Symmetries of the k-bounded partition lattice

Chris Berg[1,2,3] and Mike Zabrocki[1,3]

[1] *Fields Institute, Toronto, ON, Canada.*
[2] *Laboratoire de combinatoire et d'informatique mathématique, Université du Québec à Montréal, QC, Canada.*
[3] *York University, Toronto, ON, Canada.*

Abstract. We generalize the symmetry on Young's lattice, found by Suter, to a symmetry on the k-bounded partition lattice of Lapointe, Lascoux and Morse.

Résumé. Nous généralisons la symmetrie sur le treillis de Young, découvert par Suter, à une symétrie sur le treillis des partages bornés par k et étudié par Lapointe, Lascoux and Morse.

Keywords: core partitions, k-Schur functions, cyclic symmetry

1 Introduction

In [Su1], Suter found a dihedral symmetry which exists in Young's lattice, by taking all partitions whose bounding rectangle is contained within the staircase $(k, k-1, k-2, \ldots, 2, 1)$. He recognized that these partitions would have the same symmetries as the affine Dynkin diagram of type A_k.

While studying k-Schur functions, we noticed that the rectangles which Suter uses are the same rectangles that appear in Morse and Lapointe's paper [LM3]. The rectangles in this picture correspond to special elements of the homology of the affine Grassmannian [L1, L2, L3]. For this reason, the lattice of k-bounded partitions related to the algebra of k-Schur functions is a natural place to view a generalization of the symmetry observed by Suter.

Recent results of Berg, Bergeron, Thomas and Zabrocki [BBTZ] developed some geometric properties of the affine hyperplane arrangement. We use this geometric picture to generalize the symmetry that Suter found to the k-bounded partition lattice of Lapointe, Lascoux and Morse [LLM]. We do this by recognizing that the k-bounded partitions which are contained in a concatenation of m rectangles with a k hook is isomorphic to an m-dilation in the geometric picture.

Recently, Nathan Williams [W] has identified an isomorphism between the geometric picture presented here and the set of words of length $k+1$ on $\{0, 1, 2, \ldots, m\}$ which sum to 0 $(mod\ m+1)$ and a cyclic group action on these words.

1.1 From root systems in type A_k to the the affine Grassmannian

Let $\alpha_1, \dots, \alpha_k$ denote the simple roots of type A_k, which form a basis for a vector space V. V has a symmetric bilinear form given by:

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \\ 0 & \text{else.} \end{cases}$$

and we let $\{\Lambda_i\}_{1 \le i \le k}$ denote the basis dual to $\{\alpha_i\}_{1 \le i \le k}$ under this bilinear form. The $\mathbb{Z}$ span of the $\{\Lambda_i\}_{1 \le i \le k}$ will be called the *weights*.

For $v \in V$, we let H_v denote the hyperplane through the origin, perpendicular to v. We write H_i for H_{α_i} and $H_{v,p}$ for the points x satisfying $\langle v, x \rangle = p$.

Let s_i represent the reflection of a vector v through the hyperplane H_i so that the set of reflections $s_1, \dots, s_k$ corresponding to the roots $\alpha_1, \dots, \alpha_k$ generate a reflection group W_0 which is isomorphic to the symmetric group S_{k+1}. The corresponding (finite) root system is Φ_0 is the closure of the set of vectors $\{\alpha_i\}_{1 \le i \le k}$ under the action of W_0. The element $\phi = \alpha_1 + \cdots + \alpha_k$ is known as the *highest root* of the the root system.

The *affine arrangement* is the collection of all hyperplanes $H_{\alpha,p}$ for $\alpha \in \Phi_0$ and $p \in \mathbb{Z}$.

The *dominant chamber* is the (closed) collection of points in V which are bounded by the hyperplanes $H_{\alpha_i,0}$. We denote it by C. A weight is called *dominant* if it lies in the dominant chamber.

The *fundamental alcove* is bounded by the walls of the dominant chamber, together with the hyperplane $H_{\phi,1}$. We denote it by $A_\emptyset$.

The affine reflection group, W, has an additional generator s_0, which acts as reflection in $H_{\phi,1}$. The generators $s_0, s_1, \dots, s_k$ satisfy the affine type A Coxeter relations:

$$s_i^2 = 1 \text{ for } i \in \{0, 1, \dots, k\}$$
$$s_i s_j = s_j s_i \text{ if } i - j \neq \pm 1$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } i \in \{0, 1, \dots, k\}$$

where $i - j$ and $i + 1$ are understood to be taken modulo $k + 1$.

There is an action of W on V defined by s_i reflecting across the hyperplane H_i for $i \in \{1, 2, \dots, k\}$ and s_0 reflecting across the hyperplane $H_{\phi,1}$.

We let $A_w := w^{-1} A_\emptyset$. The collection of A_w are called the *alcoves* of the affine arrangement. The hyperplanes $H_{\alpha_i,n}$ will intersect with A_w either in the empty set, at a single weight, or in a facet of the alcove (the convex hull of k of the vertices of A_w). An alcove $A_w \subset C$ if and only if w is a minimal length coset representative of W/W_0. The set of minimal length coset representatives is denoted W^0. A permutation $w \in W^0$ is called an *affine Grassmannian permutation*.

Example 1.1 *Let $k = 4$. Then $s_4 s_1 s_0 = s_1 s_4 s_0$ is affine Grassmannian because all its reduced words end in s_0, but $s_0 s_1 s_0 = s_1 s_0 s_1$ is not.*

A partition λ is called a $(k+1)$-core if λ has no removable $(k+1)$-rim hook. Define the size of a $(k+1)$-core, $|\lambda|$, to be the number of cells (i, j) with hook smaller than $k + 1$ where the hook of a cell is $\lambda_i + \lambda'_j - i - j + 1$. Let $\mathcal{C}^{(k+1)}$ denote the set of all $(k+1)$-cores.

Example 1.2 *Let $k = 3$ and $\lambda = (4, 2, 2)$. Then λ has no removable 4-rim hooks and the size of λ is 6.*

W has an action on $\mathcal{C}^{(k+1)}$. Let the *content* of a cell (i, j) in the Young diagram of λ be the integer $j - i \mod k + 1$. If λ is a $(k+1)$-core then $s_i\lambda$ is λ union all addable cells of content i, if λ has such an addable cell, $s_i\lambda$ is λ minus all removable boxes of content i from λ if λ has such a removable box (a $(k+1)$-core cannot have both a removable box and an addable position of the same content), and $s_i\lambda = \lambda$ otherwise.

Example 1.3 *Let $k = 3$ and $\lambda = (4, 2, 2)$ as above. Then $s_1\lambda = (4, 3, 2, 1)$ and $s_3\lambda = (3, 2, 1)$.*

0	1	2	3
3	0		
2	3		

Proposition 1.4 *[Lascoux] There is a bijection between affine Grassmannian permutations of length r and the set of $(k+1)$-cores of size r by sending $w \in W^0$ to the $(k+1)$ core $w\emptyset$ obtained by w acting on the empty core.*

2 Background: Suter symmetry

For a fixed positive integer k, let $R_1 = (1^k), R_2 = (2^{k-1}), \ldots, R_k = (k)$ denote the rectangular partitions which have largest hook length equal to k. Let Y^k denote the (finite) sublattice of Young's lattice which contains everything smaller than $R_1, R_2, \ldots, R_k$, i.e. $Y^k = \{\lambda : \lambda \subset R_i \text{ for some } i\}$.

Suter [Su1] noticed that Y^k had a dihedral symmetry, coming from the usual symmetry of partition transposition, as well as a k-fold rotational symmetry, as pictured in Figure 1.

Suter defined a cyclic action on Y^k of order $k+1$, described on a Young diagram of a partition. We will not present this here; our generalization comes from a different description of this cyclic action which we now introduce.

2.1 Suter symmetry on alcoves

Since every partition in Y^k is a $(k+1)$-core, we can associate each partition $\lambda \in Y^k$ with some affine Grassmannian permutation, or equivalently, to an alcove A_w in the dominant chamber. It was noticed by Suter in [Su2] that all partitions whose hook is smaller than or equal to k are in bijection with the alcoves in the fundamental chamber bounded by $H_{\phi,2}$. The elements of Y^k, viewed as alcoves, now form a 2 fold dilation of the fundamental alcove. The fundamental alcove has a $k+1$ cyclic symmetry (cycling the vertices of the dilated alcove) and so the elements of Y^k also have this symmetry. We will generalize this version of Suter symmetry in Section 4.

3 Combinatorics of k-bounded partitions

Lapointe and Morse [LM2] introduced a bijection between $(k+1)$-cores and k-bounded partitions (a partition is k-bounded if all of it's parts are less than or equal to k). The bijection sends a $(k+1)$-core μ to the k-bounded partition λ whose i^{th} part is equal to the number of cells (i, j) in μ with hook less than $k+1$. For a $(k+1)$-core μ, we let $\mathfrak{p}(\mu)$ denote the corresponding k-bounded partition, and we will let $\mathfrak{c}$ denote the inverse map (so $\mathfrak{c}(\mathfrak{p}(\mu)) = \mu$).

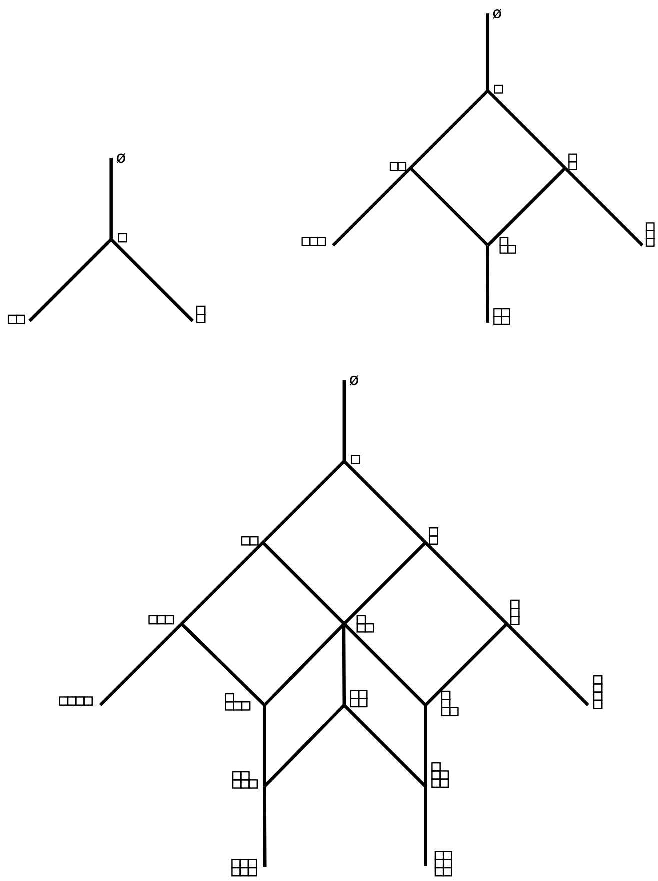

Fig. 1: Three examples of the $k + 1$ dihedral symmetry of Y^k for $k \in \{2, 3, 4\}$.

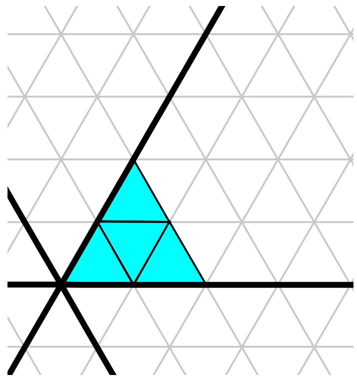

Fig. 2: A dilation of the fundamental alcove of $\tilde{A}_2$ by multiplying the edge lengths by 2. The highlighted cells are in bijection with the partitions $\{\emptyset, (1), (2), (1,1)\}$.

Lapointe, Lascoux and Morse [LLM] introduced a k-version of Young's lattice. It is a sublattice of Young's lattice whose vertices are labeled by k-bounded partitions. It is the lattice generated by the covering relation $\lambda \lessdot \mu$ if $|\lambda| + 1 = |\mu|$ and $s_i \mathfrak{c}(\lambda) = \mathfrak{c}(\mu)$ for some $i = 0, 1, \ldots, k$.

The rectangles $R_1, \ldots, R_k$ described above play an important role in the study of k-Schur functions. k-Schur functions, first introduced by Lapointe, Lascoux and Morse [LLM], were motivated in the study of Macdonald polynomials, but have since appeared in other contexts (see, in particular, [L2, L3, LS, LM3]). Each k-Schur function $s_\lambda^{(k)}$ is indexed by a k-bounded partition λ (or equivalently a $(k+1)$-core, or an affine Grassmannian permutation).

An important open problem in the study of k-Schur functions is to understand their multiplication rule. One special case is very explicitly understood, due to the following theorem of Lapointe and Morse. For two partitions λ and μ, let $\lambda \cup \mu$ denote the partition obtained by combining the parts of λ and μ and placing them into non-increasing order.

Theorem 3.1 (Lapointe, Morse [LM3]) $s_\lambda^{(k)} s_R^{(k)} = s_{\lambda \cup R}^{(k)}$ *for a rectangle* $R = R_1, \ldots, R_k$.

4 Generalized Suter symmetry

We now fix an integer $m > 1$. With Theorem 3.1 in mind, we will study all partitions contained in a product of m rectangles. Let Y_m^k denote the subposet of the k-Young's lattice which contains all partitions contained in a stack of $m-1$ of the k-rectangles (so $\lambda \in Y_m^k$ if $\lambda \subset R_{i_1} \cup R_{i_2} \cup \cdots \cup R_{i_{m-1}}$ for some $i_1, \ldots, i_{m-1}$). By this definition, $Y_2^k = Y^k$ from the beginning of Section 2. As exhibited in Figure 4, the set Y_m^k also has a $k+1$ cyclic symmetry. We will prove this by appealing to the geometric description of Suter symmetry. The collection of alcoves in the dominant chamber which are bounded by the affine hyperplane $H_{\phi,m}$ again inherits the cyclic $k+1$ symmetry of the fundamental alcove, thus proving that a cyclic $k+1$ symmetry exists on the alcoves. It remains to be shown that the alcoves in the dominant chamber bounded by the hyperplane $H_{\phi,m}$ correspond to the partitions which are contained in a product of $m-1$ rectangles. Once we have shown this, our main theorem, that Y_m^k has a cyclic $k+1$ action, will follow.

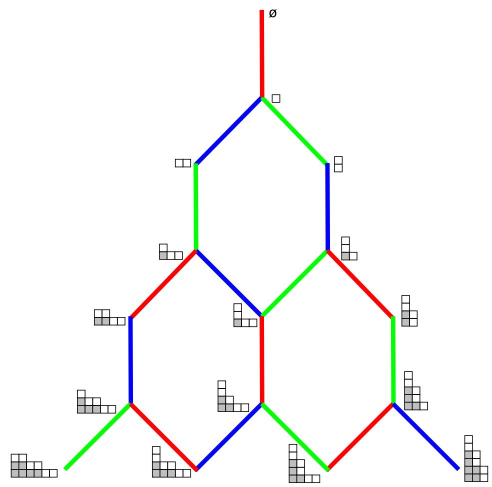

Fig. 3: The poset Y_4^2 labeled by cores which exhibits a dihedral 3-fold symmetry. A reflection in this symmetry is realized through conjugation of the 3-cores. The red indicates the cells added to the core are content $0\ (mod\ 3)$, blue at the cells are content $1\ (mod\ 3)$, green the cells are content $2\ (mod\ 3)$

5 The affine Nil-Coxeter algebra and rectangle k-Schur functions

The affine nilCoxeter algebra $\mathbb{A}$ is the algebra generated by u_i for $i \in \{0, 1, \ldots, k\}$, subject to the relations (see for instance [L1]):

$$u_i^2 = 0 \text{ for } i \in \{0, 1, \ldots, k\}$$
$$u_i u_j = u_j u_i \text{ if } i - j \neq \pm 1$$
$$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} \text{ for } i \in \{0, 1, \ldots, k\}$$

where $i - j$ and $i + 1$ are understood to be taken modulo $k + 1$.

If $s_{i_1} \ldots s_{i_m}$ is a reduced word for an element $w \in W$, we let $\mathbf{u}(w) = u_{i_1} \ldots u_{i_m}$, then $U := \{\mathbf{u}(w) : w \in W\}$ is a basis of $\mathbb{A}$.

The affine nilCoxeter algebra has an action on the free abelian group with basis the $(k+1)$-cores. Let $\nu \in \mathcal{C}^{(k+1)}$ and then define $u_i \nu$ to be the $(k+1)$-core formed by adding all addable boxes of content i if ν has at least one such addable box, and $u_i \nu$ is 0 otherwise.

Within the affine nilCoxeter algebra, Lam [L1] found elements $\mathbf{h}_i$ for $1 \leq i \leq k$ which generate a subalgebra isomorphic to the subring of symmetric functions generated by the complete homogenous symmetric functions $h_1, \ldots, h_k$.

Definition 5.1 *An element $u = u_{i_1} u_{i_2} \cdots u_{i_m} \in U$ is said to be cyclically decreasing if each of $i_1, \ldots, i_m$ are distinct, and whenever $j = i_s$ and $j + 1 = i_t$ then $t < s$ (here $j + 1$ is taken modulo $k + 1$). To a strict subset $D \subset \{0, 1, \ldots, k\}$, we let u_D denote the unique element of U which is cyclically decreasing and is a product of the generators u_m for $m \in D$.*

Example 5.2 *Let $k = 7$ and let $D = \{0, 1, 4, 7\}$. Then $u_D = u_1 u_0 u_7 u_4$.*

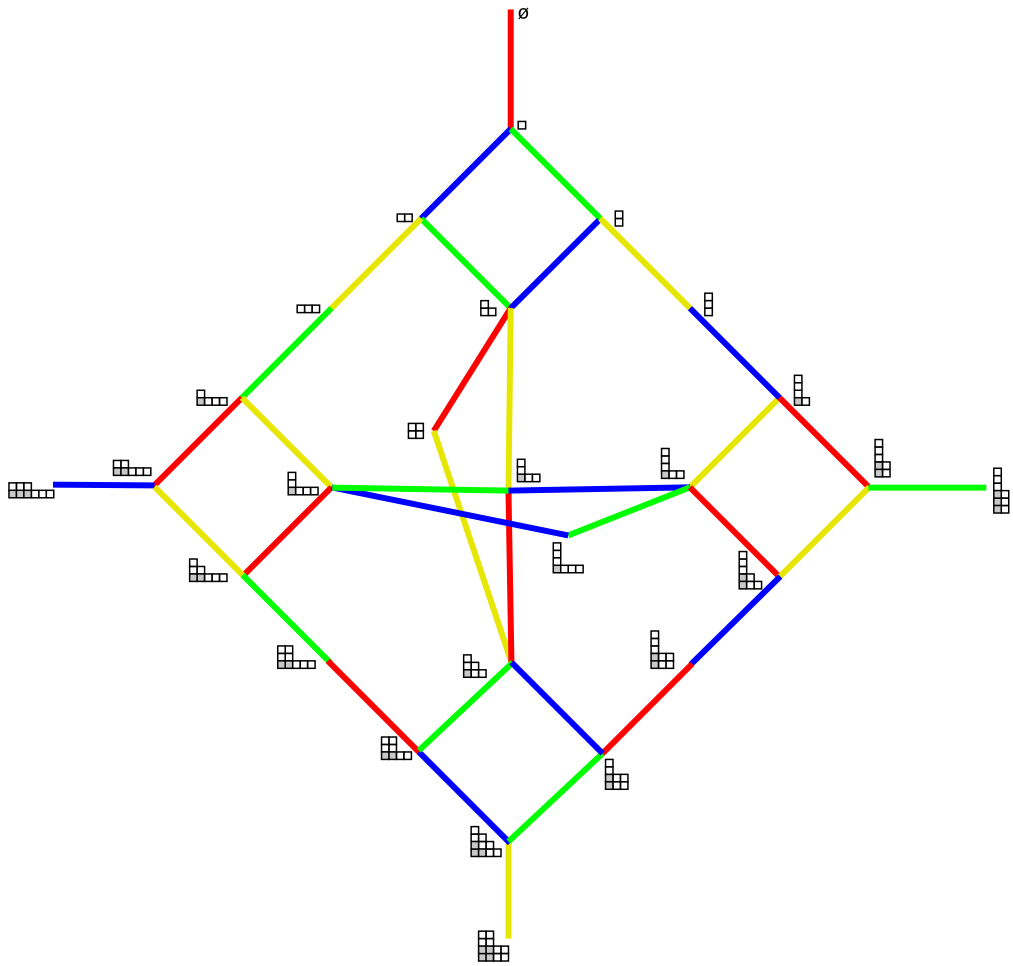

Fig. 4: The poset Y_3^3 exhibits a cyclic 4 symmetry. The vertices are labelled by 4-cores, and corresponding 3-bounded partitions are obtained by deleting shaded boxes and left justifying the partition. The edge colors correspond to the integer modulo 4 of the content of the cells being added; red is 0, blue is 1, yellow is 2 and green is 3.

Lam then defines elements $\mathbf{h}_i := \sum_{|D|=i} u_D \in \mathbb{A}$ for $i \in \{0, 1, \ldots, k\}$.

Theorem 5.3 (Lam [L1] Corollary 14) *The $\mathbf{h}_i$ for $i \in \{1, 2, \ldots k\}$ generate a subalgebra isomorphic to the ring generated by the complete homogeneous symmetric functions h_i for $i \in \{1, 2, \ldots k\}$. The isomorphism identifies $\mathbf{h}_i$ and h_i.*

One can then define the k-Schur functions.

Definition 5.4 *Let λ be a k-bounded partition. Then we define $s_\lambda^{(k)}$ to be the unique elements of the subring generated by the $\mathbf{h}_i$ which satisfy the following rule, known as the k-Pieri rule:*

$$\mathbf{h}_i s_\lambda^{(k)} = \sum_\mu s_\mu^{(k)}; \qquad s_\emptyset^{(k)} = 1.$$

where $\mu = \mathbf{u}(y)\lambda$ and y is a cyclically decreasing element of length i.

Remark 5.5 *In general, expanding $s_\lambda^{(k)} = \sum_w c_w \mathbf{u}(w)$ is an open problem, and has been shown to be equivalent to understanding the structure coefficients of k-Schur functions (called the k-Littlewood Richardson coefficients).*

5.1 Expression of rectangle k-Schur functions as pseudo-translations

In [BBTZ], the authors introduced the notion of a pseudo-translation in order to describe the expansion of k-Schur functions corresponding to $R_1, \ldots, R_k$ in $\mathbb{A}$. Pseudo-translations have since been realized by Lam and Shimozono as being translations of the extended affine Weyl group (see [LS2]).

Definition 5.6 *Let η be a weight. We say $y \in W$ is a pseudo-translation of A_w in direction η if $A_{yw} = A_w + \eta$.*

For a weight γ we let z_γ denote the pseudo-translation of the fundamental alcove $A_\emptyset$ in direction γ.

Theorem 5.7 (Berg, Bergeron, Thomas, Zabrocki [BBTZ]) *Inside $\mathbb{A}$,*

$$s_{R_i}^{(k)} = \sum_{\gamma \in W_0 \Lambda_i} \mathbf{u}(z_\gamma).$$

5.2 Alcoves with a facet on the hyperplane $H_{\phi,m}$

Let R be the k-bounded partition $R_{i_1} \cup \cdots \cup R_{i_{m-1}}$ with $i_j \in \{1, 2, \ldots, k\}$. Then $s_R^{(k)} = s_{R_{i_{m-1}}}^{(k)} \cdots s_{R_{i_1}}^{(k)}$ by Theorem 3.1. By Theorem 5.7, the k-bounded partition R corresponds to the alcove $A_\emptyset + (\Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}})$.

Lemma 5.8 *There are $\binom{m-1+k-1}{k-1}$ distinct k-bounded partitions of the form $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$.*

Proof: The partition R will be the union some number (possibly 0) of each of the different rectangles $R_1, R_2, \ldots, R_k$. Hence the number of such rectangles is the number of ways to pick a set of $m-1$ objects from a set of k elements with repetition. □

Lemma 5.9 *The alcove A corresponding to the k-bounded partition $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$ shares a facet with the wall $H_{\phi,m}$.*

Proof: The fundamental alcove $A_\emptyset$ shares a facet with the hyperplane $H_{\phi,1}$. The fundamental weights Λ_i all satisfy $\langle \Lambda_i, \phi\rangle = 1$ and are the coordinates of the vertices of this facet. Since A is a translate of the fundamental alcove, $A = A_\emptyset + (\Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}})$, the vertices of A which are not translates of the origin will have weight $v_d = \Lambda_d + \Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}}$. They will satisfy

$$\langle v_d, \phi\rangle = 1 + \sum_{j=1}^{m-1} \langle \Lambda_{i_j}, \phi\rangle = m,$$

and so will lie on the wall $H_{\phi,m}$. □

Lemma 5.10 *The number of vertices on the hyperplane $H_{\phi,m-1}$ which are in the fundamental chamber is $\binom{m-1+k-1}{k-1}$. There is a bijection between the alcoves corresponding to products of rectangles and these vertices; we identify an alcove with its unique vertex on $H_{\phi,m-1}$.*

Proof: Each vertex on $H_{\phi,m-1}$ has the form $\sum_i a_i\Lambda_i$ with a_i all non-negative integers and $\sum_i a_i = m-1$.

We conclude then that the vertices are then in bijection with non-negative integer solutions ($a_i \geq 0$) to the equation $\sum_{i=1}^{k} a_i = m-1$ and this is well known to be $\binom{m-1+k-1}{k-1}$.

For the last statement it is sufficient to remark that each alcove corresponding to partition $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$ contains the vertex $\Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}}$ which lies on $H_{\phi,m-1}$ and by Lemma 5.8 these sets have the same number of elements. □

Lemma 5.11 *Let $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$ and let $\mathcal{R}$ be the $(k+1)$-core which corresponds to the k-bounded partition R. Then $\mathcal{R}$ has only one addable residue, that is there exists a unique i for which $u_i\mathcal{R} \neq 0$.*

Proof: The only residue which is addable is $i = i_1 + \cdots + i_{m-1}$. The core $\mathcal{R}$ is obtained by appending rectangles ordered by their widths in a skew fashion, stacking the rectangles so that adjacent rectangles share neither row nor column. Cells which are on the opposite sides of a rectangle in the core will have the same residue because they are separated by a hook of k therefore only one residue is addable. The length of the first row of $\mathcal{R}$ will be $i = i_1 + \cdots + i_{m-1}$ and so it is also the residue of the addable corner. □

Corollary 5.12 *Let λ be a k-bounded partition and suppose that λ corresponds to an alcove A_w in the fundamental chamber which is bounded by $H_{\phi,m}$. Then there exists an $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$ such that $\lambda \subset R$.*

Proof: The proof is by induction on m. When $m = 1$ the statement is trivial; the only dominant alcove bounded by $H_{\phi,1}$ is the fundamental alcove, which corresponds to the empty partition $\emptyset$, which is contained in an empty product of rectangles.

Now we fix m. If A_w is bounded by $H_{\phi,m-1}$ then the statement follows by induction; we know that there is an $R' = R_{i_1} \cup R_{i_2} \cup \cdots \cup R_{i_{m-2}}$ with $\lambda \subset R'$. Therefore $\lambda \subset R' \cup R_{i_{m-1}}$ for any other $i_{m-1} \in \{1, 2, \ldots, k\}$.

Now we may assume that A_w is between $H_{\phi,m-1}$ and $H_{\phi,m}$. Therefore A_w has at least one vertex on $H_{\phi,m-1}$. Let such a vertex be $\Lambda = \sum_{j=1}^{m-1} \Lambda_{i_j}$. Let $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$ as in Lemma 5.10. We claim that $\lambda \subset R$.

Let B denote the alcove corresponding to R. By Lemma 5.11, B has a unique addable residue, which we shall denote r. This residue corresponds to crossing the hyperplane $H_{\phi,m}$, since crossing the hyperplane will increase the length of the corresponding core and we know there is only one reflection which will add box to R, by Lemma 5.11. Applications of all other generators s_i for $i \neq r$ must therefore decrease the size of the partition. Since B shares a vertex with A_w, there is an element $s_{a_1} s_{a_2} \cdots s_{a_x}$ of W_r which takes A_w to B (i.e. $A_{s_{a_1} s_{a_2} \cdots s_{a_x} w} = B$ for some $a_j \neq r$). Therefore $\lambda \subset R$, since $\lambda = s_{a_x} \cdots s_{a_1} R$. □ As a consequence of Corollary 5.12 we have the following results.

Theorem 5.13 *The set Y_m^k has a cyclic $k+1$ action.*

Proof: By Corollary 5.12, Y_m^k corresponds precisely with alcoves in the dominant chamber which are bounded by $H_{\phi,m}$. The region in the dominant chamber bounded by $H_{\phi,m}$ has the same shape as the fundamental alcove; the lengths of the edges of the fundamental alcove have been multiplied by m in $H_{\phi,m}$. Since the fundamental alcove has a cyclic $k+1$ action which is inherited from the affine Dynkin diagram, the collection of alcoves in this region inherits the cyclic $k+1$ action. □

As a corollary we also have as a consequence an enumeration of the elements in Y_m^k.

Proposition 5.14 *The number of partitions in Y_m^k is m^k.*

Proof: As noted in the previous result, Y_m^k is in bijection with the alcoves which lie inside of an m-dilation of the fundamental alcove. In a k dimensional space the volume of a region dilated by m on a side will be m^k times the original, hence there are m^k alcoves within this region. □ Others (e.g. Sommers [Som, Theorem 5.7]) have considered this dilated alcove for reasons other than the connection with k-bounded partitions and $(k+1)$-cores and so this lattice may have unexpected algebraic applications.

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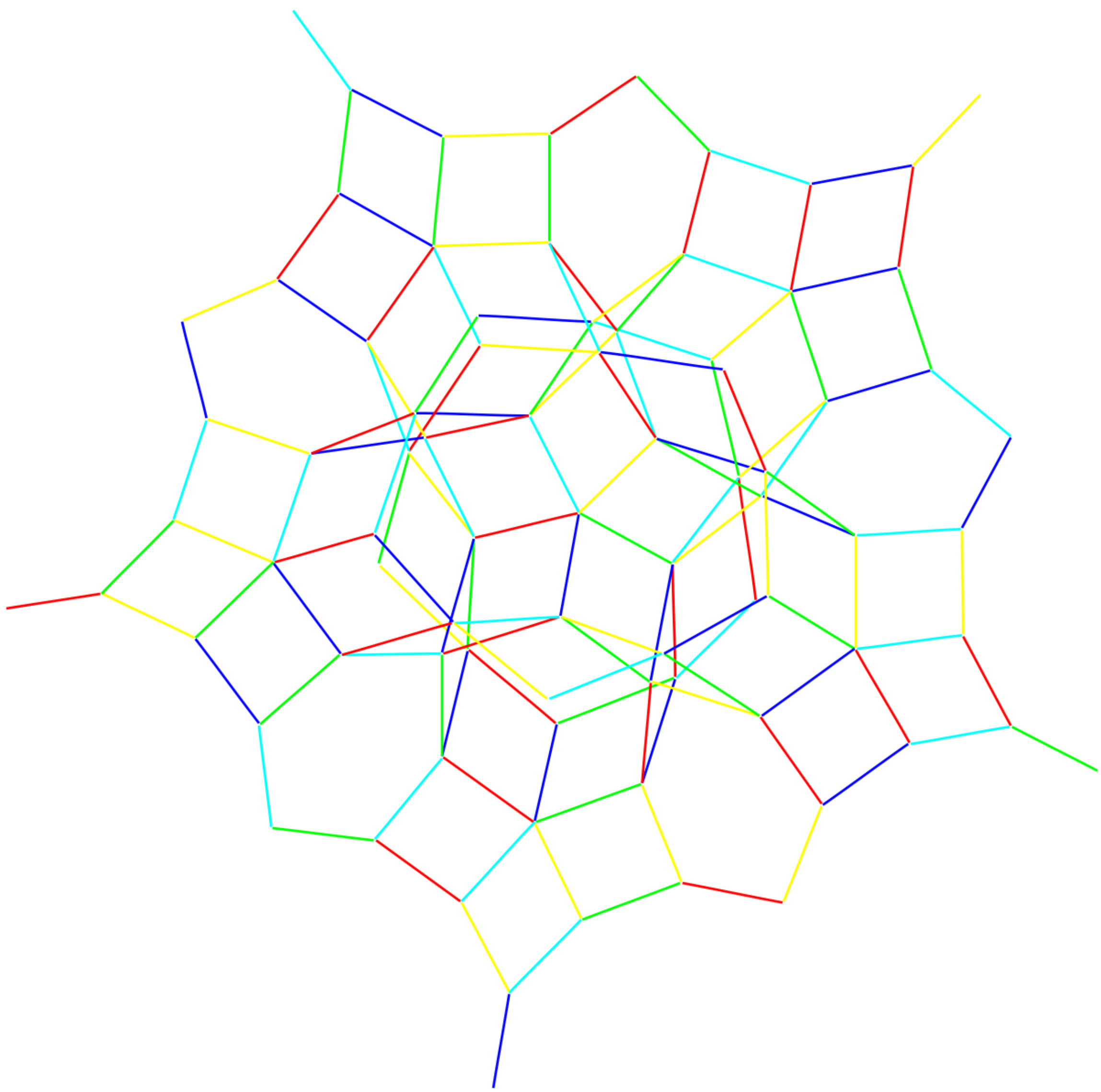

Fig. 5: Suter symmetry of type $k = 4$ and $m = 3$

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