therefore induces a probability measure over $\mathcal{A}^\infty$ i.e. the space of infinite sequences of elements in $\mathcal{A}$. This measure corresponds to the probability distribution of the random sequence $X = (X_i)_{i\ge 1}$.

Probabilistic analyses associated with dynamical sources in the context of analysis of algorithms have been carried via Ruelle operators, as seen e.g. in [24, 23, 2, 4]. In our context, for each $\alpha \in \mathcal{A}$, define $h_\alpha : I \to I_\alpha$ to be the inverse function of the restriction of T to the interval I_α. In particular:

$$h_\alpha(T(y)) = y, \text{ for all } y \in I_\alpha; \tag{4}$$

$$T(h_\alpha(y)) = y, \text{ for all } y \in I. \tag{5}$$

More generally, for each $\omega \in \mathcal{A}^*$, define:

$$h_\omega := (h_{\omega_1} \circ \ldots \circ h_{\omega_\ell}), \tag{6}$$

where ℓ denotes the length of ω i.e. $\ell = |\omega|$. Due the identities in equations (4)-(5), a straightforward inductive argument shows that $y \in h_\omega(I)$ if and only if $\sigma(T^{k-1}(y)) = \omega_k$, for all $1 \leq k \leq |\omega|$. As a result, the probability of the event *"ω is a prefix of the infinite sequence X"* may be computed as

$$p_\omega = \left| \int_{h_\omega(I)} f(y)\, dy \right| = \int_0^1 (\mathbf{G}_\omega f)(s)\, ds, \tag{7}$$

where $\mathbf{G}_\omega$ is the linear operator defined as

$$(\mathbf{G}_\omega f)(s) \quad := \quad (f \circ h_\omega)(s) \cdot |h'_\omega(s)|. \tag{8}$$

Due to the Chain rule, if $u, v \in \mathcal{A}^*$ then

$$\mathbf{G}_{uv} = (\mathbf{G}_v \circ \mathbf{G}_u). \tag{9}$$

This identity is what permits the use of the symbolic method [6] to specify the generating functions of languages associated with regular expressions in the setting of probabilistic dynamical sources. Note how the concatenation of words entails the composition of the associated transfer operators but in reverse order. This order is important because the operators do not generally commute, except for memoryless dynamical sources.

3.1 Generating operators.

Consider a language $\mathcal{L} \subset \mathcal{A}^*$ and cost function $c : \mathcal{L} \to \mathbb{R}$. The bivariate generating function associated with $\mathcal{L}$ and c is defined as

$$\mathcal{L}(z, u) := \sum_{\omega \in \mathcal{L}} p_\omega\, z^{|\omega|}\, u^{c(\omega)}.$$

In the context of symbolic combinatorics, the variable z above is said to mark lengths and u the costs of words in $\mathcal{L}$. Due to the identity in equation (7), note that

$$\mathcal{L}(z, u) = \int_0^1 \big(\mathbf{L}(z, u) f\big)(s)\, ds,$$

where

$$\mathbf{L}(z, u) := \sum_{\omega \in \mathcal{L}} z^{|\omega|}\, u^{c(\omega)}\, \mathbf{G}_\omega$$

is the so called generating or also transfer operator associated with the language $\mathcal{L}$ and cost c.

Consider languages $\mathcal{L}_1, \mathcal{L}_2 \subset \mathcal{A}^*$ and cost functions $c_i : \mathcal{L}_i \to \mathbb{R}$ for $i = 1, 2$.

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are disjoint then the generating operator associated with $(\mathcal{L}_1 \cup \mathcal{L}_2)$ is easily seen to be $\mathbf{L}_1(z, u) + \mathbf{L}_2(z, u)$. Here, the cost function implicitly used over the union of the disjoint languages is $c(\omega) = c_i(\omega)$ when $\omega \in \mathcal{L}_i$. On the other hand, regardless if the languages are disjoint or not, but under the assumption that $u_1 v_1 = u_2 v_2$, with $u_1, u_2 \in \mathcal{L}_1$ and $v_1, v_2 \in \mathcal{L}_2$, implies that $u_1 = u_2$ and $v_1 = v_2$, the generating operator associated with the product language $(\mathcal{L}_1 \times \mathcal{L}_2)$ i.e. the set of all words of the form uv, with $u \in \mathcal{L}_1$ and $v \in \mathcal{L}_2$, is $\mathbf{L}_2(z,u) \circ \mathbf{L}_1(z,u)$ by virtue of the linearity of the operators $\mathbf{G}_\omega$, for each $\omega \in \mathcal{A}^*$, and equation (9). In this case, the cost function defined implicitly over the product is $c(uv) = c_1(u) + c_2(v)$ when $u \in \mathcal{L}_1$ and $v \in \mathcal{L}_2$.

Consider the language $\mathcal{L}^*$ composed by the empty word—from now on denoted as ϵ—as well as all words formed by concatenating a finite number of words in $\mathcal{L}$ i.e.

$$\mathcal{L}^* := \cup_{n\geq 0}\mathcal{L}^n,$$

with $\mathcal{L}^0 := \{\epsilon\}$. It is the disjoint union of various Cartesian products so, under the assumption that each non-empty word $\omega \in \mathcal{L}^*$ decomposes uniquely in the form $\omega = \omega_1 \cdots \omega_n$ for certain $n \geq 1$ and $\omega_1, \ldots, \omega_n \in \mathcal{L}$; in particular, $\epsilon \notin \mathcal{L}$, the associated generating operator is the so-called quasi-inverse of $\mathbf{L}(z,u)$, namely

$$(\mathbf{I} - \mathbf{L}(z,u))^{-1} := \sum_{n\geq 0} \mathbf{L}^n(z,u).$$

Above, $\mathbf{I}$ denotes the identity operator i.e. $\mathbf{I}(f) = f$, for all function f. Furthermore, if c denotes as before the primitive cost over $\mathcal{L}$ then the cost function used over $\mathcal{L}^*$ is defined as $c^*(\epsilon) = 0$ and $c^*(\omega_1 \cdots \omega_n) = c(\omega_1) + \ldots + c(\omega_n)$, for each $n \geq 1$ and $\omega_1, \ldots, \omega_n \in \mathcal{L}$.

3.2 Spectral decompositions.

Although generating functions derived from transfer operators are not typically very explicit, spectral properties of the later may lead to asymptotic expressions for the coefficients of the former, in the same lines that asymptotic formulae for paths in graphs [6] and sojourn-times in Markov chains are derived from spectral properties (usually implied by the Perron-Froebenius theorem [16, 5]) of the associated adjacency and probability transition matrices, respectively.

To convey the main idea consider a linear operator $\mathbf{L}$. Its spectral properties depend on the Banach space of functions on which it may act, and a first technical difficulty is to identify such a space on which a strong quasi-compacity property holds. Precisely, the operator $\mathbf{L}$ is expected to have a unique eigenvalue of maximum modulus— called dominant eigenvalue —isolated from the remainder of the spectrum by a spectral gap. If λ denotes such a dominant eigenvalue, the operator admits a spectral decomposition of the form $\mathbf{L} = \lambda\,\mathbf{P} + \mathbf{R}$, where $\mathbf{P}$ is a projector over the dominant eigenspace, $(\mathbf{P} \circ \mathbf{R}) = (\mathbf{R} \circ \mathbf{P}) = 0$, and the spectral radius ρ of $\mathbf{R}$ is strictly less than $|\lambda|$. Since $\mathbf{P}$ and $\mathbf{R}$ commute, the iterates of $\mathbf{L}$ satisfy $\mathbf{L}^n = \lambda^n\,\mathbf{P} + \mathbf{R}^n$ for each $n \geq 1$. In particular, summing over all possible n leads to the following fundamental relation for the quasi-inverse of the operator $z\mathbf{L}$,

$$(\mathbf{I} - z\,\mathbf{L})^{-1} = \frac{\mathbf{P}}{1 - z\lambda} + \mathbf{R} \circ (\mathbf{I} - z\,\mathbf{R})^{-1}.$$

The spectral properties of $\mathbf{R}$ imply that the quasi-inverse operator $(\mathbf{I} - z\,\mathbf{R})^{-1}$ may be represented as the series $\sum_{n\geq 0} z^n \mathbf{R}^n$, which is absolutely convergent in the operator-norm for $|z| < 1/\rho$. Because of this, we say that the operator-valued transformation $z \to (\mathbf{I} - z\,\mathbf{R})^{-1}$ is analytic for $|z| < 1/\rho$. The quasi-inverse $(\mathbf{I} - z\,\mathbf{L})^{-1}$ therefore admits a simple pole in $z = 1/\lambda$ in the centered open disc of radius $1/\rho$. In particular, if f is in the Banach space where $\mathbf{L}$ acts on and $\mathbf{P}f \neq 0$ then, for each $0 < \epsilon < (\lambda - \rho)$, it applies that

$$\begin{aligned}[z^n]\,(\mathbf{I}-z\,\mathbf{L})^{-1}f &= \lambda^n\cdot(\mathbf{P}f)+(\mathbf{R}^n f)\\ &= \lambda^n\cdot(\mathbf{P}f)+O((\rho+\epsilon)^n)=\lambda^n\cdot(\mathbf{P}f)\cdot(1+o(1)).\end{aligned}$$

4 Scheme of proof of the main theorem

In what follows, for each $\alpha\in\mathcal{A}$, $|\omega|_\alpha$ is the number of times that α occurs as a character in ω. Furthermore, for each $m,n\geq 0$, let $\mathcal{H}_{n,m}$ denote the language of all words $\omega\in\mathcal{A}^*$ such that $|\omega|=n$ and $|\omega|_a=m$. Define

$$\mathcal{H}_m:=\bigcup_{n\geq 0}\mathcal{H}_{m,n},$$

and consider the generating functions

$$\mathcal{H}_m(z,u):=\sum_{\omega\in\mathcal{H}_m}p_\omega\,z^{|\omega|}\,u^{\Omega(\omega)};\tag{10}$$

$$\mathcal{H}(z,u):=\sum_{m\geq 0}\mathcal{H}_m(z,u)=\sum_{\omega\in\mathcal{A}^*}p_\omega\,z^{|\omega|}\,u^{\Omega(\omega)}.\tag{11}$$

Unless otherwise stated, Ω is the cost function associated with all the involved generating functions and operators ahead.

To state our first result, consider the sub-alphabets

$$\begin{aligned}\mathcal{B} &:= \mathcal{A}\setminus\{a\};\\ \mathcal{C} &:= \mathcal{A}\setminus\{a,b\}.\end{aligned}$$

Theorem 4.1 *(Bivariate generating function.)*

$$\mathcal{H}_m(z,u)=\int_0^1\Big(\mathbf{H}_m(z,u)f\Big)(s)\,ds,\tag{12}$$

where

$$\mathbf{H}_0(z,u) := (\mathbf{I}-z\,\mathbf{G}_\mathcal{B})^{-1};\tag{13}$$

$$\mathbf{H}_m(z,u) := \left\{\prod_{i=m}^{1}\mathbf{K}(z,u^{i/m})\right\}\circ(\mathbf{I}-z\,\mathbf{G}_\mathcal{B})^{-1},\ \textit{for all } m\geq 1;\tag{14}$$

and

$$\mathbf{K}(z,u):=z\Big(\mathbf{I}-z\,(\mathbf{G}_\mathcal{C}+u\,\mathbf{G}_b)\Big)^{-1}\circ\mathbf{G}_a.\tag{15}$$

Proof: When $m=0$, $\mathcal{H}_m=\mathcal{B}^*$ and the result follows from the fact that the transfer operator associated with the language $\mathcal{B}^*$ is $(\mathbf{I}-z\,\mathbf{G}_\mathcal{B})^{-1}$. On the other hand, if $m\geq 1$ then

$$\mathcal{H}_m=\mathcal{B}^*\times(\{a\}\times\mathcal{B}^*)^m.\tag{16}$$

Call each word in $\{a\}\times\mathcal{B}^*$ a module; in particular, words in $\mathcal{H}_m$ consist of m-modules. Let $\mathbf{K}_0(z,u)$ be the transfer operator associated with a single module, where z marks the length of a module as usual, but u marks now the number of matches with "b" i.e.

$$\mathbf{K}_0(z,u)=\sum_{\omega\in\{a\}\times\mathcal{B}^*} z^{|\omega|}\,u^{|\omega|_b}\,\mathbf{G}_\omega.$$

Since $\mathcal{B}=\{b\}\cup\mathcal{C}$, we obtain:

$$\mathbf{K}_0(z,u)=\sum_{n\geq 0} z^n(\mathbf{G}_\mathcal{C}+u\,\mathbf{G}_b)^n\circ(z\,\mathbf{G}_a)=z\Big(\mathbf{I}-z\,(\mathbf{G}_\mathcal{C}+u\,\mathbf{G}_b)\Big)^{-1}\circ\mathbf{G}_a=\mathbf{K}(z,u).$$

Next note that each match with the character b in the first module of a word in $\mathcal{H}_m$ contributes to a single match with the pattern (a,b). However, each b in the second module adds two new matches with (a,b). In general, each b in the i-th module adds i new matches with the pattern (a,b). The transfer operator $\mathbf{K}(z,u^{i/m})$ therefore keeps track of the length of the i-th module and the amount each b in that module contributes to a match with (a,b). The result now follows from the symbolic specification in equation (16). □

Our next goal is identify the dominant singularities of the generating functions $\mathcal{H}_m(z,u)$ in Theorem 4.1. We will see that these are determined by the singularities of the generating operators in equations (13)-(14), which we identify via a spectral analysis. Without further hypotheses on the dynamical source, it is difficult to determine the spectral properties of these operators. Following Vallée [23], we consider holomorphic dynamical sources whose precise definition is given in Annex A (see Definition 6.1). When acting on a convenient Banach space of holomorphic functions, the operators $\mathbf{G}_\mathcal{B}$, $\mathbf{G}_\mathcal{C}$, $\mathbf{G}_a$ and $\mathbf{G}_b$ are compact and admit a unique positive eigenvalue of maximum modulus. In particular, when $u>0$, the same is true for the operator $(\mathbf{G}_\mathcal{C}+u\,\mathbf{G}_b)$, whose unique eigenvalue of maximum modulus is from now on denoted as $\mu(u)$; in particular, $\mu(1)<1$ is the unique dominant eigenvalue of $\mathbf{G}_\mathcal{B}$.

By perturbation theory [13], there exist a complex neighborhood $\mathcal{U}_1$ of the positive real axis such that, for all $u\in\mathcal{U}_1$, $(\mathbf{G}_\mathcal{C}+u\,\mathbf{G}_b)$ admits also a unique eigenvalue of maximum modulus, denoted as $\mu(u)$ as well, that is isolated from the remainder of the spectrum by a spectral gap. This entails a spectral decomposition of the form:

$$\begin{aligned}(\mathbf{G}_\mathcal{C}+u\,\mathbf{G}_b)^n &= \mu(u)^n\,\mathbf{P}_u+\mathbf{R}_u^n,\\ \big(\mathbf{I}-z\,(\mathbf{G}_\mathcal{C}-u\,\mathbf{G}_b)\big)^{-1} &= \frac{\mathbf{P}_u}{1-z\mu(u)}+\mathbf{R}_u\circ(\mathbf{I}-z\mathbf{R}_u)^{-1},\end{aligned}$$

for all $u\in\mathcal{U}_1$ and $n\geq 1$, where $\mathbf{P}_u$ is the projection over the eigenspace relative to $\mu(u)$, $(\mathbf{P}_u\circ\mathbf{R}_u)=(\mathbf{R}_u\circ\mathbf{P}_u)=0$, and the spectral radius of $\mathbf{R}_u$, denoted as $\rho(u)$, is strictly less than $|\mu(u)|$; in particular, $\|\mathbf{R}_u^n\|_\infty=O((\rho(u)+\epsilon)^n)$, for each $\epsilon>0$. We emphasize that $\mathbf{P}_u$ and $\mathbf{R}_u$ are compact, $\mu(u)$ depends analytically on u, and $\rho(u)$ is a continuous function of u.

Fix $u\in\mathcal{U}_1$. Due to the above observations, the operator $\mathbf{K}(z,u)$ defined in equation (15) can be alternatively rewritten as

$$\mathbf{K}(z,u)=\frac{z\,(\mathbf{P}_u\circ\mathbf{G}_a)}{1-z\,\mu(u)}+z\,\mathbf{R}_u(\mathbf{I}-z\,\mathbf{R}_u)^{-1}\circ\mathbf{G}_a.$$

Since the quasi-inverse $(\mathbf{I} - z\,\mathbf{R}_u)^{-1}$ is analytic for z inside the centered disc of radius $1/\rho(u)$, the operator $\mathbf{K}(z,u)$—which in principle is defined only for $|z| < 1/|\mu(u)|$—admits an analytic extension to the disk $|z| < 1/\rho(u)$ punctured at the simple pole $z = 1/\mu(u)$.

Proposition 4.2 *(Dominant singularities of $\mathcal{H}_m(z,u)$.) There exists a complex neighborhood $\mathcal{U}_2 \subset \mathcal{U}_1$ of* 1 *such that* $\sup_{u\in\mathcal{U}_2} \mu(u) = \bar{\mu} < 1$ *and, for each $m \geq 0$ and $u \in \mathcal{U}_2$, $\mathcal{H}_m(z,u)$ is analytic for $|z| < 1/\bar{\mu}$.*

Our next goal is to make this generating function more explicit so as to have a better handle of the generating function $\mathcal{H}(z,u)$ near its singularities (see equation (11)). This is key if one aims to settle the asymptotic normality of the random variable Ω_n via the identity

$$\mathbb{E}\big(u^{\Omega_n}\big) = [z^n]\mathcal{H}(z,u).$$

As previously, $\mathbf{K}(z,u)$ is compact and admits for real-pairs (z,u), with $u \in \mathcal{U}_2$ and $|z| < 1/\bar{\mu}$, a unique positive eigenvalue of maximum modulus which we denote as $\lambda(z,u)$. Furthermore, again due to perturbation theory, $\mathbf{K}(z,u)$ has in fact a unique dominant eigenvalue—also to be denoted as $\lambda(z,u)$—isolated from the rest of the spectrum for all (z,u) in a certain complex open neighborhood $\mathcal{Z}_3 \times \mathcal{U}_3$ of $(0,1/\bar{\mu}) \times \{1\}$, with $\mathcal{U}_3 \subset \mathcal{U}_2$. Hence, there are operators $\mathbf{P}(z,u)$ and $\mathbf{R}(z,u)$ such that

$$\mathbf{K}(z,u) = \lambda(z,u)\cdot[\mathbf{P}(z,u) + \mathbf{R}(z,u)]\,,$$

for all (z,u) in the open neighborhood $\mathcal{Z}_3 \times \mathcal{U}_3$. Above, $\mathbf{P}(z,u)$ is the projection over the eigenspace relative to $\lambda(z,u)$, $\mathbf{P}(z,u)\circ\mathbf{R}(z,u) = \mathbf{R}(z,u)\circ\mathbf{P}(z,u) = 0$, and the spectral radius of $\mathbf{R}(z,u)$, to be denoted as $\rho(z,u)$, is strictly less than 1. Note that all the operators or spectral objects are analytic in (z,u) due to the analyticity of $\mathbf{K}(z,u)$.

We may first use Lemma 7.2, Lemma 7.1 and Proposition 6.6 (see annexes), to represent $\mathbf{H}_m$ more explicitly as follows:

$$\begin{aligned} \prod_{j=m}^{1} \mathbf{K}(z,u^{j/m})[f] &= \prod_{j=1}^{m} \lambda(z,u^{j/m}) \prod_{j=m}^{1} \mathbf{P}(z,u^{j/m})[f]\,(1+O(1/m))\,, \\ &= A(z,u)^m(\mathbf{B}(z,u)[f] + \mathbf{C}_m(z,u)[f]), \end{aligned} \tag{17}$$

with $\|\mathbf{C}_m(z,u)\| = O(1/m)$, $\mathbf{B}(z,u)$ a bounded linear operator and

$$A(z,u) = \exp\left(\int_0^1 \log\lambda(z,u^t)\,dt\right). \tag{18}$$

(The function $A(z,u)$ is obtained from the Euler-Maclaurin formula when applied to $\prod_{j=1}^m \lambda(z,u^{j/m})$.)

Back to the definition of $\mathcal{H}_m(z,u)$, we can now conclude with the Corollary 6.7 (see Annex A) that says the following: on a convenient complex neighborhood $\mathcal{W}_1$ of $u = 1$ and $0 < z < 1/\bar{\mu}$, there exist analytic functions A, B and C_m such that

$$\mathcal{H}_m(z,u) = A(z,u)^m[B(z,u) + C_m(z,u)],\ \text{with}\ \|C_m\| = O(1/m),$$

with a uniform constant in the O-term. Summing over all the possible m's gives

$$\mathcal{H}(z,u) = \frac{B(z,u)}{1 - A(z,u)} + O\left(\log\frac{1}{1 - |A(z,u)|}\right). \tag{19}$$

The final step in the analysis requires extracting the coefficients of this generating function.

It is known that $A(1,1) = \lambda(1,1) = 1$ and the implicit function theorem implies the existence of an analytic function σ, defined on a convenient complex-neighborhood of $u = 1$, such that $A(\sigma(u), u) = 1$. Furthermore, when u is real, $\sigma(u)$ is the only singularity of both parts of relation (19) inside the disk of convergence $|z| < \sigma(u)$. Note that for complex u, it is difficult if not impossible to find such disk due to the lack of information on the generating function $C_m(z, u)$.

The coefficient extraction of the meromorphic part on the right hand-side of equation (19) is very classical and we obtain

$$[z^n]\frac{B(z,u)}{1-A(z,u)} = \frac{B(\sigma(u),u)}{A'_z(\sigma(u),u)}\sigma(u)^{-(n+1)}\left(1+O(\bar{\rho}^n)\right),$$

with $\bar{\rho} < 1$ and the O-term uniform in u. On the other hand, the coefficient extraction of the second term can be performed using a saddle point strategy. Up to technical details described in Annex A, we obtain

$$\begin{aligned}
[z^n]\sum_{m\geq 0} A(z,u)^m C_m(z,u) &= [z^n]\sum_{m\geq \phi n} A(z,u)^m C_m(z,u) + O(\tilde{\rho}^n), \\
&= O(\sigma(u)^{-1} n^{-\frac{1}{2}+\xi}\log n),
\end{aligned}$$

where $\tilde{\rho} < 1$.

Combining both extractions, we finally obtain that

$$\mathbb{E}[u^{\Omega_n}] = \alpha(u)\,\beta(u)^n\,(1+O(\epsilon_n)),$$

uniformly for u real in a neighborhood of 1, with $\epsilon_n \to 0$ as $n \to \infty$. This quasi-power property entails the convergence of the Laplace transforms of $(\Omega_n - \bar{\beta}'(0)\cdot n)/(\bar{\beta}''(0)\cdot n)^{1/2}$ to the Laplace transform of a Gaussian law, where $\bar{\beta}(s) := \log\beta(e^s)$, as long as the variability condition $\bar{\beta}''(0) \neq 0$ holds. We prove the later in Annex A.

5 Conclusions and open problems

For the first time, we prove that the number of occurrences of a bi-modular pattern (after a normalization) in a random text produced by a correlated source admits a Gaussian limit law. Of course, the strong restriction on the structure of the pattern (that avoids correlations and auto-correlations) has simplified a lot the analysis. Unfortunately, due to the lack of information on the bivariate generating function $\mathcal{H}(z,u)$ associated with the cost Ω when u is complex (see equation (11)), we did not succeed in applying results such as Hwang's quasi-power theorem [12] to obtain information on the speed of convergence to the Gaussian distribution. It may still be possible to fill this gap using a Berry-Esseen like inequality for Laplace transforms [22].

On the other hand, the normalization of the cost function $C(\omega)$ by the number of occurrences of the character a in a word ω (see equations (1)-(3)) has greatly simplified our analysis. Indeed, the expression in equation (17) involves operators of the form $\mathbf{K}(z, u^s)$ which are amenable to a spectral perturbation analysis because this normalization keeps the term u^s in a bounded neighborhood of 1. A more direct analysis of the cost $C(\omega)$ may be also possible when normalizing by $|\omega|$ instead of $|\omega|_a$. However, this creates technical difficulties we have not been able to fully address as the generating function $A(z,u)$ in equation (18) would now also depend on the ratio $|\omega|_a/|\omega| = m/n$ and would have to be replaced by

$$A_{m,n}(z,u) = \exp\left(n\int_0^{\frac{m}{n}} \log\lambda(z,u^t)\,dt\right).$$

The extraction of coefficients is far more intricate in this case. Similar difficulties arise when attempting to analyze joint distributions via the trivariate generating function:

$$\sum_{\omega\in\mathcal{A}^*} p_\omega\, z^{|\omega|}\, u^{C(\omega)}\, v^{|\omega|_a},$$

associated with the costs $C(\omega)$ and $|\omega|_a$.

To conclude, we note that extensions of Theorem 2.1 are foreseeable to all bi-modular hidden patterns in which the modules cannot overlap (neither with themselves nor with each other). Due to the proof of Theorem 4.1, any such case should be reduced in principle to identifying the asymptotically dominant singularity of an operator of the form $\prod_{i=m}^{1}\mathbf{K}(z,u^{i/m})$, with $m\geq 1$, which we now know how. We hope however that the techniques used to solve this "toy–problem" may be further developed to deal with auto-correlated patterns in the context of correlated sources.

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6 Annex A

6.1 Holomorphic dynamical sources

In what follows, we use the notation of § 3. For the sake of simplicity, we consider complete (holomorphic) dynamical sources as defined by Vallée in [23]. In this context, some spectral properties of operators are almost immediate (due to compacity and nuclearity) whereas the same properties have to be proved in more general settings, such as the one considered in [3]. The gap is essentially technical rather than difficult and hence we focus on a general strategy for studying occurrences of modular patterns under holomorphic dynamical sources.

Definition 6.1 *Let $\mathcal{S} = (I, T, \mathcal{A}, (I_\alpha)_{\alpha\in\mathcal{A}}, \sigma, f)$ be a probabilistic dynamical source whose associated set of inverse branches is $\{h_\alpha,\ \alpha \in \mathcal{A}\}$. $\mathcal{S}$ is said holomorphic if there exists an open complex neighborhood $\mathcal{V}$ of $[0, 1]$ on which:*

1. *the inverse branches h_α extend to holomorphic functions on $\mathcal{V}$;*
2. *for each $\alpha \in \mathcal{A}$, h_α maps $\mathcal{V}$ strictly inside $\mathcal{V}$ i.e. $h_\alpha(\overline{\mathcal{V}}) \subset \mathcal{V}$;*
3. *for each $\alpha \in \mathcal{A}$, there exists $0 < \delta_\alpha < 1$ such that $|h'_\alpha|$ can be extended to an holomorphic function $\widetilde{h}_\alpha$ over all of $\mathcal{V}$ that satisfies $0 < |\widetilde{h}_\alpha(z)| < \delta_\alpha$, for all $z \in \mathcal{V}$; and*
4. *the series $\sum_{\alpha\in\mathcal{A}} \delta_\alpha$ converges.*

We note that $\overline{\mathcal{V}}$ above denotes the adherence (also called closure) of $\mathcal{V}$ in the complex plane. In the sequel, we will always suppose that $\mathcal{S}$ is a holomorphic probabilistic dynamical source.

6.2 Functional space and first spectral properties

All the generating functions in this paper involve operators whose analyticity is closely related to their spectrum. However, the spectrum of an operator depends on the functional space on which it acts. A convenient space for holomorphic dynamical sources is the space $\mathcal{A}_\infty(\mathcal{V})$ of all functions that are analytic on $\mathcal{V}$ and continuous on $\overline{\mathcal{V}}$. Endowed with the sup-norm $\|\cdot\|$, this is a Banach space.

For each word $\omega \in \mathcal{A}^\star$, the component operator $\mathbf{G}_\omega$ defined by $\mathbf{G}_\omega[f] = \widetilde{h}_\omega \cdot (f \circ h_\omega)$ is also called a composition operator. Such operators are widely studied by the authors of [21, 20], who prove the following:

Proposition 6.2 *For each $\omega \in \mathcal{A}^\star$, the operator $\mathbf{G}_\omega : \mathcal{A}_\infty(\mathcal{V}) \to \mathcal{A}_\infty(\mathcal{V})$ is well-defined, compact, and nuclear of order 0.*

The notion of nuclearity was introduced by Grothendieck [8]. A nuclear operator "looks like" a matrix in the sense that one can e.g. define its trace and an analogue of the characteristic polynomial, also known as the Fredholm determinant. An immediate consequence of Proposition 6.2 is the following corollary.

Corollary 6.3

(i) *For all complex u, the operator $(\mathbf{G}_\mathcal{C} + u\,\mathbf{G}_b)$ is compact and nuclear of order 0.*

(ii) *If $\mu(u)$ denotes one of the dominant eigenvalues of $(\mathbf{G}_\mathcal{C} + u\,\mathbf{G}_b)$ then the operator $\mathbf{K}(z,u)$ is analytic in (z,u) for $|z| < (1/|\mu(u)|)$. In addition, $\mathbf{K}(z,u)$ is compact and nuclear of order 0.*

Operators with a unique dominant eigenvalue isolated from the remainder of the spectrum by a spectral gap are fundamental in all dynamical analyses. The compactness property ensures the existence of a spectral gap. The uniqueness of the dominant eigenvalue is then obtained for $z > 0$ and $u > 0$.

Proposition 6.4

(i) *For all $u > 0$, the operator $(\mathbf{G}_\mathcal{C} + u\,\mathbf{G}_b)$ admits a unique dominant eigenvalue $\mu(u)$, which happens to be simple and strictly positive.*

(ii) *For $u > 0$ and $0 < z < (1/\mu(u))$, the operator $\mathbf{K}(z,u)$ admits a unique dominant eigenvalue $\lambda(z,u)$, which happens to be also simple and strictly positive. The associated eigenfunction, $\psi_{z,u}$, is strictly positive over $(\mathcal{V} \cap \mathbb{R})$.*

(iii) *There exists a complex neighborhood $\mathcal{W}_0$ of $\{(z,u)|\ u > 0, 0 < z < (1/\mu(u))\}$ such that the operator $(\mathbf{G}_\mathcal{C} + u\,\mathbf{G}_b)$ (resp. $\mathbf{K}(z,u)$) admits a unique dominant eigenvalue $\mu(u)$ (resp. $\lambda(z,u)$), which happens to be simple and isolated from the remainder of the spectrum by a spectral gap. Furthermore, the eigenfunction $\psi_{z,u}$ relative to $\lambda(z,u)$ is non-zero over $(\mathcal{V} \cap \mathbb{R})$.*

Proof: (Sketch.) The proof is the same as the one of Proposition 2 in [23]. In a first step, the result is obtained for the real Banach space $\mathcal{A}_{\infty,\mathbb{R}}(\mathcal{V})$ of functions in $\mathcal{A}_\infty(\mathcal{V})$ that are real-valued over $(\mathcal{V}\cap\mathbb{R})$. The subspace of $\mathcal{A}_{\infty,\mathbb{R}}(\mathcal{V})$ of functions that are non-negative over $(\mathcal{V} \cap \mathbb{R})$ is a proper and reproducing cone. Now, the operators under consideration in the proposition are u_0-positive for the constant function $u_0 = 1$ with respect to the order induced by the cone. The compactness and a Perron-Frobenius type Theorem due to Krasnoselsky [15] are the main ingredients to conclude the first part of the proof. The result is then transfered to the complex Banach space $\mathcal{A}_\infty(\mathcal{V})$ by using the nuclearity property and the trace of the operators. The extension to the complex neighborhood is just a consequence of the perturbation theory due to Kato [13]. □

Proposition 6.4 describes the operator $\mathbf{K}(z,u)$ when z and u are close to the real axis. In § 6.3, we show how this proposition and the spectral decomposition lead to alternative expressions of the generating functions $\mathcal{H}_m(z,u)$ and $\mathcal{H}(z,u)$ for z and u near the real axis. But, for a complete analysis, additional information when z is close to or far from the real axis is also needed. This is the aim of the next proposition.

Proposition 6.5 *Consider $z_0 > 0$ and $u_0 > 0$ in $\mathcal{W}_0$ (see part (iii) in Proposition 6.4).*

(i) The functions $z \to \lambda(z,u_0)$ and $u \to \lambda(z_0,u)$ are strictly increasing along $(\mathcal{V} \cap \mathbb{R}_+)$.

(ii) $|\lambda(z,u_0)| \le \lambda(|z|,u_0)$, and the inequality is strict if $z \neq |z|$.

(iii) $\lambda(1,1) = 1$.

(iv) The function $|\lambda(z,u)|$ can be bounded in terms of $\lambda(|z|,u)$, for all $(z,u) \in \mathcal{W}_0$ in a neighborhood of $(1,1)$. More precisely, there exist $\epsilon > 0$, a positive constant $\kappa_\epsilon > 0$, and a complex neighborhood $\mathcal{Z}_\epsilon$ of $z = 1$ such that for each $z \in \mathcal{Z}_\epsilon$ and $u \in]1-\epsilon, 1+\epsilon[$, $(z,u) \in \mathcal{W}_0$ and $|\lambda(z,u)| \le \exp(-\kappa_\epsilon |\arg(z)|^2) \cdot \lambda(|z|,u)$, where $\arg(z)$ denotes the principal argument of z.

Proof: (i) Consider $z < t$ and the eigenfunctions ψ_{z,u_0} and ψ_{t,u_0} such that, for each $x \in (\mathcal{V} \cap \mathbb{R})$, $\psi_{z,u_0}(x) \le \psi_{t,u_0}(x)$ and $\psi_{z,u_0}(x_0) = \psi_{t,u_0}(x_0)$, for some $x_0 \in (\mathcal{V} \cap \mathbb{R})$. It then applies that:

$$\begin{aligned}
\lambda(z,u_0)^n\, \psi_{z,u_0}(x_0) &= \mathbf{K}(z,u_0)^n[\psi_{z,u_0}](x_0), \\
&= \sum_{h \in \mathcal{K}^n} \widetilde{h}(x_0)\, (\psi_{z,u_0} \circ h)(x_0)\, z^{|h|}\, u^{c(h)}, \\
&\le \sum_{h \in \mathcal{K}^n} \widetilde{h}(x_0)\, (\psi_{t,u_0} \circ h)(x_0)\, t^{|h|}\, u^{c(h)}, \\
&= \mathbf{K}(t,u_0)^n[\psi_{t,u_0}] = \lambda(t,u_0)^n\, \psi_{t,u_0}(x_0).
\end{aligned}$$

The definition of x_0 entails that $\lambda(z,u_0) \le \lambda(t,u_0)$. Now, equality occurs above only when, for each $h \in \mathcal{K}^n$, $(\psi_{z,u_0} \circ h)(x_0) = (\psi_{t,u_0} \circ h)(x_0)$ and $t^{|h|} = z^{|h|}$. This shows that $z = t$; in particular, $z \to \lambda(z,u_0)$ is strictly increasing. The result for $u \to \lambda(z_0,u)$ is obtained by a similar argument, which completes the proof of claim (i).

(ii) Consider $r > 0$ and $u > 0$ such that $(r,u) \in \mathcal{W}_0$. For $0 < \theta < 2\pi$, let $\psi_{re^{i\theta},u}$ denote an eigenfunction of $\mathbf{K}(re^{i\theta},u)$ relative to one of the dominant eigenvalues $\lambda(re^{i\theta},u)$. We also suppose that for all $x \in (\mathcal{V} \cap \mathbb{R})$, $|\psi_{re^{i\theta},u}(x)| \le \psi_{r,u}(x)$ with equality for at least one $x_0 \in (\mathcal{V} \cap \mathbb{R})$. Like previously, we have:

$$
\begin{aligned}
|\lambda(re^{i\theta},u)|^n\,|\psi_{re^{i\theta},u}(x_0)| &= |\mathbf{K}(re^{i\theta},u)^n[\psi_{re^{i\theta},u}](x_0)|, \\
&\le \sum_{h\in\mathcal{K}^n} \widetilde{h}(x_0)\,|(\psi_{re^{i\theta},u}\circ h)(x_0)|\,r^{|h|}\,u^{c(h)}, \qquad (20)\\
&\le \sum_{h\in\mathcal{K}^n} \widetilde{h}(x_0)\,(\psi_{r,u}\circ h)(x_0)\,r^{|h|}\,u^{c(h)}, \qquad (21)\\
&= \mathbf{K}(r,u)^n[\psi_{r,u}](x_0) = \lambda(r,u)^n\,\psi_{r,u}(x_0).
\end{aligned}
$$

This proves the bound $|\lambda(re^{i\theta},u)| \le \lambda(r,u)$.

Next suppose that $|\lambda(re^{i\theta},u)| = \lambda(r,u)$ i.e. the inequalities in equations (20)-(21) are in fact equalities. Then, for each $h \in \mathcal{K}^\star$, $|(\psi_{re^{i\theta},u}\circ h)(x_0)| = (\psi_{r,u}\circ h)(x_0)$. Since any point in I_a is the limit of a sequence $h(x_0)$, we conclude that $|\psi_{re^{i\theta},u}(x)| = |\psi_{r,u}(x)|$, for all $x \in I_a$. Equivalently, if we define $\alpha(x) := \psi_{re^{i\theta},u}(x)/\psi_{r,u}(x)$ then $|\alpha(x)| = 1$, for all $x \in I_a$. Under the assumption that equality holds in equations (20)-(21), this implies that

$$\left|\sum_{h\in\mathcal{K}^n} a_h(x)\right| = \sum_{h\in\mathcal{K}^n} |a_h(x)|,$$

for all $x \in I_a$, where $a_h(x) := \tilde{h}(x)\,(\psi_{re^{i\theta},u}\circ h)(x)\,(re^{i\theta})^{|h|}\,u^{c(h)}$. In particular, there exists $\beta(x)$, with $|\beta(x)| = 1$ for $x \in I_a$, such that $a_h(x) = \beta(x)|a_h(x)|$. Furthermore, it follows for each $h \in \mathcal{K}$ and $x \in I_a$ that

$$\beta(x) = \frac{a_h(x)}{|a_h(x)|} = \frac{(\psi_{re^{i\theta},u}\circ h)(x)}{(\psi_{r,u}\circ h)(x)}\,e^{i\theta|h|} = (\alpha\circ h)(x)\,e^{i\theta|h|}.$$

But note that $\beta(x)$ is continuous and independent of the inverse branches. In particular, when x is fixed, the argument of $\beta(x)$ is constant. On the other hand, note that the argument of $\beta(x)$ depends on the depth of h. (If $h = (h_1\circ\ldots\circ h_k)$, with $k \in \mathbb{N}^\star$ and $h_i \in \mathcal{K}$ for each i, we refer to k as the depth of h.) Since all the depths are possible when h goes through $\mathcal{K}$ (the $\gcd$ of all the possible depth is 1), the above identity entails that $\theta = 0$, thus proving claim (ii).

(iii) The operator $\mathbf{G}_{\mathcal{A}}$ is usually called the density transformer since it transforms probability density functions into probability density functions. It follows that 1 is the unique dominant eigenvalue of $\mathbf{G}_{\mathcal{A}}$. Next, observe that

$$(\mathbf{I} - u\,\mathbf{K}(1,1))^{-1} \circ (\mathbf{I} - \mathbf{G}_{\mathcal{B}})^{-1} = (\mathbf{I} - (\mathbf{G}_{\mathcal{B}} + u\,\mathbf{G}_a))^{-1},$$

because $\mathcal{B}^\star\,(a\,\mathcal{B}^\star)^\star = \mathcal{A}^\star$. The right-hand side above admits a singularity when $(\mathbf{G}_{\mathcal{B}} + u\,\mathbf{G}_a)$ has 1 as eigenvalue. But when $u = 1$, this operator corresponds to a density transformer whose dominant eigenvalue is 1. Hence, $\mathbf{K}(1,1)^n = [u^n](\mathbf{I} - (\mathbf{G}_{\mathcal{B}} + u\,\mathbf{G}_a))^{-1} \circ (\mathbf{I} - \mathbf{G}_{\mathcal{B}})$ is asymptotically the residue operator of $(\mathbf{I} - (\mathbf{G}_{\mathcal{B}} + u\,\mathbf{G}_a))^{-1} \circ (\mathbf{I} - \mathbf{G}_{\mathcal{B}})$ when $u = 1$, which is asymptotically constant. This is only possible when $\lambda(1,1) = 1$, which shows claim (iii).

For the proof of claim (iv), first remark that

$$\left|\frac{\lambda(re^{i\theta},u)}{\lambda(r,u)}\right| = \exp\left(-\bar{\lambda}''(s_0,t_0)\frac{\theta^2}{2} + O(\theta^3)\right),$$

with $\bar{\lambda}(s,t) = \log\lambda(e^s, e^t)$, $e^{s_0} = r$ and $e^{t_0} = u$. We will prove in Proposition 6.11 that $\bar{\lambda}$ is strictly convex. This completes the proof. □

6.3 Alternative expressions for the generating functions

The generating function $\mathcal{H}_m(z,u)$ involves a product of operators. When $u = 1$, the product simplifies to

$$\prod_{j=m}^{1} \mathbf{K}(z, u^{j/m}) = \mathbf{K}(z,1)^m = \lambda(z,1)^m\, \mathbf{P}(z,1) + \mathbf{R}(z,1)^m,$$

where $\mathbf{P}(z,u) \circ \mathbf{R}(z,u) = \mathbf{R}(z,u) \circ \mathbf{P}(z,u) = 0$, and $\mathbf{P}(z,u)$ is the projector over the dominant eigenspace relative to the eigenvector $\psi_{z,u}$ (see Proposition 6.4). When u is close to 1, we can expect the product on the left-hand side above to remain close to some m-th power. The next proposition makes this heuristic more precise.

Proposition 6.6 *Consider $\mathcal{W}_1 = \mathcal{Z}_1 \times \mathcal{U}_1$ an open subset of $\mathcal{W}_0$, with $\mathcal{U}_1 = (1-\epsilon, 1+\epsilon)$, for some $\epsilon > 0$, and such that $\overline{\mathcal{W}_1}$ is a compact subset of $\mathcal{W}_0$. For all $m \in \mathbb{N}^\star$ and $(z,u) \in \mathcal{W}_1$, there exist a complex number $A(z,u)$, a bounded linear operator $\mathbf{B}(z,u)$ and a linear application $\mathbf{C}_m(z,u)$ defined on $\mathcal{A}_\infty(\mathcal{V})$ such that*

$$\prod_{j=m}^{1} \mathbf{K}(z, u^{j/m})[f] = A(z,u)^m \Big(\mathbf{B}(z,u)[f]\psi_{z,u} + \mathbf{C}_m(z,u)[f]\Big), \text{ with } \|\mathbf{C}_m(z,u)\| = O\left(\frac{1}{m}\right).$$

All these objects are analytic in (z,u) and the constant in the O-term is uniform for $(z,u) \in \overline{\mathcal{W}_1}$.

Proof: The proof is given in Annex B. □

Except for $\mathbf{C}_m$, all the objects in the proposition are explicit. As a matter of fact, if we define $\phi(z,u,v)$ by $\mathbf{P}(z,u)[\psi_{z,v}] = \phi(z,u,v)\,\psi_{z,u}$, and $\mathbf{B}(z)$ by $\mathbf{P}(z,0)[f] = \mathbf{B}(z)[f]\psi_{z,0}$ then

$$\begin{aligned} A(z,u) &= \exp\left(\int_0^1 \log\lambda(z,u^t)\,dt\right), \qquad (22)\\ \mathbf{B}(z,u)[f] &= \mathbf{B}(z)[f]\,\exp\left(-\log u \int_0^1 u^t \phi'_v(z,u^t,u^t)\,dt\right). \end{aligned}$$

An immediate consequence of Proposition 6.6 is an alternative expression of the generating functions.

Corollary 6.7 *With the same notations of Proposition 6.6, for all $m \in \mathbb{N}^\star$ and $(z,u) \in \mathcal{W}_1$, there exist analytic functions $A(z,u)$, $B(z,u)$ and $C_m(z,u)$ such that*

$$\mathcal{H}_m(z,u) = A(z,u)^m\,(B(z,u) + C_m(z,u)), \text{ with } \|C_m\| = O\left(\frac{1}{m}\right). \qquad (23)$$

The constant in the O-term is uniform for $(z,u) \in \overline{\mathcal{W}_1}$. Furthermore, if $|A(z,u)| < 1$ then

$$\mathcal{H}(z,u) = \frac{B(z,u)}{1-A(z,u)} + O\left(\log\frac{1}{1-|A(z,u)|}\right). \qquad (24)$$

Of course the function $A(z,u)$ is given by (22), and

$$\begin{aligned} B(z,u) &= \int_0^1 \mathbf{B}(z,u) \circ (\mathbf{I} - z\mathbf{G}_{\mathcal{B}})^{-1}[f](t)\,dt, \\ C_m(z,u) &= \int_0^1 \mathbf{C}_m(z,u) \circ (\mathbf{I} - z\mathbf{G}_{\mathcal{B}})^{-1}[f](t)\,dt. \end{aligned}$$

6.4 Singularity analysis

When $u = 1$, $A(z,1)$ equals to the eigenvalue $\lambda(z,1)$ and Proposition 6.5-(iii) entails that $A(1,1) = 1$. Since $A'_z(1,1) \cdot A'_u(1,1) \neq 0$, the implicit function theorem implies that there exist a complex neighborhood $\mathcal{Z}_2$ of $z = 1$, a complex neighborhood $\mathcal{U}_2$ of $u = 1$, an analytic function $\sigma : \mathcal{U}_2 \to \mathcal{Z}_2$ such that, for each $u \in \mathcal{U}_2$, $z = \sigma(u)$ is the unique solution in $\mathcal{Z}_2$ to the equation $A(z,u) = 1$. Combining the existence of this function σ with Proposition 6.5–(i) and (ii), we obtain the following description of the singularities of the generating functions.

Proposition 6.8 *There exist a real neighborhood $\mathcal{U}_3$ of $u = 1$ such that, for each $u \in \mathcal{U}_3$, the generating function $\mathcal{H}(z,u)$ is analytic for $|z| < \sigma(u)$. In addition, $z = \sigma(u)$ is the unique singularity of $\mathcal{H}(z,u)$ on $|z| = \sigma(u)$.*

Next we extract the coefficients. Due to equation (23), the generating function $\mathcal{H}(z,u)$ can be decomposed into a dominant and remainder term, where the later is a linear combination of terms of the form $A(z,u)^m C_m(z,u)$, with $m \geq 0$. Due to the lack of information on the singularities outside the disk of convergence $|z| = \sigma(u)$ (see previous proposition), we use a saddle-point method to estimate coefficients of the remainder term. This method is well adapted for obtaining asymptotic formulas for the coefficients $A(z,u)^m$, when m is large, although the remainder term in $\mathcal{H}(z,u)$ also involves small powers of $A(z,u)$. The next lemma is crucial to disregard small values of m.

Lemma 6.9 *There exist $\epsilon > 0$, $0 < \phi < 1$ and $0 < \rho < 1$ such that for all $(1-\epsilon) < u < (1+\epsilon)$,*

$$[z^n] \sum_{m \leq \phi n} \mathcal{H}_m(z,u) = O(\rho^n).$$

Proof: Consider $\epsilon > 0$ and $(1-\epsilon) < u < (1+\epsilon)$. Since $\Omega_n \leq n$, we obtain that

$$[z^n] \sum_{m \leq \phi n} \mathcal{H}_m(z,u) \leq (1+\epsilon)^n \, \mathbb{P}_n \left[|\omega|_a \leq \phi n \right].$$

Next note that the operator $(\mathbf{G}_{\mathcal{B}} + v\,\mathbf{G}_a)$ is compact and admits a unique positive dominant eigenvalue $\gamma(v)$ when $v \geq 0$. Furthermore, $v \to \gamma(v)$ is strictly increasing, with $\gamma(1) = 1$, and in a certain open neighborhood $\mathcal{V}_0$ of $v = 1$, there exist a finite constant $M \geq 0$ (independent of v) such that

$\|(\mathbf{G}_\mathcal{B} + v\,\mathbf{G}_a)^k\| \leq M\gamma(v)^k$, for all $k \geq 0$. Hence, for $m \leq \phi n$, we obtain

$$\begin{aligned}
\mathbb{P}_n\left[|\omega|_a = m\right] &= \sum_{\omega \in \mathcal{A}^n : |\omega|_a = m} p_\omega, \\
&\leq \frac{1}{r^m} \sum_{\omega \in \mathcal{A}^n} p_\omega r^{|\omega|_a}, \\
&= \frac{1}{r^m} \int_0^1 (\mathbf{G}_\mathcal{B} + r\,\mathbf{G}_a)^n [f](t) dt \leq M\|f\| \frac{\gamma(r)^n}{r^m},
\end{aligned}$$

uniformly for all $r > 0$ in $\mathcal{V}_0$; in particular, for $0 < r < 1$, we find

$$\mathbb{P}_n\left[|\omega|_a \leq \phi n\right] \leq \frac{M\|f\|}{1-r} \frac{\gamma(r)^n}{r^{\phi n}}.$$

If ϕ is chosen such that $\phi < \gamma'(1)/2$, the derivative $[\gamma(r)/r^\phi]'$ taken at $r = 1$ is greater than $\gamma'(1)/2$, which is strictly positive. Hence, up to perhaps a smaller neighborhood $\mathcal{V}_0$ of 1, for all in $\overline{\mathcal{V}}_0$ except for $r = 1$, we have $\gamma(r)/r^\phi < 1$. In particular, if we choose

$$c := \inf \left\{ \frac{\gamma(r)}{r^\phi} \mid 0 < r < 1,\ r \in \overline{\mathcal{V}}_0 \right\},$$

then $c < 1$ and $\mathbb{P}_n\left[|\omega|_a \leq \phi n\right] = O(c^n)$. To conclude, ϵ must be chosen so that $\rho = (1+\epsilon)c < 1$. $\square$

Proposition 6.10 *There exist a real neighborhood $\mathcal{U}_4$ of $u = 1$ such that, for each $u \in \mathcal{U}_4$ and $\xi > 0$, $\mathbb{E}_n[u^{\Omega_n}] = \alpha(u) \cdot \beta(u)^n (1 + O(n^{-\frac{1}{2}+\xi} \log n))$, where the constant in the O-term is uniform for all $u \in \mathcal{U}_4$, and*

$$\begin{aligned}
\alpha(u) &= \frac{B(\sigma(u), u)}{\sigma(u)\, A'_z(\sigma(u), u)}, \\
\beta(u) &= \sigma(u)^{-1}.
\end{aligned}$$

Proof: According to equation (24), $\mathcal{H}(z, u)$ is composed by a meromorphic part and a remainder part. The extraction of the coefficients of the meromorphic part satisfies:

$$[z^n] \frac{B(z,u)}{1 - A(z,u)} = \alpha(u) \cdot \beta(u)^n \left(1 + O(\rho_1^n)\right),$$

for some uniform $\rho_1 < 1$ on a neighborhood of $u = 1$.

To study the remainder part, a saddle point method is used. We also suppose that the conditions of Lemma 6.9 are verified for some ϵ, ϕ and ρ. According to the aforementioned lemma, the analysis of the generating function

$$R(z,u) := \sum_{m \geq \phi n} A(z,u)^m C_m(z,u)$$

is sufficient. Up to a smaller ϵ, consider $\mathcal{Z}_\epsilon$ like in Proposition 6.5. The continuity of the function δ entails that there exist $\epsilon' < \epsilon$, $\eta > 0$ and $\tau > 0$ such that $\sigma([1-\epsilon', 1+\epsilon']) \subset]1-\eta, 1+\eta[\subset \mathcal{Z}_{\eta,\tau} \subset \mathcal{Z}_\epsilon$, with

$\mathcal{Z}_{\eta,\tau} = \{z \mid |z-1| < 2\eta,\ \arg(z) < \tau\}$. For $u \in [1-\epsilon', 1+\epsilon']$, $\mathcal{H}(z,u)$ is analytic in the centered disc of radius $\sigma(u)$. We apply the Cauchy's formula for coefficients with a circle $\mathcal{C}$ of radius $(\sigma(u) - 1/n)$ in order to obtain an upper bound. The contour $\mathcal{C}$ is split into three parts:

$$\begin{aligned}
\mathcal{C}_1 &= \left\{ z \text{ s.t. } |z| = \sigma(u) - \frac{1}{n}, |\arg(z)| < \epsilon_n \right\}, \\
\mathcal{C}_2 &= \left\{ z \text{ s.t. } |z| = \sigma(u) - \frac{1}{n}, \epsilon_n \le |\arg(z)| < \tau \right\}, \\
\mathcal{C}_3 &= \left\{ z \text{ s.t. } |z| = \sigma(u) - \frac{1}{n}, \tau \le |\arg(z)| \le \pi \right\},
\end{aligned}$$

where $\epsilon_n = n^{-\frac{1}{2}+\xi}$.

Contour $\mathcal{C}_1$: the derivative $A'_z(z,u)$ is non zero for $(1-\epsilon') < u < (1+\epsilon')$ and $z \in (\mathcal{Z}_{\eta,\tau} \cap \mathbb{R})$ since the same is true for $\lambda'_z(z,u)$. In particular, there exists a finite constant $K_1 > 0$ such that for each (z,u) as before, $|A(z,u) - 1| \ge K_1 |z - \sigma(u)|$. Now, for $z \in \mathcal{C}_1$, Proposition 6.5-(iv) leads to the following upper bound, $|A(z,u)| \le \exp(-\kappa_\epsilon |\arg(z)|^2)\, A(\sigma(u) - 1/n, u)$. Recall that there exists a uniform constant K_2 such that $\|C_m\| \le K_2/m$. Then for $r = \sigma(u) - 1/n$, one has

$$\left| \frac{1}{2\pi} \int_{-\epsilon_n}^{\epsilon_n} \frac{R(re^{i\theta}, u)}{r^n e^{in\theta}} d\theta \right| \le \frac{1}{2\pi} \frac{2\epsilon_n}{r^n}\, K_2 \log\left(\frac{1}{1 - A(r,u)} \right) = O\left(\frac{\epsilon_n \log n}{\sigma(u)^n} \right),$$

where the constant in the O-term is uniform for $|u-1| < \epsilon'$.

Contour $\mathcal{C}_2$: Once more, we apply Proposition 6.5–(iv). Using the same notation as before, we obtain

$$\begin{aligned}
\left| \frac{1}{2\pi} \int_{\epsilon_n}^{\tau} \frac{R(re^{i\theta}, u)}{r^n e^{in\theta}} d\theta \right| &\le \frac{1}{2\pi} \frac{1}{r^n} \sum_{m \ge \phi n} \frac{A(r,u)^m}{m} \int_{\epsilon_n}^{\tau} \exp(-m\kappa_\epsilon \theta^2) d\theta, \\
&= O\left(\frac{\exp(-\kappa_\epsilon \phi n \epsilon_n^2)}{\sigma(u)^n} \log \frac{1}{1 - A(r,u)} \right) = O\left(\frac{\exp(-\kappa_\epsilon n^{-2\xi})}{\sigma(u)^n} \log n \right),
\end{aligned}$$

where the constant in the O-term is uniform for $|u-1| < \epsilon'$.

Contour $\mathcal{C}_3$: Proposition 6.5–(ii) entails that up to smaller ϵ' and η (not τ), there exist $k \in \mathbb{N}$ such that for all $i \le (m-k)$, $\|\mathbf{K}(z, u^{(i+k)/m} \circ \ldots \circ \mathbf{K}(z, u^{i/m})\| \le \|\mathbf{K}(z, u^{i/m})^k\| \le (1-\bar{\rho})$, with $\bar{\rho} > 0$ and all (z,u) such that $|z-1| < \eta$, $|\arg(z)| > \tau$ and $|u-1| < \epsilon'$. Then uniformly in u, we obtain the bound

$$\left| \frac{1}{2\pi} \int_{|\theta| > \tau} \frac{R(re^{i\theta}, u)}{r^n e^{in\theta}} d\theta \right| = O\left((1-\bar{\rho})^{\phi n} \right).$$

The lemma now follows from the sum of the integrals over the three contours. □

6.5 Gaussian limit law

Our results so far are valid only for real values of u. The use of continuity theorems for characteristic functions is then out of the scope of this work. It is well known, however, that point-wise convergence of the Laplace transform of a random variable over an open interval of the form $(-\epsilon, \epsilon)$ entails a convergence in law. The completion of our proof of Theorem 2.1 is therefore based on the continuity theorem of the bilateral Laplace Transform [6].

Precisely, motivated by Proposition 6.10, define $U(s) := -\log \sigma(e^s)$ and $V(s) := \log \alpha(e^s)$. Clearly, $U(s)$ is analytic for s in a neighborhood of $s = 0$, and it follows from the proposition that the normalized random variable $\Omega_n^\star = \big(\Omega_n - n\,U'(0)\big)/\sqrt{n\,U''(0)}$ satisfies:

$$\mathbb{E}_n[e^{s\Omega_n^\star}] = \frac{s^2}{2} + O\left(\frac{1}{\sqrt{n}} + \frac{\log n}{n^{\frac{1}{2}-\xi}}\right),$$

over a certain real-neiborhood of $s = 0$, as long as $U''(0) = -\big(\sigma''(1) + \sigma'(1) - \sigma'(1)^2\big) \neq 0$.

Theorem 2.1 is finally a consequence of the following result.

Proposition 6.11 *The variability condition $\sigma''(1) + \sigma'(1) - \sigma'(1)^2 \neq 0$ holds.*

Proof: In what follows, we associate with each function $S(z,u)$ of (z,u) the function $\bar{S}(s,t) := S(e^s, e^t)$. As soon as $S(z,u)$ is analytic around $(z,u) = (1,1)$ with nonnegative coefficients, the Cauchy-Schwartz inequality entails the following convexity inequality:

$$\log \bar{S}\left(\frac{s_1+s_2}{2}, \frac{t_1+t_2}{2}\right) \leq \frac{\log \bar{S}(s_1,t_1) + \log \bar{S}(s_2,t_2)}{2}.$$

In particular, the functions $s \to \log \bar{S}(s,t)$, $t \to \log \bar{S}(s,t)$ and $t \to \log \bar{S}(qt,t)$, with $q > 0$ a constant, are convex (concave up).

By considering the generating function relative to the operator $\mathbf{K}(z,u)^m$, the functions $s \to \log \bar{\lambda}(s,t)$, $t \to \log \bar{\lambda}(s,t)$ and $t \to \log \bar{\lambda}(qt,t)$, with $q > 0$, must be convex. The strict convexity is obtained using the same kind of arguments as in Proposition 6.5 (manipulating spectral objects).

Due to the definition of $A(z,u)$ in function of $\lambda(z,u)$, the functions $s \to \log \bar{A}(s,t)$, $t \to \log \bar{A}(s,t)$ and $t \to \log \bar{A}(qt,t)$ for $q > 0$ are also proved to be strictly convex. To conclude with the proof of the proposition, we just use the strict convexity with the following equalities

$$\sigma''(1) + \sigma'(1) - \sigma'(1)^2 = \bar{\sigma}''(0) = -\frac{L''(0)}{\bar{A}'_s(0,0)},$$

with $L(t) = \log \bar{A}(\bar{\sigma}'(0)t, t)$ and $\bar{\sigma}(t) = \log \sigma(e^t)$. □

7 Annex B

In this section we finally prove Proposition 6.6. For this it is useful to introduce a notation for the product (i.e. composition) of various operators. More precisely, if $\mathbf{O}(z,u)$ is an operator then we define

$$[\mathbf{O}]_\beta = \prod_{\ell=\alpha+\beta}^{\alpha+1} \mathbf{O}(z, u^{\frac{\ell}{m}}) = \mathbf{O}(z, u^{\frac{\alpha+\beta}{m}}) \circ \ldots \circ \mathbf{O}(z, u^{\frac{\alpha+1}{m}}),$$

for some appropriately selected integers $m \geq 1$ and $\alpha, \beta \geq 0$ such that $(\alpha+\beta) \leq m$.

In this section, we mainly use the spectral decomposition:

$$\mathbf{K}(z,u) = \lambda(z,u)[\mathbf{P}(z,u) + \mathbf{R}(z,u)], \tag{25}$$

where $\mathbf{P}(z,u)$ is the projection over the eigenspace relative to $\lambda(z,u)$, $\mathbf{R}(z,u)$ has spectral radius strictly less than one, $\mathbf{P}(z,u) \circ \mathbf{P}(z,u) = \mathbf{P}(z,u)$, and $\mathbf{P}(z,u) \circ \mathbf{R}(z,u) = \mathbf{R}(z,u) \circ \mathbf{P}(z,u) = 0$.

Sketch of the proof. Inserting the above spectral decomposition into the product $\prod_{j=m}^{1} \mathbf{K}(z, u^{j/m})$ leads to an expression of the form

$$\prod_{j=m}^{1} \mathbf{K}(z, u^{j/m}) = \prod_{j=1}^{m} \lambda(z, u^{j/m}) \times \left(\prod_{j=m}^{1} \mathbf{P}(z, u^{j/m}) + \text{remainder terms} \right), \tag{26}$$

where each term in the remainder is the product of m $\mathbf{R}$- and $\mathbf{P}$-type operators, with at least one $\mathbf{R}$-type factor. Lemma 7.3 shows that the overall contribution of the remainder terms is negligible as m tends to infinity. To show this, we are led to studying products of $\mathbf{P}$-type and $\mathbf{R}$-type operators as well as interchanged factors of this type. Lemma 7.1 describes precisely what the product of m $\mathbf{P}$-type operators looks like asymptotically, whereas the first inequality in Lemma 7.2 gives an upper bound on the norm of products of $\mathbf{R}$-type operators. On the other hand, the relations:

$$\begin{aligned} \mathbf{R}(z,u) \circ \mathbf{P}(z,v) &= [\mathbf{R}(z,u) - \mathbf{R}(z,v)] \circ \mathbf{P}(z,v); \\ \mathbf{P}(z,u) \circ \mathbf{R}(z,v) &= [\mathbf{P}(z,u) - \mathbf{P}(z,v)] \circ \mathbf{R}(z,v); \end{aligned}$$

which are implied from the spectral decomposition in equation (25), are fundamental to prove that remainder terms in equation (26) are effectively negligible. Indeed, each time a product of $\mathbf{P}$-type operators is followed by a product of $\mathbf{R}$-type operators (or vice-versa), the above relations can be used with Lemma 7.2 to show that a factor of order $1/m$ appears. Lemma 7.1 and the Euler-Maclaurin formula are then the final ingredients in the proof of Proposition 6.6.

Lemma 7.1 *For each compact set $\mathcal{C} \subset \mathcal{W}_0$, there exist $K_1 > 1$ such that*

$$\|[\mathbf{P}]_j\| \le K_1,$$

for all $(z,u) \in \mathcal{C}$ and $j \ge 1$. Furthermore, let $\phi(z,u,v)$ be defined by $\mathbf{P}(z,u)[\psi_{z,v}] = \phi(z,u,v)\psi_{z,u}$ for $(z,u),(z,v) \in \mathcal{W}_0$, and let $\mathbf{c}_{z,u}$ be the linear form such that $\mathbf{P}(z,u)[f] = \mathbf{c}_{z,u}[f]\,\psi_{z,u}$, for all $f \in \mathcal{A}^\infty(\mathcal{V})$. We have

$$\left\| \prod_{j=m}^{1} \mathbf{P}(z, u^{j/m}) - \mathbf{c}_{z,1}\,\psi_{z,u}\, \exp\left\{ -\log u \cdot \int_0^1 u^t \phi'_v(z, u^t, u^t) dt \right\} \right\| = O\left(\frac{\log u}{m} \right),$$

where the constant in the big-O is uniform for $(z,u) \in \mathcal{C}$.

Proof: Consider $0 \le \alpha < \beta \le m$. Then, applying recursively the definition of $\phi(z,u,v)$, we obtain

$$\begin{aligned} \prod_{j=\beta}^{\alpha} \mathbf{P}(z, u^{j/m})[f] &= \mathbf{c}_{z,u^{\alpha/m}}[f] \left[\prod_{j=\beta}^{\alpha+1} \mathbf{P}(z, u^{j/m}) \right] [\psi_{z,u^{\alpha/m}}], \\ &= \mathbf{c}_{z,u^{\alpha/m}}[f] \psi_{z,u^{\beta/m}} \prod_{j=\alpha+1}^{\beta} \phi(z, u^{j/m}, u^{(j-1)/m}). \end{aligned}$$

Note that $\phi(z, u^{j/m}, u^{j/m}) = 1$. Furthermore, the product of ϕ functions above can be simplified using the exp-log transformation and the Taylor expansions of $\log(1-w)$ about the point $w = 0$, as well as of $w = 1 - \phi(z, u^{j/m}, u^{(j-1)/m})$ about $(z, u^{j/m}, u^{j/m})$. We obtain

$$\begin{aligned}
&\prod_{j=\alpha+1}^{\beta} \phi(z, u^{j/m}, u^{(j-1)/m}) \\
&\quad = \exp\left\{ -\frac{\log u}{m} \sum_{j=\alpha+1}^{\beta} u^{j/m} \phi'_v(z, u^{j/m}, u^{j/m}) \right\} \cdot \left(1 + O\Big(\frac{\log^2 u}{m}\Big)\right), \\
&\quad = \exp\left\{ -\log u \int_{\alpha/m}^{\beta/m} u^t \phi'_v(z, u^t, u^t) dt + O\Big(\frac{(\beta-\alpha)\log u}{m^2}\Big) \right\} \cdot \left(1 + O\Big(\frac{\log^2 u}{m}\Big)\right), \\
&\quad = \exp\left\{ -\log u \int_{\alpha/m}^{\beta/m} u^t \phi'_v(z, u^t, u^t) dt \right\} \cdot \left(1 + O\Big(\frac{\log u}{m}\Big)\right),
\end{aligned}$$

uniformly for $(z, u) \in \mathcal{C}$ and $0 \le \alpha < \beta \le m$. □

Lemma 7.2 *For each compact set $\mathcal{C} \subset \mathcal{W}_0$, there exist finite constants $K_2, K_3, K_4 > 1$ and $0 < \rho < 1$ such that*

$$\begin{aligned}
\|[\mathbf{R}]_j\| &\le K_2\, \rho^j, \\
\|\mathbf{P}(z, u^{j/m}) - \mathbf{P}(z, u^{(j-1)/m})\| &\le K_3/m, \\
\|\mathbf{R}(z, u^{j/m}) - \mathbf{R}(z, u^{(j-1)/m})\| &\le K_4/m,
\end{aligned}$$

for all $(z, u) \in \mathcal{C}$ and $j \ge 1$.

Proof: The last two inequalities are just the consequence of the analyticity of the operators $\mathbf{R}(z, u)$ and $\mathbf{P}(z, u)$ with respect to (z, u).

To show the first inequality, we start by showing that there is an integer $j_0 \ge 1$ such that

$$\|\mathbf{R}(z, u)^{j_0}\| < 1,$$

for all $(z, u) \in \mathcal{C}$. Following an argument by contradiction, we suppose otherwise. Then, for each integer $j \ge 1$, there exist $(z_j, u_j) \in \mathcal{C}$ such that $\|\mathbf{R}(z_j, u_j)^j\| \ge 1$. Because $\mathcal{C}$ is compact, we can extract a subsequence (z'_j, u'_j) that converges to certain point $(z_0, u_0) \in \mathcal{C}$. Since the spectral radius of $\mathbf{R}(z_0, u_0)$ is strictly less than 1, there exists $j_1 \ge 1$ such that $\|\mathbf{R}(z_0, u_0)^{j_1}\| < 1/2$. Further, due to the continuity of $\|\mathbf{R}(z, u)^{j_1}\|$ as a function of (z, u), there exists a neighborhood $\mathcal{V}_0$ of (z_0, u_0) such that $\|\mathbf{R}(z, u)^{j_1}\| < 3/4$, for all $(z, u) \in \mathcal{V}_0$. In particular, there exists a constant $\alpha > 0$ such that $\|\mathbf{R}(z, u)^j\| \le \alpha(3/4)^{\lfloor j/j_1 \rfloor}$, for all $(z, u) \in \mathcal{V}_0$ and integer $j \ge 1$. Since the right-hand side of this inequality is strictly less than 1 for all sufficiently large j, we have reached a contradiction. Hence, there is an integer $j_0 \ge 1$ such that $\|\mathbf{R}(z, u)^{j_0}\| < 1$ for all $(z, u) \in \mathcal{C}$. By continuity over the compact set $\mathcal{C}$, there also exist a positive real number $\rho_{\mathcal{C}} < 1$ such that $\|\mathbf{R}(z, u)^{j_0}\| < \rho_{\mathcal{C}}$ for all $(z, u) \in \mathcal{C}$. Furthermore, there are finite constants $\rho = \rho_{\mathcal{C}}^{1/j_0} < 1 < c$ such that

$$\|\mathbf{R}(z, u)^j\| < c\rho^j, \tag{27}$$

for all $(z, u) \in \mathcal{C}$ and $j \geq 1$.

To conclude the proof of the lemma, we aim to bound the norm of a product of the form

$$\prod_{j=\alpha+\beta}^{\alpha+1} \mathbf{R}(z, u^{j/m}), \tag{28}$$

with $m \geq 1$ and $\alpha, \beta \geq 0$ such that $(\alpha + \beta) \leq m$. For this, we first consider $\beta = j_0$, where j_0 is as in the previous paragraph. Before multiplying out, we rewrite each operator in the product as $\mathbf{R}(z, u^{\alpha/m}) + \{\mathbf{R}(z, u^{j/m}) - \mathbf{R}(z, u^{\alpha/m})\}$. After multiplying out, we recognize a summation of terms, each of which involves j_0 factors of the form $\mathbf{R}(z, u^{\alpha/m})$ or $\{\mathbf{R}(z, u^{j/m}) - \mathbf{R}(z, u^{\alpha/m})\}$, with $\alpha < j \leq (\alpha + j_0)$. Note that if a term involves k factors of the later form then its norm is bounded by $(j_0 K_4/m)^k c^{k+1} \rho^{j_0 - k}$, because of the inequality in equation (27) and the fact that $\|\mathbf{R}(z, u^{j/m}) - \mathbf{R}(z, u^{\alpha/m})\| \leq j_0 K_4/m$. Since the only admissible values for k are integers between 0 and j_0, we obtain that:

$$\left\| \mathbf{R}(z, u^{\alpha/m})^{j_0} - \prod_{j=\alpha+j_0}^{\alpha+1} \mathbf{R}(z, u^{j/m}) \right\| \leq \sum_{k=1}^{j_0} \binom{j_0}{k} \left(\frac{j_0 K_4}{m} \right)^k c^{k+1} \rho^{j_0 - k} = c\rho^{j_0} \left[\left(1 + \frac{j_0 c K_4}{m\rho} \right)^{j_0} - 1 \right].$$

Next, consider $\epsilon > 0$ such that $(\rho_{\mathcal{C}} + \epsilon) < 1$. Clearly, there exist m_0 such that for each $m \geq m_0$, the right hand-side of the above inequality is strictly less than ϵ. As a result:

$$\left\| \prod_{j=\alpha+j_0}^{\alpha+1} \mathbf{R}(z, u^{j/m}) \right\| < (\rho_{\mathcal{C}} + \epsilon) < 1,$$

for all $m \geq m_0$ and $(z, u) \in \mathcal{C}$. Finally, to bound the norm of the product in equation (28), write $\beta = (q \cdot j_0 + r)$ using Euclidean division, and decompose the product into q parts, each corresponding to some operator $[\mathbf{R}]_{j_0}$, and a remainder part involving at most $(j_0 - 1)$ factors. Precisely, for all $m \geq m_0$ and $(z, u) \in \mathcal{C}$, we obtain

$$\left\| \prod_{j=\alpha+\beta}^{\alpha+1} \mathbf{R}(z, u^{j/m}) \right\| < \bar{K}^{j_0} (\rho_{\mathcal{C}} + \epsilon)^q \leq \bar{K}^{j_0} (\rho_{\mathcal{C}} + \epsilon)^{(1-j_0)/j_0} \times (\rho_{\mathcal{C}} + \epsilon)^{\beta/j_0},$$

with

$$\bar{K} = \max\left(1, \sup_{(z,u)\in\mathcal{C}} \|\mathbf{R}(z, u)\|\right),$$

whereas, for all $m < m_0$ and $(z, u) \in \mathcal{C}$, we obtain

$$\left\| \prod_{j=\alpha+\beta}^{\alpha+1} \mathbf{R}(z, u^{j/m}) \right\| < \bar{K}^{m_0}.$$

The first inequality of the lemma now easily follows from the previous inequalities. □

Lemma 7.3 *For each compact set $\mathcal{C} \subset \mathcal{W}_0$, it applies that*

$$\prod_{j=m}^{1} \mathbf{K}(z, u^{j/m}) = \prod_{j=m}^{1} \lambda(z, u^{j/m}) \times \left(\prod_{j=m}^{1} \mathbf{P}(z, u^{j/m}) + O\Big(\frac{1}{m}\Big) \right),$$

uniformly for $(z, u) \in \mathcal{C}$.

Proof: Fix an integer $m \geq 1$. Due to the decomposition in equation (25), we obtain:

$$\prod_{j=m}^{1} \mathbf{K}(z, u^{j/m}) = \prod_{j=m}^{1} \lambda(z, u^{j/m}) \prod_{j=m}^{1} \Big(\mathbf{P}(z, u^{j/m}) + \mathbf{R}(z, u^{j/m}) \Big), \tag{29}$$

$$= \prod_{j=m}^{1} \lambda(z, u^{j/m}) \left(\prod_{j=m}^{1} \mathbf{P}(z, u^{j/m}) + \sum_{k} \sum_{i_0,\ldots,i_k,j_1,\ldots,j_k} [\mathbf{P}]_{i_0} \circ [\mathbf{R}]_{j_1} \circ \cdots \circ [\mathbf{R}]_{j_k} \circ [\mathbf{P}]_{i_k} \right), \tag{30}$$

where $1 \leq k \leq \lceil m/2 \rceil$; $i_0, i_k \geq 0$; $i_1, \ldots, i_{k-1}, j_1, \ldots, j_k \geq 1$; and $(i_0 + \cdots + i_k + j_1 + \cdots + j_k) = m$.

Next, to understand the summation above, consider the bracket operators $[\mathbf{P}]_i = \prod_{\ell=a+i}^{a+1} \mathbf{P}(z, u^{\ell/m})$ and $[\mathbf{R}]_j = \prod_{\ell=b+j}^{b+1} \mathbf{R}(z, u^{\ell/m})$ for appropriately selected integers $a, b \geq 0$. Observe that if $i, j \geq 1$ and $a = (b + j)$ then

$$\mathbf{P}^{[i]} \circ \mathbf{R}^{[j]} = \prod_{\ell=a+i}^{a+2} \mathbf{P}(z, u^{\ell/m}) \circ \Big(\mathbf{P}(z, u^{(a+1)/m}) - \mathbf{P}(z, u^{a/m}) \Big) \circ \prod_{\ell=b+j}^{b+1} \mathbf{R}(z, u^{\ell/m}),$$

because $\mathbf{P}(z, u^{a/m}) \circ \mathbf{R}(z, u^{a/m}) = \mathbf{R}(z, u^{a/m}) \circ \mathbf{P}(z, u^{a/m}) = 0$. In particular, if the notation $[\mathbf{P}]_{i\Delta}$ is used to mean that the most-right factor in $[\mathbf{P}]_i$ i.e. the operator $\mathbf{P}(z, u^{(a+1)/m})$ is replaced by the difference $\mathbf{P}(z, u^{(a+1)/m}) - \mathbf{P}(z, u^{a/m})$, the above shows that

$$[\mathbf{P}]_i \circ [\mathbf{R}]_j = [\mathbf{P}]_{i\Delta} \circ [\mathbf{R}]_j. \tag{31}$$

Similarly, when $i, j \geq 1$ and $b = (a + i)$ we find that:

$$[\mathbf{R}]_j \circ [\mathbf{P}]_i = [\mathbf{R}]_{j\Delta} \circ [\mathbf{P}]_i. \tag{32}$$

By applying recursively the identities in equations (31)-(32), the remainder terms in equation (29) may be also expressed as

$$[\mathbf{P}]_{i_0} \circ [\mathbf{R}]_{j_1} \circ \cdots \circ [\mathbf{R}]_{j_k} \circ [\mathbf{P}]_{i_k}$$

$$= \begin{cases} [\mathbf{P}]_{i_0\Delta} \circ [\mathbf{R}]_{j_1\Delta} \circ \cdots \circ [\mathbf{P}]_{i_{k-1}\Delta} \circ [\mathbf{R}]_{j_k\Delta} \circ [\mathbf{P}]_{i_k} & , \text{ if } i_0 > 0 \text{ and } i_k > 0; \\ [\mathbf{P}]_{i_0\Delta} \circ [\mathbf{R}]_{j_1\Delta} \circ \cdots \circ [\mathbf{P}]_{i_{k-1}\Delta} \circ [\mathbf{R}]_{j_k} & , \text{ if } i_0 > 0 \text{ and } i_k = 0; \\ [\mathbf{R}]_{j_1\Delta} \circ \cdots \circ [\mathbf{P}]_{i_{k-1}\Delta} \circ [\mathbf{R}]_{j_k\Delta} \circ [\mathbf{P}]_{i_k} & , \text{ if } i_0 = 0 \text{ and } i_k > 0; \\ [\mathbf{R}]_{j_1\Delta} \circ \cdots \circ [\mathbf{P}]_{i_{k-1}\Delta} \circ [\mathbf{R}]_{j_k} & , \text{ if } i_0 = 0 \text{ and } i_k = 0. \end{cases} \tag{33}$$

Due to Lemma 7.2, note that $\|[\mathbf{P}]_{i\Delta}\| \leq K_1K_3/m$ and $\|[\mathbf{R}]_{j\Delta}\| \leq K_2K_4\rho^{j-1}/m$. Inserting these inequalities into the relations in equation (33), we obtain the upper-bound

$$\|[\mathbf{P}]_{i_0} \circ [\mathbf{R}]_{j_1} \circ \cdots \circ [\mathbf{R}]_{j_k} \circ [\mathbf{P}]_{i_k}\| \;\leq\; c^2 \times \left([\![i_0 = 0]\!] + \frac{[\![i_0 > 0]\!]}{m} \right) \times \left([\![i_k = 0]\!] + \frac{[\![i_k > 0]\!]}{m} \right) \times \left(\prod_{\ell=1}^{k-1} \frac{c}{m^2} \right) \times \left(\prod_{\ell=1}^{k} \rho^{j_\ell - 1} \right),$$

where we have defined $c := K_1K_2K_3K_4 \geq 1$.

Motivated by the identity in equation (30), for a fixed value of k, we aim now to add up the terms on the right-hand side of the above inequality over all the admissible values for $i_0, \ldots, i_k, j_1, \ldots, j_k$. This corresponds to extracting the coefficient of z^m of $c^2 \cdot I_0^2(z) \cdot (I(z))^{k-1}(J(z))^k$, where

$$\begin{aligned} I_0(z) &:= \sum_{i \geq 0} \left([\![i = 0]\!] + \frac{[\![i > 0]\!]}{m} \right) z^i = 1 + \frac{z}{m(1-z)}; \\ I(z) &:= \frac{c}{m^2} \sum_{i \geq 1} z^i = \frac{cz}{m^2(1-z)}; \\ J(z) &:= \sum_{j \geq 1} \rho^{j-1} z^j = \frac{z}{1 - \rho z}. \end{aligned}$$

In particular, adding up over all $k \geq 1$, we obtain

$$\sum_k \sum_{i_0,\ldots,i_k,j_1,\ldots,j_k} \|[\mathbf{P}]_{i_0} \circ [\mathbf{R}]_{j_1} \circ \cdots \circ [\mathbf{R}]_{j_k} \circ [\mathbf{P}]_{i_k}\| \leq [z^m]\, S(z),$$

where

$$S(z) := \frac{c^2 \cdot I_0^2(z) \cdot J(z)}{1 - I(z) \cdot J(z)}.$$

This rational generating function admits three-poles at $z = 1$, $z = z_0$ and $z = z_1$, with $z_0 = 1 + O(1/m^2)$ and $z_1 = \rho^{-1} \times (1 + O(1/m^2))$. Due to Cauchy's formula and the Residue theorem, there is a constant $r > 1$ such that

$$\begin{aligned} [z^m]\, S(z) &= -\mathrm{Res}\left(\frac{S(z)}{z^{m+1}}, z = 1 \right) - \mathrm{Res}\left(\frac{S(z)}{z^{m+1}}, z = z_0 \right) + O(r^{-m}), \\ &= -\frac{1}{c} + \frac{1}{c} \cdot \left(1 + O\Big(\frac{1}{m}\Big) \right) + O(r^{-m}) = O\Big(\frac{1}{m}\Big). \end{aligned}$$

This completes the proof of the lemma. □

We finally prove Proposition 6.6.

According to Lemma 7.1, there exists a bounded linear form $\mathbf{B}_0(z, u)$ such that

$$\prod_{j=m}^{1} \mathbf{P}(z, u^{j/m}) = \mathbf{B}_0(z, u) \left(1 + O\left(\frac{1}{m} \right) \right).$$

Furthermore by the Euler-Maclaurin formula, we obtain

$$\prod_{j=1}^{m} \lambda(z, u^{j/m}) = A(z,u)^m B(z,u) \left(1 + O\left(\frac{1}{m}\right)\right),$$

with

$$\begin{aligned} A(z,u) &:= \exp\left(\int_0^1 \log \lambda(z, u^t)\, dt\right); \\ B(z,u) &:= \frac{\lambda(z,u)}{\lambda(z,1)}. \end{aligned}$$

This completes the proof of Proposition 6.6 with

$$\mathbf{B}(z,u) = \frac{\lambda(z,u)}{\lambda(z,1)}\, \mathbf{B}_0(z,u).$$