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Bases for modules of differential operators of order 2 on the classical Coxeter arrangements

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Abstract. It is well-known that the derivation modules of Coxeter arrangements are free. Holm began to study the freeness of modules of differential operators on hyperplane arrangements. In this paper, we study the cases of the Coxter arrangements of type A, B and D. In this case, we prove that the modules of differential operators of order 2 are free. We give examples of all the 3-dimensional classical Coxeter arrangements. Two keys for the proof are "Cauchy–Sylvester's theorem on compound determinants" and "Saito–Holm's criterion".

Résumé. Il est connu que les modules de la dérivation d'arrangements de Coxeter sont libres. Holm a commencé à étudier les modules libres des opérateurs différentiels sur des compositions d'hyperplans. Dans cet article, nous étudions les cas des compositions de Coxter les types A, B et D. Dans ce cas, nous prouvons que les modules d'opérateurs différentiels d'ordre 2 sont libres. Nous donnons des exemples de toutes les compositions de Coxeter classiques de dimension 3. Les deux points clefs pour la preuve sont le théorème de Cauchy–Sylvester sur déterminants composés et le critère de Saito–Holm.

Keywords: Coxeter arrangement, Cauchy–Sylvester's compound determinants, Schur functions

1 Introduction

This article is an extended abstract of (N). Details of proofs are omitted.

Let K be a field of characteristic zero, and let V be an ℓ-dimensional vector space over K. Let $\{x_1, \dots, x_\ell\}$ be a basis for the dual space V^*, and let $S := \mathrm{Sym}(V^*) \simeq K[x_1, \dots, x_\ell]$ be the polynomial ring. A central (hyperplane) arrangement $\mathscr{A}$ is a finite collectin of affine hyperplanes which contain the origin in V. For each hyperplane $H \in \mathscr{A}$ fix a linear form $p_H \in V^*$ such that $\ker(p_H) = H$, and put $Q(\mathscr{A}) := \prod_{H \in \mathscr{A}} p_H$. We call $Q(\mathscr{A})$ a defining polynomial of $\mathscr{A}$. The study of hyperplane arrangement has been depeloped by many reaserchers. In particular, the study of the freeness of the module of $\mathscr{A}$-delevations is one of the most important study of hyperplane arrangements. Recently, Holm began to study the module of $\mathscr{A}$-differential operators.

Let $D^{(m)}(S) := \bigoplus_{|\boldsymbol{\alpha}|=m} S\partial^{\boldsymbol{\alpha}}$ be the module of differential operators (of order m) of S, where $\boldsymbol{\alpha} \in \mathbb{N}^\ell$ is a multi-index. A nonzero element $\theta = \sum_{|\boldsymbol{\alpha}|=m} f_{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}} \in D^{(m)}(S)$ is homogeneous of degree i if $f_{\boldsymbol{\alpha}}$ is zero or homogeneous of degree i for each $\boldsymbol{\alpha}$. In this case, we write $\deg(\theta) = i$. For a multi-index $\boldsymbol{\alpha}$, we put

$$x_{\boldsymbol{\alpha}} := (x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_\ell, \ldots, x_\ell),$$

where the number of x_i is α_i. We define the module $D^{(m)}(\mathscr{A})$ of $\mathscr{A}$-differential operators of order m as follows:

$$D^{(m)}(\mathscr{A}) := \left\{ \theta \in D^{(m)}(S) \mid \theta(Q(\mathscr{A})S) \subseteq Q(\mathscr{A})S \right\}.$$

In the case $m = 1$, $D^{(1)}(\mathscr{A})$ is the module of $\mathscr{A}$-derivations. We say $\mathscr{A}$ to be free if $D^{(1)}(\mathscr{A})$ is a free S-module.

Holm generalized some elementally propaties which hold in the case $m = 1$ into the case order m general. For example, the module of $\mathscr{A}$-differential operators is the intersection of the modules of differential operators which preserving the ideal generated by p_H:

$$D^{(m)}(\mathscr{A}) = \bigcap_{H \in \mathscr{A}} D^{(m)}(p_H S),$$

where $D^{(m)}(p_H S) = \left\{ \theta \in D^{(m)}(S) \mid \theta(p_H x^{\boldsymbol{\alpha}}) \in p_H S \text{ for any } |\boldsymbol{\alpha}| = m - 1 \right\}$ for $H \in \mathscr{A}$. In the case $m = 1$, the property above is well-known (see for example (OT1, Proposition 4.8)).

An another important property is a criterion for knowing whether given elements form a basis for $D^{(m)}(\mathscr{A})$. We say this criterion Saito–Holm's criterion. We introduce Saito–Holm's criterion in Section 2.

Moreover, Holm showed a expression of the ring of differential operators of the coordinate ring of arrangements. Let $\mathscr{D}(R)$ denote the ring of differential operators of a commutative K-algebra R. Then Holm (H2) proved that the ring of differential operators of $S/Q(\mathscr{A})S$ express a quotient of the direct sum of the modules of $\mathscr{A}$-differential operators as an S-module:

$$\mathscr{D}(S/Q(\mathscr{A})S) \simeq \frac{\bigoplus_{m \geq 0} D^{(m)}(\mathscr{A})}{Q(\mathscr{A})\mathscr{D}(S)}.$$

One of the aim of studying the module of $\mathscr{A}$-differential operators is to express the ring of differential operators of the coordinate rings of arrangements. In this article, we consider that freeness and basis for the module of $\mathscr{A}$-differential operators of order 2 when $\mathscr{A}$ is the classical Coxeter arrangement.

The classical Coxeter arrangements $\mathcal{A}_{\ell-1}$, $\mathcal{B}_\ell$ and $\mathcal{D}_\ell$ of type A, B and D are defined as

$$\begin{aligned}
\mathcal{A}_{\ell-1} &:= \{H_{ij} = \{x_i - x_j = 0\} \mid 1 \leq i < j \leq \ell\}, \\
\mathcal{B}_\ell &:= \{H_i = \{x_i = 0\} \mid i = 1, \ldots, \ell\} \\
&\quad \cup \left\{H_{ij}^{\pm 1} = \{x_i \pm x_j = 0\} \mid 1 \leq i < j \leq \ell\right\}, \\
\mathcal{D}_\ell &:= \left\{H_{ij}^{\pm 1} = \{x_i \pm x_j = 0\} \mid 1 \leq i < j \leq \ell\right\}.
\end{aligned}$$

It is well-known that the Coxeter arrangements are free (see Theorem 6.60 in (OT1)). Coxeter arrangemets were studied by Orlik-Terao (OT2), Solomon-Terao (ST), Terao (T) and so on.

There exists a well-known basis for $D^{(1)}(\mathscr{A})$ when $\mathscr{A}$ is one of the classical Coxeter arrangements (see for example (JS)). The aim of this paper is to prove that the modules of differential operators of order

2 on the classical Coxeter arrangements are free by constructing bases. For this purpose, we introduce Cauchy–Sylvester's theorem on compound determinants and Saito–Holm's criterion in Section 2.

We will show that the modules of differential operators of order 2 on the classical Coxeter arrangements are free in Section 3 and 4. We also give examples of $\mathcal{A}_2$, $\mathcal{B}_3$ and $\mathcal{D}_3$.

2 Preliminaries

In this section, we explain Saito–Holm's criterion and Cauchy–Sylvester's theorem on compound determinants. Throughout this paper, assume $\ell \geq m$.

First, we introduce Saito–Holm's criterion. Put $s_m := \binom{\ell+m-1}{m}$ and $t_m := \binom{\ell+m-2}{m-1}$, and set

$$\{\boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(s_m)}\} = \{\boldsymbol{\alpha} \in \mathbb{N}^\ell \mid |\boldsymbol{\alpha}| = m\},$$

where $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_\ell$ for a multi-index $\boldsymbol{\alpha} \in \mathbb{N}^\ell$. For operators $\theta_1, \ldots, \theta_{s_m} \in D^{(m)}(\mathcal{A})$, define the coefficient matrix $M_m(\theta_1, \ldots, \theta_{s_m})$ of the operators $\theta_1, \ldots, \theta_{s_m}$ as follows:

$$M_m(\theta_1, \ldots, \theta_{s_m}) := \left(\theta_i\left(\frac{x^{\boldsymbol{\alpha}^{(j)}}}{\boldsymbol{\alpha}^{(j)}!}\right)\right)_{1 \leq i,j \leq s_m},$$

where $\boldsymbol{\alpha}! = \alpha_1! \cdots \alpha_\ell!$. Thus the (i,j)-entry of the coefficient matrix is the polynomial coefficient of $\partial^{\boldsymbol{\alpha}^{(j)}}$ in θ_i.

The following criterion was originally given by Saito (S) in the case $m = 1$, and was generalized by Holm (H1) into the case m general.

Proposition 2.1 (Saito–Holm's criterion) *Let $\theta_1, \ldots, \theta_{s_m} \in D^{(m)}(\mathscr{A})$ be homogeneous operators. Then the following two conditions are equivalent:*

(1) $\det M_m(\theta_1, \ldots, \theta_{s_m}) = cQ^{t_m}$ *for some* $c \in K^\times$.

(2) $\theta_1, \ldots, \theta_{s_m}$ *form a basis for* $D^{(m)}(\mathscr{A})$ *over* S.

When $D^{(m)}(\mathscr{A})$ is a free S-module, we define the exponents of $D^{(m)}(\mathscr{A})$ to be the multiset of degrees of a homogeneous basis $\{\theta_1, \ldots, \theta_{s_m}\}$ for $D^{(m)}(\mathscr{A})$, which is denoted by $\exp D^{(m)}(\mathscr{A})$:

$$\exp D^{(m)}(\mathscr{A}) = \{\deg(\theta_1), \ldots, \deg(\theta_{s_m})\}.$$

Next, we explain Cauchy–Sylvester's theorem on compound determinants. In the rest of this section, we will follow the notation of the paper by Ito and Okada (IO) as far as possible. We denote by $\succ$ the lexicographic order on $\mathbb{Z}^m$. That is, for $\mu = (\mu_1, \ldots, \mu_m)$ and $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{Z}^m$, we write $\mu \succ \nu$ if there exist an index k such that

$$\mu_1 = \nu_1, \ldots, \mu_{k-1} = \nu_{k-1}, \text{ and } \mu_k > \nu_k.$$

Put

$$Z := \{\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{Z}^m \mid 1 \leq \mu_1 < \mu_2 < \cdots < \mu_m \leq \ell\}.$$

Then Z is a totally ordered subset of $\mathbb{Z}^m$. Put $x_\mu := (x_{\mu_1}, \dots, x_{\mu_m}) \in S^m$.

Let $A = (a_{i,j})_{1 \le i,j \le \ell}$ be a square matrix of order ℓ. For $\mu, \nu \in Z$ put

$$A_{\mu,\nu} := \left(a_{\mu_i,\nu_j}\right)_{1 \le i,j \le m}.$$

We define the m-th compound matrix $A^{(m)}$ by

$$A^{(m)} := \left(\det A_{\mu,\nu}\right)_{\mu,\nu \in Z},$$

where the rows and columns are arranged in the increasing order on Z.

The following was obtained by Cauchy and Sylvester (see for example (IO, Proposition 3.1)).

Proposition 2.2 (Cauchy–Sylvester) *Let $A = (a_{i,j})_{1 \le i,j \le \ell}$ be a square matrix. Then the determinant of the m-th compound matrix $A^{(m)}$ is given by*

$$\det A^{(m)} = (\det A)^{\binom{\ell-1}{m-1}}. \tag{1}$$

Put

$$\Lambda := \{\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m \mid \ell - m \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m \ge 0\}.$$

We regard Λ as a totally ordered subset of $\mathbb{Z}^m$ by the order $\succ$. Then the map

$$Z \ni (\mu_1, \dots, \mu_m) \longmapsto (\ell - m + 1 - \mu_1, \ell - m + 2 - \mu_2, \dots, \ell - \mu_m) \in \Lambda$$

is a bijection between Λ and Z, and this bijection reverses the ordering on Λ and Z.

For $\lambda \in \Lambda$, we define the following symmetric polynomials and a Laurent polynomial:

$$s_\lambda^{\mathcal{A}} := \frac{\det(t_i^{\lambda_j+m-j})_{1 \le i,j \le m}}{\det(t_i^{m-j})_{1 \le i,j \le m}} \in S[t_1, \dots, t_m], \tag{2}$$

$$s_\lambda^{\mathcal{B}} := \frac{\det(t_i^{2(\lambda_j+m-j)+1})_{1 \le i,j \le m}}{\det(t_i^{2(m-j)})_{1 \le i,j \le m}} \in S[t_1, \dots, t_m], \tag{3}$$

$$s_\lambda^{\mathcal{D}} := \frac{\det(t_i^{2(\lambda_j+m-j)-1})_{1 \le i,j \le m}}{\det(t_i^{2(m-j)})_{1 \le i,j \le m}} \in S[t_1^{\pm 1}, \dots, t_m^{\pm 1}]. \tag{4}$$

The polynomial $s_\lambda^{\mathcal{A}}$ is the Schur polynomial corresponding to the partition λ. We remark that $s_\lambda^{\mathcal{D}}$ is a symmetric polynomial if $\lambda_m \ge 1$. Now the degrees of these Laurent polynomials are as follows:

$$\deg s_\lambda^{\mathcal{A}} = |\lambda|, \quad \deg s_\lambda^{\mathcal{B}} = 2|\lambda| + m, \quad \deg s_\lambda^{\mathcal{D}} = 2|\lambda| - m, \tag{5}$$

where $|\lambda| := \lambda_1 + \cdots + \lambda_m$.

Proposition 2.3 *We have the following determinant identities:*

$$\det\left(s_\lambda^{\mathcal{A}}(x_\mu)\right)_{\substack{\lambda\in\Lambda\\ \mu\in Z}} = \left[\prod_{1\le i<j\le \ell}(x_i-x_j)\right]^{\binom{\ell-2}{m-1}}, \tag{6}$$

$$\det\left(s_\lambda^{\mathcal{B}}(x_\mu)\right)_{\substack{\lambda\in\Lambda\\ \mu\in Z}} = (x_1\cdots x_\ell)^{\binom{\ell-1}{m-1}}\left[\prod_{1\le i<j\le \ell}(x_i^2-x_j^2)\right]^{\binom{\ell-2}{m-1}}, \tag{7}$$

$$\det\left(s_\lambda^{\mathcal{D}}(x_\mu)\right)_{\substack{\lambda\in\Lambda\\ \mu\in Z}} = \frac{1}{(x_1\cdots x_\ell)^{\binom{\ell-1}{m-1}}}\left[\prod_{1\le i<j\le \ell}(x_i^2-x_j^2)\right]^{\binom{\ell-2}{m-1}}. \tag{8}$$

3 Type A and B

Let $\mathscr{A}$ be an arbitrary arrangement. By (H2, Proposition 2.3) and (H2, Theorem 2.4), we have

$$D^{(m)}(\mathscr{A}) = \bigcap_{H\in\mathscr{A}} D^{(m)}(p_H S), \tag{9}$$

where $D^{(m)}(p_H S) = \left\{\theta\in D^{(m)}(S) \mid \theta(p_H x^{\boldsymbol{\alpha}})\in p_H S \text{ for any } |\boldsymbol{\alpha}| = m-1\right\}$ for $H\in\mathscr{A}$.

Recall that the defining polynomials of Coxeter arrangements $\mathcal{A}_{\ell-1}$ and $\mathcal{B}_\ell$ of types A and B are

$$Q(\mathcal{A}_{\ell-1}) = \prod_{1\le i<j\le\ell}(x_i-x_j),$$
$$Q(\mathcal{B}_\ell) = x_1\cdots x_\ell\prod_{1\le i<j\le\ell}(x_i^2-x_j^2).$$

We introduce some operators in $D^{(m)}(\mathcal{A}_{\ell-1})$ and $D^{(m)}(\mathcal{B}_\ell)$. By using these operators, we construct bases for the modules $D^{(2)}(\mathcal{A}_{\ell-1})$ and $D^{(2)}(\mathcal{B}_\ell)$ of differential operators of order 2 on $\mathcal{A}_{\ell-1}$ and $\mathcal{B}_\ell$.

Let $k = 1,\dots,\ell$, and put $h_k^{\mathcal{A}} := (x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_\ell)$ and $h_k^{\mathcal{B}} := x_k(x_k^2-x_1^2)\cdots(x_k^2-x_{k-1}^2)(x_k^2-x_{k+1}^2)\cdots(x_k^2-x_\ell^2)$. We define operators $\eta_k^{\mathcal{A}}$ and $\eta_k^{\mathcal{B}}$ in $D^{(m)}(S)$ as follows:

$$\eta_k^{\mathcal{A}} := h_k^{\mathcal{A}}\frac{1}{m!}\partial_k^m,\quad \eta_k^{\mathcal{B}} := h_k^{\mathcal{B}}\frac{1}{m!}\partial_k^m.$$

Then $\deg\eta_k^{\mathcal{A}} = \ell-1$ and $\deg\eta_k^{\mathcal{B}} = 2\ell-1$.

Proposition 3.1 *For $k = 1,\dots,\ell$, we have that $\eta_k^{\mathcal{A}}\in D^{(m)}(\mathcal{A}_{\ell-1})$ and $\eta_k^{\mathcal{B}}\in D^{(m)}(\mathcal{B}_\ell)$.*

For a Laurent polynomial $f(t_1,\dots,t_m)\in S[t_1^{\pm1},\dots,t_m^{\pm1}]$ satisfying $f(x_{\boldsymbol{\alpha}})\in S$ for any $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| = m$, we define an operator

$$\theta_f := \sum_{|\boldsymbol{\alpha}|=m} f(x_{\boldsymbol{\alpha}})\frac{1}{\boldsymbol{\alpha}!}\partial^{\boldsymbol{\alpha}}.$$

We call a Laurent polynomial $f(t_1, \dots, t_m)$ is symmetric if

$$f(t_1, \dots, t_i, \dots, t_j, \dots, t_m) = f(t_1, \dots, t_j, \dots, t_i, \dots, t_m)$$

for all pairs (i, j).

Lemma 3.2 *Assume that $f(t_1, \dots, t_m)$ is a symmetric Laurent polynomial. Then we have that $\theta_f \in D^{(m)}(\mathcal{A}_{\ell-1})$.*

For $\lambda \in \Lambda$, define operators

$$\theta_\lambda^{\mathcal{A}} := \sum_{|\boldsymbol{\alpha}|=m} s_\lambda^{\mathcal{A}}(x_{\boldsymbol{\alpha}}) \frac{1}{\boldsymbol{\alpha}!} \partial^{\boldsymbol{\alpha}}, \quad \theta_\lambda^{\mathcal{B}} := \sum_{|\boldsymbol{\alpha}|=m} s_\lambda^{\mathcal{B}}(x_{\boldsymbol{\alpha}}) \frac{1}{\boldsymbol{\alpha}!} \partial^{\boldsymbol{\alpha}}.$$

Then $\deg \theta_\lambda^{\mathcal{A}} = |\lambda|, \deg \theta_\lambda^{\mathcal{B}} = 2|\lambda| + m$ by the formula (5).

Proposition 3.3 *For $\lambda \in \Lambda$, we have $\theta_\lambda^{\mathcal{A}} \in D^{(m)}(\mathcal{A}_{\ell-1})$ and $\theta_\lambda^{\mathcal{B}} \in D^{(m)}(\mathcal{B}_\ell)$.*

Theorem 3.4 *Let $m = 2$.*

(1) *The set*

$$C_{\mathcal{A}} := \left\{\eta_i^{\mathcal{A}} \mid i = 1, \dots \ell\right\} \cup \left\{\theta_\lambda^{\mathcal{A}} \mid \lambda \in \Lambda\right\}$$

forms an S-basis for $D^{(2)}(\mathcal{A}_{\ell-1})$. Hence

$$\exp D^{(2)}(\mathcal{A}_{\ell-1}) = \{\ell - 1, \dots, \ell - 1\} \cup \{|\lambda| \mid \lambda \in \Lambda\}.$$

(2) *The set*

$$C_{\mathcal{B}} := \left\{\eta_i^{\mathcal{B}} \mid i = 1, \dots \ell\right\} \cup \left\{\theta_\lambda^{\mathcal{B}} \mid \lambda \in \Lambda\right\}$$

forms an S-basis for $D^{(2)}(\mathcal{B}_\ell)$. Hence

$$\exp D^{(2)}(\mathcal{B}_\ell) = \{2\ell - 1, \dots, 2\ell - 1\} \cup \{2|\lambda| + 2 \mid \lambda \in \Lambda\}.$$

We give examples of $\mathcal{A}_2$ and $\mathcal{B}_3$. It is convenient to write $f \doteq g$ for $f, g \in S$ if $f = cg$ for some $c \in K^\times$.

Example 3.5 *Let $\ell = 3$, and $m = 2$. In this case, $s_2 = \binom{3+2-1}{2} = 6$ and $t_2 = \binom{3+2-2}{2-1} = 3$. Then*

$$\Lambda = \{(\lambda_1, \lambda_2) \mid 1 \geq \lambda_1 \geq \lambda_2 \geq 0\} = \{(1,1), (1,0), (0,0)\}.$$

First, we consider $D^{(2)}(\mathcal{A}_2)$. The Schur polynomials are as follows:

$$s_{(1,1)}^{\mathcal{A}} = t_1 t_2, \quad s_{(1,0)}^{\mathcal{A}} = t_1 + t_2, \quad s_{(0,0)}^{\mathcal{A}} = 1.$$

Thus we obtain the operators of the set $C_{\mathcal{A}}$:

$$\begin{aligned}
\eta_1^{\mathcal{A}} &= (x_1 - x_2)(x_1 - x_3)\frac{1}{2}\partial_1^2,\\
\eta_2^{\mathcal{A}} &= (x_2 - x_1)(x_2 - x_3)\frac{1}{2}\partial_2^2,\\
\eta_3^{\mathcal{A}} &= (x_3 - x_1)(x_3 - x_2)\frac{1}{2}\partial_3^2,\\
\theta_{(1,1)}^{\mathcal{A}} &= x_1^2\frac{1}{2}\partial_1^2 + x_2^2\frac{1}{2}\partial_2^2 + x_3^2\frac{1}{2}\partial_3^2 + x_1x_2\partial_1\partial_2 + x_1x_3\partial_1\partial_3 + x_2x_3\partial_2\partial_3,\\
\theta_{(1,0)}^{\mathcal{A}} &= 2x_1\frac{1}{2}\partial_1^2 + 2x_2\frac{1}{2}\partial_2^2 + 2x_3\frac{1}{2}\partial_3^2 + (x_1 + x_2)\partial_1\partial_2 + (x_1 + x_3)\partial_1\partial_3 + (x_2 + x_3)\partial_2\partial_3,\\
\theta_{(0,0)}^{\mathcal{A}} &= \frac{1}{2}\partial_1^2 + \frac{1}{2}\partial_2^2 + \frac{1}{2}\partial_3^2 + \partial_1\partial_2 + \partial_1\partial_3 + \partial_2\partial_3.
\end{aligned}$$

Then the determinant of the coefficient matrix of operators above is

$$\begin{aligned}
&\det M_2\left(\eta_1^{\mathcal{A}}, \eta_2^{\mathcal{A}}, \eta_3^{\mathcal{A}}, \theta_{(1,1)}^{\mathcal{A}}, \theta_{(1,0)}^{\mathcal{A}}, \theta_{(0,0)}^{\mathcal{A}}\right)\\
&= \begin{vmatrix}
(x_1 - x_2)(x_1 - x_3) & 0 & 0 & \frac{1}{2}x_1^2 & x_1 & \frac{1}{2}\\
0 & (x_2 - x_1)(x_2 - x_3) & 0 & \frac{1}{2}x_2^2 & x_2 & \frac{1}{2}\\
0 & 0 & (x_3 - x_1)(x_3 - x_2) & \frac{1}{2}x_3^2 & x_2 & \frac{1}{2}\\
0 & 0 & 0 & x_1x_2 & x_1 + x_2 & 1\\
0 & 0 & 0 & x_1x_3 & x_1 + x_3 & 1\\
0 & 0 & 0 & x_2x_3 & x_2 + x_3 & 1
\end{vmatrix}\\
&= -(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2 \begin{vmatrix}
x_1x_2 & x_1 + x_2 & 1\\
x_1x_3 & x_1 + x_3 & 1\\
x_2x_3 & x_2 + x_3 & 1
\end{vmatrix}\\
&\doteq Q(\mathcal{A}_2)^3
\end{aligned}$$

by Proposition 2.3. Hence the operators $\eta_1^{\mathcal{A}}, \eta_2^{\mathcal{A}}, \eta_3^{\mathcal{A}}, \theta_{(1,1)}^{\mathcal{A}}, \theta_{(1,0)}^{\mathcal{A}}, \theta_{(0,0)}^{\mathcal{A}}$ form an S-basis for $D^{(2)}(\mathcal{A}_2)$ by Proposition 2.1.

Next, we give a basis for $D^{(2)}(\mathcal{B}_3)$. After calculating polynomials $s_\lambda^{\mathcal{B}}$, we obtain

$$s_{(1,1)}^{\mathcal{B}} = t_1^3t_2^3, \quad s_{(1,0)}^{\mathcal{B}} = t_1t_2(t_1^2 + t_2^2), \quad s_{(0,0)}^{\mathcal{B}} = t_1t_2.$$

Then

$$
\begin{aligned}
\eta_1^{\mathcal{B}} &= x_1(x_1^2 - x_2^2)(x_1^2 - x_3^2)\frac{1}{2}\partial_1^2,\\
\eta_2^{\mathcal{B}} &= x_2(x_2^2 - x_1^2)(x_2^2 - x_3^2)\frac{1}{2}\partial_2^2,\\
\eta_3^{\mathcal{B}} &= x_3(x_3^2 - x_1^2)(x_3^2 - x_2^2)\frac{1}{2}\partial_3^2,\\
\theta_{(1,1)}^{\mathcal{B}} &= x_1^6\frac{1}{2}\partial_1^2 + x_2^6\frac{1}{2}\partial_2^2 + x_3^6\frac{1}{2}\partial_3^2 + x_1^3x_2^3\partial_1\partial_2 + x_1^3x_3^3\partial_1\partial_3 + x_2^3x_3^3\partial_2\partial_3,\\
\theta_{(1,0)}^{\mathcal{B}} &= 2x_1^4\frac{1}{2}\partial_1^2 + 2x_2^4\frac{1}{2}\partial_2^2 + 2x_3^4\frac{1}{2}\partial_3^2\\
&\quad + x_1x_2(x_1^2 + x_2^2)\partial_1\partial_2 + x_1x_3(x_1^2 + x_3^2)\partial_1\partial_3 + x_2x_3(x_2^2 + x_3^2)\partial_2\partial_3,\\
\theta_{(0,0)}^{\mathcal{B}} &= x_1^2\frac{1}{2}\partial_1^2 + x_2^2\frac{1}{2}\partial_2^2 + x_3^2\frac{1}{2}\partial_3^2 + x_1x_2\partial_1\partial_2 + x_1x_3\partial_1\partial_3 + x_2x_3\partial_2\partial_3.
\end{aligned}
$$

Thus the determinant of the coefficient matrix of operators above is

$$
\begin{aligned}
&\det M_2\left(\eta_1^{\mathcal{B}}, \eta_2^{\mathcal{B}}, \eta_3^{\mathcal{B}}, \theta_{(1,1)}^{\mathcal{B}}, \theta_{(1,0)}^{\mathcal{B}}, \theta_{(0,0)}^{\mathcal{B}}\right)\\
=&\begin{vmatrix}
x_1(x_1^2 - x_2^2)(x_1^2 - x_3^2) & 0 & 0 & \frac{1}{2}x_1^6 & x_1^4 & \frac{1}{2}x_1^2\\
0 & x_2(x_2^2 - x_1^2)(x_2^2 - x_3^2) & 0 & \frac{1}{2}x_2^6 & x_2^4 & \frac{1}{2}x_2^2\\
0 & 0 & x_3(x_3^2 - x_1^2)(x_3^2 - x_2^2) & \frac{1}{2}x_3^6 & x_3^4 & \frac{1}{2}x_3^2\\
0 & 0 & 0 & x_1^3x_2^3 & x_1x_2(x_1^2 + x_2^2) & x_1x_2\\
0 & 0 & 0 & x_1^3x_3^3 & x_1x_3(x_1^2 + x_3^2) & x_1x_3\\
0 & 0 & 0 & x_2^3x_3^3 & x_2x_3(x_2^2 + x_3^2) & x_2x_3
\end{vmatrix}\\
=&-x_1x_2x_3(x_1^2 - x_2^2)^2(x_1^2 - x_3^2)^2(x_2^2 - x_3^2)^2\begin{vmatrix}
x_1^3x_2^3 & x_1x_2(x_1^2 + x_2^2) & x_1x_2\\
x_1^3x_3^3 & x_1x_3(x_1^2 + x_3^2) & x_1x_3\\
x_2^3x_3^3 & x_2x_3(x_2^2 + x_3^2) & x_2x_3
\end{vmatrix}\\
\doteq&\, Q(\mathcal{B}_2)^3.
\end{aligned}
$$

4 Type D

In this section, we assume $m = 2$, and we construct a basis for $D^{(2)}(\mathcal{D}_\ell)$. Recall the defining polynomial $Q(\mathcal{D}_\ell) = \prod_{1\le i<j\le \ell}(x_i^2 - x_j^2)$ of the Coxeter arrangement of type D.

Set

$$
\begin{aligned}
\Lambda' &:= \{\lambda = (\lambda_1, \lambda_2) \mid \ell - 2 \ge \lambda_1 \ge \lambda_2 \ge 1\},\\
\Lambda'' &:= \{\lambda = (\lambda_1, \lambda_2) \mid \ell - 2 \ge \lambda_1 \ge 0, \lambda_2 = 0\}.
\end{aligned}
$$

Then $\Lambda = \Lambda^{'} \cup \Lambda^{''}$. Put $\lambda^{(0)} := (0,0)$. We define operators $\theta_\lambda^{\mathcal{D}}$ as follows:

$$\theta_\lambda^{\mathcal{D}} := \sum_{|\boldsymbol{\alpha}|=2} s_\lambda^{\mathcal{D}}(x_{\boldsymbol{\alpha}}) \frac{1}{\boldsymbol{\alpha}!}\partial^{\boldsymbol{\alpha}} \quad \text{if } \lambda \in \Lambda^{'},$$

$$\theta_\lambda^{\mathcal{D}} := (x_1 \cdots x_\ell) \sum_{|\boldsymbol{\alpha}|=2} s_\lambda^{\mathcal{D}}(x_{\boldsymbol{\alpha}}) \frac{1}{\boldsymbol{\alpha}!}\partial^{\boldsymbol{\alpha}} \quad \text{if } \lambda \in \Lambda^{''} \setminus \{\lambda^{(0)}\},$$

$$\theta_\lambda^{\mathcal{D}} := (x_1 \cdots x_\ell)^2 \sum_{|\boldsymbol{\alpha}|=2} s_\lambda^{\mathcal{D}}(x_{\boldsymbol{\alpha}}) \frac{1}{\boldsymbol{\alpha}!}\partial^{\boldsymbol{\alpha}} \quad \text{if } \lambda = \lambda^{(0)}.$$

If $\lambda \in \Lambda^{'}$, then we have

$$s_\lambda^{\mathcal{D}} = \frac{\det(t_i^{2(\lambda_j - 1 + 2 - j)+1})_{1\le i,j\le 2}}{\det(t_i^{2(2-j)})_{1\le i,j\le 2}} = s_{\lambda-\mathbf{1}}^{\mathcal{B}},$$

where $\lambda - \mathbf{1} = (\lambda_1 - 1, \lambda_2 - 1)$.

If $\lambda \in \Lambda^{''} \setminus \{\lambda^{(0)}\}$, then

$$s_\lambda^{\mathcal{D}} = \frac{t_1^{2\lambda_1+1}\cdot t_2^{-1} - t_2^{2\lambda_1+1}\cdot t_1^{-1}}{t_1^2 - t_2^2} = \frac{1}{t_1 t_2}\sum_{j=0}^{\lambda_1} t_1^{2j} t_2^{2(\lambda_1 - j)}.$$

Thus $(x_1 \cdots x_\ell) s_\lambda^{\mathcal{D}}(x_{\boldsymbol{\alpha}})$ is a polynomial for any multi-index $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| = 2$.

We have

$$\theta_{\lambda^{(0)}}^{\mathcal{D}} = (x_1 \cdots x_\ell)^2 \left(\sum_{i=1}^{\ell} \frac{1}{2x_i^2}\partial_i^2 + \sum_{1\le i<j\le \ell} \frac{1}{x_i x_j}\partial_i \partial_j \right).$$

Hence $\theta_\lambda^{\mathcal{D}}$ for any $\lambda \in \Lambda$. The degrees of these operators are as follows:

$$\deg \theta_\lambda^{\mathcal{D}} = 2|\lambda| - 2 = 2\lambda_1 + 2\lambda_2 - 2 \quad \text{if} \quad \lambda \in \Lambda^{'},$$
$$\deg \theta_\lambda^{\mathcal{D}} = 2\lambda_1 - 2 + \ell \quad \text{if} \quad \lambda \in \Lambda^{''} \setminus \{\lambda^{(0)}\},$$
$$\deg \theta_\lambda^{\mathcal{D}} = 2\ell - 2 \quad \text{if} \quad \lambda = \lambda^{(0)}.$$

Proposition 4.1 *For $\lambda \in \Lambda$, we have $\theta_\lambda^{\mathcal{D}} \in D^{(2)}(\mathcal{D}_\ell)$.*

We introduce other operators $h_k^{\mathcal{D}}$ of $D^{(2)}(\mathcal{D}_\ell)$. For $k = 1, \ldots, \ell$ put $h_k^{\mathcal{D}} := (x_k^2 - x_1^2)\cdots(x_k^2 - x_{k-1}^2)(x_k^2 - x_{k+1}^2)\cdots(x_k^2 - x_\ell^2)$, and define

$$\eta_k^{\mathcal{D}} := \frac{h_k^{\mathcal{D}}}{2x_k}\partial_k^2 - (-1)^{\ell-1}\frac{1}{x_k}\theta_{\lambda^{(0)}}^{\mathcal{D}}.$$

The coefficient of ∂_k^2 in $\eta_k^{\mathcal{D}}$ is

$$\frac{h_k^{\mathcal{D}}}{2x_k} - (-1)^{\ell-1}\frac{(x_1\cdots x_\ell)^2}{2x_k \cdot x_k^2} = \frac{h_k^{\mathcal{D}} - (-1)^{\ell-1}(x_1 \cdots x_{k-1}x_{k+1}\cdots x_\ell)^2}{2x_k} \in S.$$

Hence we obtain $\eta_k^{\mathcal{D}} \in D^{(2)}(S)$, and $\deg \eta_k^{\mathcal{D}} = 2\ell - 2$.

Proposition 4.2 *For $k = 1, \dots, \ell$, we have that $\eta_k^{\mathcal{D}} \in D^{(2)}(\mathcal{D}_\ell)$.*

Theorem 4.3 *Assume $m = 2$. The set*

$$C_{\mathcal{D}} := \left\{\eta_i^{\mathcal{D}} \mid i = 1, \dots \ell\right\} \cup \left\{\theta_\lambda^{\mathcal{D}} \mid \lambda \in \Lambda\right\}$$

forms an S-basis for $D^{(2)}(\mathcal{D}_\ell)$. Hence

$$\begin{aligned}\exp D^{(2)}(\mathcal{D}_\ell) =& \{2\ell - 2, \dots, 2\ell - 2\} \cup \{2\lambda_1 + 2\lambda_2 - 2 \mid \ell - 2 \geq \lambda_1 \geq \lambda_2 \geq 1\} \\ & \cup \{2\lambda_1 - 2 + \ell \mid \ell - 2 \geq \lambda_1 \geq 1\} \cup \{2\ell - 2\}.\end{aligned}$$

We give a example of $\mathcal{D}_3$.

Example 4.4 *Let $\ell = 3, m = 2$. We have*

$$s^{\mathcal{D}}_{(1,1)} = t_1 t_2, \quad s^{\mathcal{D}}_{(1,0)} = \frac{t_1^2 + t_2^2}{t_1 t_2}, \quad s^{\mathcal{D}}_{(0,0)} = \frac{1}{t_1 t_2}.$$

Then

$$\begin{aligned}
\eta_1^{\mathcal{D}} &= (x_1^3 - x_1x_2^2 - x_1x_3^2)\frac{1}{2}\partial_1^2 - x_1x_3^2\frac{1}{2}\partial_2^2 - x_1x_2^2\frac{1}{2}\partial_3^2 - x_2x_3^2\partial_1\partial_2 - x_2^2x_3\partial_1\partial_3 - x_1x_2x_3\partial_2\partial_3,\\
\eta_2^{\mathcal{D}} &= -x_2x_3^2\frac{1}{2}\partial_1^2 + (-x_1^2x_2 + x_2^3 - x_2x_3^2)\frac{1}{2}\partial_2^2 - x_1^2x_2\frac{1}{2}\partial_3^2 - x_1x_3^2\partial_1\partial_2 - x_1x_2x_3\partial_1\partial_3 - x_1^2x_3\partial_2\partial_3,\\
\eta_3^{\mathcal{D}} &= -x_2^2x_3\frac{1}{2}\partial_1^2 - x_1^2x_3\frac{1}{2}\partial_2^2 + (-x_1^2x_3 - x_2x_3^2 + x_3^3)\frac{1}{2}\partial_3^2 - x_1x_2x_3\partial_1\partial_2 - x_1x_2^2\partial_1\partial_3 - x_1^2x_2\partial_2\partial_3,\\
\theta^{\mathcal{D}}_{(1,1)} &= x_1^2\frac{1}{2}\partial_1^2 + x_2^2\frac{1}{2}\partial_2^2 + x_3^2\frac{1}{2}\partial_3^2 + x_1x_2\partial_1\partial_2 + x_1x_3\partial_1\partial_3 + x_2x_3\partial_2\partial_3,\\
\theta^{\mathcal{D}}_{(1,0)} &= 2(x_1x_2x_3)\frac{1}{2}\partial_1^2 + 2(x_1x_2x_3)\frac{1}{2}\partial_2^2 + 2(x_1x_2x_3)\frac{1}{2}\partial_3^2,\\
&\quad + x_3(x_1^2 + x_2^2)\partial_1\partial_2 + x_2(x_1^2 + x_3^2)\partial_1\partial_3 + x_1(x_2^2 + x_3^2)\partial_2\partial_3,\\
\theta^{\mathcal{D}}_{(0,0)} &= x_2^2x_3^2\frac{1}{2}\partial_1^2 + x_1^2x_3^2\frac{1}{2}\partial_2^2 + x_1^2x_2^2\frac{1}{2}\partial_3^2 + x_1x_2x_3^2\partial_1\partial_2 + x_1x_2^2x_3\partial_1\partial_3 + x_1^2x_2x_3\partial_2\partial_3.
\end{aligned}$$

We have the following determinant identity:

$$
\begin{aligned}
&\det M_2\left(\eta_1^{\mathcal{D}},\eta_2^{\mathcal{D}},\eta_3^{\mathcal{D}},\theta_{(1,1)}^{\mathcal{D}},\theta_{(1,0)}^{\mathcal{D}},\theta_{(0,0)}^{\mathcal{D}}\right)\\
\doteq&\begin{vmatrix}
\frac{1}{2}(x_1x_2^2+x_1x_3^2-x_1^3) & \frac{1}{2}x_2x_3^2 & \frac{1}{2}x_2^2x_3 & \frac{1}{2}x_1^2 & x_1x_2x_3 & \frac{1}{2}x_2^2x_3^2\\
\frac{1}{2}x_1x_3^2 & \frac{1}{2}(x_1^2x_2-x_2^3+x_2x_3^2) & \frac{1}{2}x_1^2x_3 & \frac{1}{2}x_2^2 & x_1x_2x_3 & \frac{1}{2}x_1^2x_3^2\\
\frac{1}{2}x_1x_2^2 & \frac{1}{2}x_1^2x_2 & \frac{1}{2}(x_1^2x_3+x_2x_3^2-x_3^3) & \frac{1}{2}x_3^2 & x_1x_2x_3 & \frac{1}{2}x_1^2x_2^2\\
x_2x_3^2 & x_1x_3^2 & x_1x_2x_3 & x_1x_2 & x_3(x_1^2+x_2^2) & x_1x_2x_3^2\\
x_2^2x_3 & x_1x_2x_3 & x_1x_2^2 & x_1x_3 & x_2(x_1^2+x_3^2) & x_1x_2^2x_3\\
x_1x_2x_3 & x_1^2x_3 & x_1^2x_2 & x_2x_3 & x_1(x_2^2+x_3^2) & x_1^2x_2x_3
\end{vmatrix}\\
\doteq&\begin{vmatrix}
\frac{(x_1^2-x_2^2)(x_1^2-x_3^2)}{x_1} & 0 & 0 & \frac{1}{2}x_1^2 & x_1x_2x_3 & \frac{1}{2}x_2^2x_3^2\\
0 & \frac{(x_2^2-x_1^2)(x_2^2-x_3^2)}{x_2} & 0 & \frac{1}{2}x_2^2 & x_1x_2x_3 & \frac{1}{2}x_1^2x_3^2\\
0 & 0 & \frac{(x_3^2-x_1^2)(x_3^2-x_2^2)}{x_3} & \frac{1}{2}x_3^2 & x_1x_2x_3 & \frac{1}{2}x_1^2x_2^2\\
0 & 0 & 0 & x_1x_2 & x_3(x_1^2+x_2^2) & x_1x_2x_3^2\\
0 & 0 & 0 & x_1x_3 & x_2(x_1^2+x_3^2) & x_1x_2^2x_3\\
0 & 0 & 0 & x_2x_3 & x_1(x_2^2+x_3^2) & x_1^2x_2x_3
\end{vmatrix}\\
\doteq&\frac{(x_1^2-x_2^2)^2(x_1^2-x_3^2)^2(x_2^2-x_3^2)^2}{(x_1x_2x_3)}(x_1x_2x_3)^3\begin{vmatrix}
x_1x_2 & \frac{x_1^2+x_2^2}{x_1x_2} & \frac{1}{x_1x_2}\\
x_1x_3 & \frac{x_1^2+x_3^2}{x_1x_3} & \frac{1}{x_1x_3}\\
x_2x_3 & \frac{x_2^2+x_3^2}{x_2x_3} & \frac{1}{x_2x_3}
\end{vmatrix}\\
\doteq&Q(\mathcal{D}_2)^3
\end{aligned}
$$

by Proposition 2.3. Hence the operators $\eta_1^{\mathcal{D}},\eta_2^{\mathcal{D}},\eta_3^{\mathcal{D}},\theta_{(1,1)}^{\mathcal{D}},\theta_{(1,0)}^{\mathcal{D}},\theta_{(0,0)}^{\mathcal{D}}$ *form an* S*-basis for* $D^{(2)}(\mathcal{D}_3)$ *by Proposition 2.1.*

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