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Symmetries of the $k$-bounded partition lattice

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Abstract. We generalize the symmetry on Young’s lattice, found by Suter, to a symmetry on the $k$-bounded partition lattice of Lapointe, Lascoux and Morse.

Résumé. Nous généralisons la symétrie sur le treillis de Young, découvert par Suter, à une symétrie sur le treillis des partages bornés par $k$ et étudié par Lapointe, Lascoux and Morse.

Keywords: core partitions, $k$-Schur functions, cyclic symmetry

1 Introduction

In [Su1], Suter found a dihedral symmetry which exists in Young’s lattice, by taking all partitions whose bounding rectangle is contained within the staircase $(k, k - 1, k - 2, \ldots, 2, 1)$. He recognized that these partitions would have the same symmetries as the affine Dynkin diagram of type $A_k$.

While studying $k$-Schur functions, we noticed that the rectangles which Suter uses are the same rectangles that appear in Morse and Lapointe’s paper [LM3]. The rectangles in this picture correspond to special elements of the homology of the affine Grassmannian [L1, L2, L3]. For this reason, the lattice of $k$-bounded partitions related to the algebra of $k$-Schur functions is a natural place to view a generalization of the symmetry observed by Suter.

Recent results of Berg, Bergeron, Thomas and Zabrocki [BBTZ] developed some geometric properties of the affine hyperplane arrangement. We use this geometric picture to generalize the symmetry that Suter found to the $k$-bounded partition lattice of Lapointe, Lascoux and Morse [LLM]. We do this by recognizing that the $k$-bounded partitions which are contained in a concatenation of $m$ rectangles with a $k$ hook is isomorphic to an $m$-dilation in the geometric picture.

Recently, Nathan Williams [W] has identified an isomorphism between the geometric picture presented here and the set of words of length $k + 1$ on $\{0, 1, 2, \ldots, m\}$ which sum to $0 \ (mod \ m + 1)$ and a cyclic group action on these words.
1.1 From root systems in type $A_k$ to the the affine Grassmannian

Let $\alpha_1, \ldots, \alpha_k$ denote the simple roots of type $A_k$, which form a basis for a vector space $V$. $V$ has a symmetric bilinear form given by:

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j + 1, \\ 0 & \text{else.} \end{cases}$$

and we let $\{\Lambda_i\}_{1 \leq i \leq k}$ denote the basis dual to $\{\alpha_i\}_{1 \leq i \leq k}$ under this bilinear form. The $\mathbb{Z}$ span of the $\{\Lambda_i\}_{1 \leq i \leq k}$ will be called the weights.

For $v \in V$, we let $H_v$ denote the hyperplane through the origin, perpendicular to $v$. We write $H_i$ for $H_{\alpha_i}$ and $H_{\alpha_i, p}$ for the points $x$ satisfying $\langle x, v \rangle = p$.

Let $s_i$ represent the reflection of a vector $v$ through the hyperplane $H_i$ so that the set of reflections $s_1, \ldots, s_k$ corresponding to the roots $\alpha_1, \ldots, \alpha_k$ generate a reflection group $W_0$ which is isomorphic to the symmetric group $S_{k+1}$. The corresponding (finite) root system is $\Phi_0$ is the closure of the set of vectors $\{\alpha_i\}_{1 \leq i \leq k}$ under the action of $W_0$. The element $\phi = \alpha_1 + \cdots + \alpha_k$ is known as the highest root of the root system.

The affine arrangement is the collection of all hyperplanes $H_{\alpha, p}$ for $\alpha \in \Phi_0$ and $p \in \mathbb{Z}$.

The dominant chamber is the (closed) collection of points in $V$ which are bounded by the hyperplanes $H_{\alpha, 0}$. We denote it by $C$. A weight is called dominant if it lies in the dominant chamber.

The fundamental alcove is bounded by the walls of the dominant chamber, together with the hyperplane $H_{\phi, 1}$. We denote it by $A_0$.

The affine reflection group, $W$, has an additional generator $s_0$, which acts as reflection in $H_{\phi, 1}$. The generators $s_0, s_1, \ldots, s_k$ satisfy the affine type A Coxeter relations:

$$s_i^2 = 1 \text{ for } i \in \{0, 1, \ldots, k\}$$

$$s_is_j = s_js_i \text{ if } i - j \neq \pm 1$$

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \text{ for } i \in \{0, 1, \ldots, k\}$$

where $i - j$ and $i + 1$ are understood to be taken modulo $k + 1$.

There is an action of $W$ on $V$ defined by $s_i$ reflecting across the hyperplane $H_i$ for $i \in \{1, 2, \ldots, k\}$ and $s_0$ reflecting across the hyperplane $H_{\phi, 1}$.

We let $A_w := w^{-1}A_0$. The collection of $A_w$ are called the alcoves of the affine arrangement. The hyperplanes $H_{\alpha_i, n}$ will intersect with $A_w$ either in the empty set, at a single weight, or in a facet of the alcove (the convex hull of $k$ of the vertices of $A_w$). An alcove $A_w \subset C$ if and only if $w$ is a minimal length coset representative of $W/W_0$. The set of minimal length coset representatives is denoted $W^0$. A permutation $w \in W^0$ is called an affine Grassmannian permutation.

**Example 1.1** Let $k = 4$. Then $s_4s_1s_0 = s_1s_4s_0$ is affine Grassmannian because all its reduced words end in $s_0$, but $s_0s_1s_0 = s_1s_0s_1$ is not.

A partition $\lambda$ is called a $(k + 1)$-core if $\lambda$ has no removable $(k + 1)$-rim hook. Define the size of a $(k + 1)$-core, $|\lambda|$, to be the number of cells $(i, j)$ with hook smaller than $k + 1$ where the hook of a cell is $\lambda_i + \lambda_j' - i - j + 1$. Let $c^{(k+1)}$ denote the set of all $(k + 1)$-cores.

**Example 1.2** Let $k = 3$ and $\lambda = (4, 2, 2)$. Then $\lambda$ has no removable $4$-rim hooks and the size of $\lambda$ is 6.
W has an action on $C^{(k+1)}$. Let the content of a cell $(i,j)$ in the Young diagram of $\lambda$ be the integer $j - i \mod k + 1$. If $\lambda$ is a $(k+1)$-core then $s_i \lambda$ is $\lambda$ union all addable cells of content $i$, if $\lambda$ has such an addable cell, $s_i \lambda$ is $\lambda$ minus all removable boxes of content $i$ from $\lambda$ if $\lambda$ has such a removable box (a $(k+1)$-core cannot have both a removable box and an addable position of the same content), and $s_i \lambda = \lambda$ otherwise.

Example 1.3 Let $k = 3$ and $\lambda = (4, 2, 2)$ as above. Then $s_1 \lambda = (4, 3, 2, 1)$ and $s_3 \lambda = (3, 2, 1)$.

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Proposition 1.4 [Lascoux] There is a bijection between affine Grassmannian permutations of length $r$ and the set of $(k+1)$-cores of size $r$ by sending $w \in W$ to the $(k+1)$ core $w(\emptyset)$ obtained by $w$ acting on the empty core.

2 Background: Suter symmetry

For a fixed positive integer $k$, let $R_1 = (1^k), R_2 = (2^{k-1}), \ldots, R_k = (k)$ denote the rectangular partitions which have largest hook length equal to $k$. Let $Y^k$ denote the (finite) sublattice of Young’s lattice which contains everything smaller than $R_1, R_2, \ldots, R_k$, i.e. $Y^k = \{ \lambda : \lambda \subset R_i \text{ for some } i \}$.

Suter [Su1] noticed that $Y^k$ had a dihedral symmetry, coming from the usual symmetry of partition transposition, as well as a $k$-fold rotational symmetry, as pictured in Figure 1.

Suter defined a cyclic action on $Y^k$ of order $k + 1$, described on a Young diagram of a partition. We will not present this here; our generalization comes from a different description of this cyclic action which we now introduce.

2.1 Suter symmetry on alcoves

Since every partition in $Y^k$ is a $(k+1)$-core, we can associate each partition $\lambda \in Y^k$ with some affine Grassmannian permutation, or equivalently, to an alcove $A_w$ in the dominant chamber. It was noticed by Suter in [Su2] that all partitions whose hook is smaller than or equal to $k$ are in bijection with the alcoves in the fundamental chamber bounded by $H_{\phi,2}$. The elements of $Y^k$, viewed as alcoves, now form a 2 fold dilation of the fundamental alcove. The fundamental alcove has a $k+1$ cyclic symmetry (cycling the vertices of the dilated alcove) and so the elements of $Y^k$ also have this symmetry. We will generalize this version of Suter symmetry in Section 4.

3 Combinatorics of $k$-bounded partitions

Lapointe and Morse [LM2] introduced a bijection between $(k+1)$-cores and $k$-bounded partitions (a partition is $k$-bounded if all of its parts are less than or equal to $k$). The bijection sends a $(k+1)$-core $\mu$ to the $k$-bounded partition $\lambda$ whose $i^{th}$ part is equal to the number of cells $(i,j)$ in $\mu$ with hook less than $k + 1$. For a $(k+1)$-core $\mu$, we let $p(\mu)$ denote the corresponding $k$-bounded partition, and we will let $c$ denote the inverse map (so $c(p(\mu)) = \mu$).
Fig. 1: Three examples of the $k + 1$ dihedral symmetry of $\gamma^k$ for $k \in \{2, 3, 4\}$. 
Lapointe, Lascoux and Morse [LLM] introduced a $k$-version of Young’s lattice. It is a sublattice of Young’s lattice whose vertices are labeled by $k$-bounded partitions. It is the lattice generated by the covering relation $\lambda \lessdot \mu$ if $|\lambda| + 1 = |\mu|$ and $s_i c(\lambda) = c(\mu)$ for some $i = 0, 1, \ldots, k$.

The rectangles $R_1, \ldots, R_k$ described above play an important role in the study of $k$-Schur functions. $k$-Schur functions, first introduced by Lapointe, Lascoux and Morse [LLM], were motivated in the study of Macdonald polynomials, but have since appeared in other contexts (see, in particular, [L2, L3, LS, LM3]). Each $k$-Schur function $s^{(k)}_\lambda$ is indexed by a $k$-bounded partition $\lambda$ (or equivalently a $(k+1)$-core, or an affine Grassmannian permutation).

An important open problem in the study of $k$-Schur functions is to understand their multiplication rule. One special case is very explicitly understood, due to the following theorem of Lapointe and Morse.

**Theorem 3.1 (Lapointe, Morse [LM3])**

$$s^{(k)}_\lambda s^{(k)}_R = s^{(k)}_{\lambda \cup R}$$ for a rectangle $R = R_1, \ldots, R_k$.

## 4 Generalized Suter symmetry

We now fix an integer $m > 1$. With Theorem 3.1 in mind, we will study all partitions contained in a product of $m$ rectangles. Let $Y_m$ denote the subposet of the $k$-Young’s lattice which contains all partitions contained in a stack of $m-1$ of the $k$-rectangles (so $\lambda \in Y_m$ if $\lambda \subset R_{i_1} \cup R_{i_2} \cup \cdots \cup R_{i_{m-1}}$ for some $i_1, \ldots, i_{m-1}$). By this definition, $Y_m^k = Y^k$ from the beginning of Section 2. As exhibited in Figure 4 the set $Y_m^k$ also has a $k+1$ cyclic symmetry. We will prove this by appealing to the geometric description of Suter symmetry. The collection of alcoves in the dominant chamber which are bounded by the affine hyperplane $H_{\phi,m}$ again inherits the cyclic $k+1$ symmetry of the fundamental alcove, thus proving that a cyclic $k+1$ symmetry exists on the alcoves. We remains to be shown that the alcoves in the dominant chamber bounded by the hyperplane $H_{\phi,m}$ correspond to the partitions which are contained in a product of $m-1$ rectangles. Once we have shown this, our main theorem, that $Y_m^k$ has a cyclic $k+1$ action, will follow.
Fig. 3: The poset $Y_2^4$ labeled by cores which exhibits a dihedral 3-fold symmetry. A reflection in this symmetry is realized through conjugation of the 3-cores. The red indicates the cells added to the core are content 0 ($\mod 3$), blue at the cells are content 1 ($\mod 3$), green the cells are content 2 ($\mod 3$).

5 The affine Nil-Coxeter algebra and rectangle $k$-Schur functions

The affine nilCoxeter algebra $A$ is the algebra generated by $u_i$ for $i \in \{0, 1, \ldots, k\}$, subject to the relations (see for instance [L1]):

$$u_i^2 = 0 \text{ for } i \in \{0, 1, \ldots, k\}$$

$$u_iu_j = u_ju_i \text{ if } i - j \neq \pm 1$$

$$u_iu_{i+1}u_i = u_{i+1}u_iu_{i+1} \text{ for } i \in \{0, 1, \ldots, k\}$$

where $i - j$ and $i + 1$ are understood to be taken modulo $k + 1$.

If $s_{i_1} \cdots s_{i_m}$ is a reduced word for an element $w \in W$, we let $u(w) = u_{i_1} \cdots u_{i_m}$, then $U := \{u(w) : w \in W\}$ is a basis of $A$.

The affine nilCoxeter algebra has an action on the free abelian group with basis the $(k+1)$-cores. Let $\nu \in C^{(k+1)}$ and then define $u_{i,\nu}$ to be the $(k+1)$-core formed by adding all addable boxes of content $i$ if $\nu$ has at least one such addable box, and $u_{i,\nu}$ is 0 otherwise.

Within the affine nilCoxeter algebra, Lam [L1] found elements $h_i$ for $1 \leq i \leq k$ which generate a subalgebra isomorphic to the subring of symmetric functions generated by the complete homogenous symmetric functions $h_1, \ldots, h_k$.

**Definition 5.1** An element $u = u_{i_1}u_{i_2} \cdots u_{i_m} \in U$ is said to be cyclically decreasing if each of $i_1, \ldots, i_m$ are distinct, and whenever $j = i_s$ and $j + 1 = i_t$ then $t < s$ (here $j + 1$ is taken modulo $k + 1$). To a strict subset $D \subset \{0, 1, \ldots, k\}$, we let $u_D$ denote the unique element of $U$ which is cyclically decreasing and is a product of the generators $u_m$ for $m \in D$.

**Example 5.2** Let $k = 7$ and let $D = \{0, 1, 4, 7\}$. Then $u_D = u_1u_0u_7u_4$. 


Fig. 4: The poset $Y^3_3$ exhibits a cyclic 4 symmetry. The vertices are labelled by 4-cores, and corresponding 3-bounded partitions are obtained by deleting shaded boxes and left justifying the partition. The edge colors correspond to the integer modulo 4 of the content of the cells being added; red is 0, blue is 1, yellow is 2 and green is 3.
Lam then defines elements $h_i := \sum_{|D|=i} u_D \in A$ for $i \in \{0, 1, \ldots, k\}$.

**Theorem 5.3** (Lam [L1] Corollary 14) The $h_i$ for $i \in \{1, 2, \ldots, k\}$ generate a subalgebra isomorphic to the ring generated by the complete homogeneous symmetric functions $h_i$ for $i \in \{1, 2, \ldots, k\}$. The isomorphism identifies $h_i$ and $h_i$.

One can then define the $k$-Schur functions.

**Definition 5.4** Let $\lambda$ be a $k$-bounded partition. Then we define $s_\lambda^{(k)}$ to be the unique elements of the subring generated by the $h_i$ which satisfy the following rule, known as the $k$-Pieri rule:

$$h_i s_\lambda^{(k)} = \sum_{\mu} s_\mu^{(k)} s_\emptyset^{(k)} = 1.$$ 

where $\mu = u(y)\lambda$ and $y$ is a cyclically decreasing element of length $i$.

**Remark 5.5** In general, expanding $s_\lambda^{(k)} = \sum_w c_w u(w)$ is an open problem, and has been shown to be equivalent to understanding the structure coefficients of $k$-Schur functions (called the $k$-Littlewood Richardson coefficients).

### 5.1 Expression of rectangle $k$-Schur functions as pseudo-translations

In [BBTZ], the authors introduced the notion of a pseudo-translation in order to describe the expansion of $k$-Schur functions corresponding to $R_1, \ldots, R_k$ in $A$. Pseudo-translations have since been realized by Lam and Shimozono as being translations of the extended affine Weyl group (see [LS2]).

**Definition 5.6** Let $\eta$ be a weight. We say $y \in W$ is a pseudo-translation of $A_w$ in direction $\eta$ if $A_w y = A_w + \eta$.

For a weight $\gamma$ we let $z_\gamma$ denote the pseudo-translation of the fundamental alcove $A_\emptyset$ in direction $\gamma$.

**Theorem 5.7** (Berg, Bergeron, Thomas, Zabrocki [BBTZ]) Inside $A$,

$$s_{R_i}^{(k)} = \sum_{\gamma \in W_{A_i}} u(z_\gamma).$$

### 5.2 Alcoves with a facet on the hyperplane $H_{\phi,m}$

Let $R$ be the $k$-bounded partition $R_1 \cup \cdots \cup R_{m-1}$ with $i_j \in \{1, 2, \ldots, k\}$. Then $s_R^{(k)} = s_R^{(k)} \cdots s_R^{(k)}$ by Theorem 3.1. By Theorem 5.7, the $k$-bounded partition $R$ corresponds to the alcove $A_\emptyset + (\Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}})$.

**Lemma 5.8** There are $\binom{m-1+k-1}{k-1}$ distinct $k$-bounded partitions of the form $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$.

**Proof:** The partition $R$ will be the union some number (possibly 0) of each of the different rectangles $R_1, R_2, \ldots, R_k$. Hence the number of such rectangles is the number of ways to pick a set of $m - 1$ objects from a set of $k$ elements with repetition. \(\square\)

**Lemma 5.9** The alcove $A$ corresponding to the $k$-bounded partition $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$ shares a facet with the wall $H_{\phi,m}$. 


**Proof:** The fundamental alcove $A_\emptyset$ shares a facet with the hyperplane $H_{\emptyset,1}$. The fundamental weights $\Lambda_i$ all satisfy $\langle \Lambda_i, \phi \rangle = 1$ and are the coordinates of the vertices of this facet. Since $A$ is a translate of the fundamental alcove, $A = A_\emptyset + (\Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}})$, the vertices of $A$ which are not translates of the origin will have weight $v_d = \Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}}$. They will satisfy

$$\langle v_d, \phi \rangle = 1 + \sum_{j=1}^{m-1} \langle \Lambda_{i_j}, \phi \rangle = m,$$

and so will lie on the wall $H_{\emptyset,m}$. □

**Lemma 5.10** The number of vertices on the hyperplane $H_{\emptyset,m-1}$ which are in the fundamental chamber is $\binom{m-1+k-1}{k-1}$. There is a bijection between the alcoves corresponding to products of rectangles and these vertices; we identify an alcove with its unique vertex on $H_{\emptyset,m-1}$.

**Proof:** Each vertex on $H_{\emptyset,m-1}$ has the form $\sum_i a_i \Lambda_i$ with $a_i$ all non-negative integers and $\sum_i a_i = m - 1$.

We conclude that the vertices are in bijection with non-negative integer solutions $(a_i \geq 0)$ to the equation $\sum_{i=1}^{k} a_i = m - 1$ and this is well known to be $\binom{m-1+k-1}{k-1}$.

For the last statement it is sufficient to remark that each alcove corresponding to partition $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$ contains the vertex $\Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}}$ which lies on $H_{\emptyset,m-1}$ and by Lemma 5.8 these sets have the same number of elements. □

**Lemma 5.11** Let $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$ and let $\mathcal{R}$ be the $(k + 1)$-core which corresponds to the $k$-bounded partition $R$. Then $\mathcal{R}$ has only one addable residue, that is there exists a unique $i$ for which $w_i \mathcal{R} \neq 0$.

**Proof:** The only residue which is addable is $i = i_1 + \cdots + i_{m-1}$. The core $\mathcal{R}$ is obtained by appending rectangles ordered by their widths in a skew fashion, stacking the rectangles so that adjacent rectangles share neither row nor column. Cells which are on the opposite sides of a rectangle in the core will have the same residue because they are separated by a hook of $k$ therefore only one residue is addable. The length of the first row of $\mathcal{R}$ will be $i = i_1 + \cdots + i_{m-1}$ and so it is also the residue of the addable corner. □

**Corollary 5.12** Let $\lambda$ be a $k$-bounded partition and suppose that $\lambda$ corresponds to an alcove $A_\lambda$ in the fundamental chamber which is bounded by $H_{\emptyset,m}$. Then there exists an $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$ such that $\lambda \subset R$.

**Proof:** The proof is by induction on $m$. When $m = 1$ the statement is trivial; the only dominant alcove bounded by $H_{\emptyset,1}$ is the fundamental alcove, which corresponds to the empty partition $\emptyset$, which is contained in an empty product of rectangles.

Now we fix $m$. If $A_\lambda$ is bounded by $H_{\emptyset,m-1}$ then the statement follows by induction; we know that there is an $R' = R_{i_1} \cup R_{i_2} \cup \cdots \cup R_{i_{m-2}}$ with $\lambda \subset R'$. Therefore $\lambda \subset R' \cup R_{i_{m-1}}$ for any other $i_{m-1} \in \{1, 2, \ldots, k\}$.
Now we may assume that $A_w$ is between $H_{\phi,m-1}$ and $H_{\phi,m}$. Therefore $A_w$ has at least one vertex on $H_{\phi,m-1}$. Let such a vertex be $\Lambda = \sum_{j=1}^{m-1} \Lambda_{ij}$. Let $R = R_{a_1} \cup \cdots \cup R_{a_{m-1}}$ as in Lemma 5.10. We claim that $\lambda \subset \mathcal{R}$.

Let $B$ denote the alcove corresponding to $R$. By Lemma 5.11, $B$ has a unique addable residue, which we shall denote $r$. This residue corresponds to crossing the hyperplane $H_{\phi,m}$, since crossing the hyperplane will increase the length of the corresponding core and we know there is only one reflection which will add box to $R$, by Lemma 5.11. Applications of all other generators $s_i$ for $i \neq r$ must therefore decrease the size of the partition. Since $B$ shares a vertex with $A_w$, there is an element $s_{a_1}s_{a_2}\cdots s_{a_x}$ of $W$, which takes $A_w$ to $B$ (i.e. $A_{s_{a_1}s_{a_2}\cdots s_{a_x}w} = B$ for some $a_j \neq r$). Therefore $\lambda \subset \mathcal{R}$, since $\lambda = s_{a_x}\cdots s_{a_1}R$.

As a consequence of Corollary 5.12 we have the following results.

**Theorem 5.13** The set $Y_{km}^k$ has a cyclic $k + 1$ action.

**Proof:** By Corollary 5.12, $Y_{km}^k$ corresponds precisely with alcoves in the dominant chamber which are bounded by $H_{\phi,m}$. The region in the dominant chamber bounded by $H_{\phi,m}$ has the same shape as the fundamental alcove; the lengths of the edges of the fundamental alcove have been multiplied by $m$ in $H_{\phi,m}$. Since the fundamental alcove has a cyclic $k + 1$ action which is inherited from the affine Dynkin diagram, the collection of alcoves in this region inherits the cyclic $k + 1$ action.

As a corollary we also have as a consequence an enumeration of the elements in $Y_{km}^k$.

**Proposition 5.14** The number of partitions in $Y_{km}^k$ is $m^k$.

**Proof:** As noted in the previous result, $Y_{km}^k$ is in bijection with the alcoves which lie inside of an $m$-dilation of the fundamental alcove. In a $k$ dimensional space the volume of a region dilated by $m$ on a side will be $m^k$ times the original, hence there are $m^k$ alcoves within this region.

□ As a consequence of Corollary 5.12 we have the following results.

Others (e.g. Sommers [Som]) have considered this dilated alcove for reasons other than the connection with $k$-bounded partitions and $(k + 1)$-cores and so this lattice may have unexpected algebraic applications.

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Fig. 5: Suter symmetry of type $k = 4$ and $m = 3$
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