The down operator and expansions of near rectangular k-Schur functions

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Abstract. We prove that the Lam-Shimozono “down operator” on the affine Weyl group induces a derivation of the affine Fomin-Stanley subalgebra of the affine nilCoxeter algebra. We use this to verify a conjecture of Berg, Bergeron, Pon and Zabrocki describing the expansion of k-Schur functions of “near rectangles” in the affine nilCoxeter algebra. Consequently, we obtain a combinatorial interpretation of the corresponding k-Littlewood–Richardson coefficients.


Keywords: symmetric functions, k-Schur functions, affine Schubert calculus, dual graded graphs

1 Introduction

k-Schur functions were first introduced by Lapointe, Lascoux and Morse [13] in the study of Macdonald polynomials. Since then, their study has flourished (see for instance [9, 12, 10, 14, 15, 16]). In particular they have been realized as Schubert classes for the homology of the affine Grassmannian. This was done by identifying the algebra of k-Schur functions with the affine Fomin-Stanley subalgebra of the affine nilCoxeter algebra \( A \) [8]. A natural question is to ask for the expansion of a k-Schur function in terms of the standard basis of \( A \), which is indexed by affine permutations.

An important related problem is to describe the multiplicative structure constants of the k-Schur functions, called the k-Littlewood–Richardson coefficients due to the similarity with the classical problem of multiplying Schur functions. Lam [1] pointed out that the k-Littlewood–Richardson coefficients are the same coefficients that appear in the expansion of a k-Schur function in the standard basis of \( A \) (see Section 4.1). Hence, results that give such expansions also give information about the k-Littlewood–Richardson coefficients. This paper is one such example; others are [7, 11, 3, 2].

In early 2011, Berg, Bergeron, Pon and Zabrocki conjectured an expansion for k-Schur functions indexed by a k-rectangle \( R \) minus its unique removable cell. Their conjecture combined ideas coming from...

This paper initiates the study of operators on the affine nilCoxeter algebra that stabilize the affine Fomin-Stanley subalgebra. We study one particular family of operators introduced by Lam and Shimozono [12] and prove that they are derivations of the affine Fomin-Stanley subalgebra (Theorem 3.5). As an application, we prove the conjecture of Berg, Bergeron, Pon and Zabrocki and provide a combinatorial interpretation for the corresponding $k$-Littlewood–Richardson coefficients (Theorem 4.6). Further properties of such operators and their applications to $k$-Schur functions will be developed in a companion article.

2 k-Combinatorics

In this section, we recall the required terminology associated to the affine type $A$ root system, the affine Weyl group, the connection with bounded partitions and core partitions and the definition of $k$-Schur functions. We work with the affine type $A$ root system $A_k^{(1)}$. Much of this introduction is borrowed from [2] which in turn was borrowed from [26].

2.1 Affine symmetric group

$I = \{0, 1, \ldots, k\}$ will denote the set of nodes of the corresponding Dynkin diagram. We say two nodes $i, j \in I$ are adjacent if $i - j = \pm 1 \pmod{k+1}$.

We let $W$ denote the affine symmetric group with generators $s_i$ for $i \in I$, and relations $s_i^2 = 1$, $s_is_j = s_js_i$, when $i$ and $j$ are not adjacent, and $s_is_js_i = s_js_is_j$ when $i$ and $j$ are adjacent. An element of the affine symmetric group may be expressed as a word in the generators $s_i$. Given the relations above, an element of the affine symmetric group may have multiple reduced words, words of minimal length which express that element. The length of $w$, denoted $\ell(w)$, is the number of generators in any reduced word of $w$.

The Bruhat order on affine symmetric group elements is a partial order where $v < w$ if there is a reduced word for $v$ that is a subword of a reduced word for $w$. If $v < w$ and $\ell(v) = \ell(w) - 1$, we write $v \triangleleft w$. There is another order on $W$, called the left weak order, which is defined by the covering relation $v \triangleleft w$ if $w = s_i v$ for some $i$ and $\ell(v) = \ell(w) - 1$.

For $j \in I$, we denote by $W_j$ the subgroup of $W$ generated by the elements $s_i$ with $i \neq j$. We denote by $W^j$ the set of minimal length representatives of the cosets $W/W_j$.

2.2 Roots and weights

Associated to the affine Dynkin diagram of type $A_k^{(1)}$ we have a root datum, which consists of a free $\mathbb{Z}$-module $\mathfrak{h}$, its dual lattice $\mathfrak{h}^* = \text{Hom}(\mathfrak{h}, \mathbb{Z})$, a pairing $\langle \cdot , \cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{Z}$ given by $\langle \mu, \lambda \rangle = \lambda(\mu)$, and sets of linearly independent elements $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ and $\{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h}$ such that

$$\langle \alpha_i^\vee, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } i \text{ and } j \text{ are adjacent}; \\ 0 & \text{else}. \end{cases} \quad (1)$$

The $\alpha_i$ are known as simple roots, and the $\alpha_i^\vee$ are simple coroots. The spaces $\mathfrak{h}_R = \mathfrak{h} \otimes \mathbb{R}$ and $\mathfrak{h}_R^\vee = \mathfrak{h}^* \otimes \mathbb{R}$ are the coroot and root spaces, respectively.
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Given a simple root $\alpha_i$, we have actions of $W$ on $h_\mathbb{R}$ and $h_\mathbb{R}^+$ defined by the action of the generators of $W$ as

$$s_i(\lambda) = \lambda - (\alpha_i^\vee, \lambda)\alpha_i \quad \text{for } i \in I, \lambda \in h_\mathbb{R}^+;$$
$$s_i(\mu) = \mu - (\mu, \alpha_i)\alpha_i^\vee \quad \text{for } i \in I, \mu \in h_\mathbb{R}.$$  \hfill (2)

The action of $W$ satisfies

$$\langle w(\mu), w(\lambda) \rangle = \langle \mu, \lambda \rangle$$ \hfill (4)

for all $\mu \in h_\mathbb{R}$, $\lambda \in h_\mathbb{R}^+$ and $w \in W$.

The set of real roots is $\Phi_\text{re} = W \cdot \{ \alpha_i \mid i \in I \}$. Given a real root $\alpha = w(\alpha_i)$, we have an associated coroot $\alpha^\vee = w(\alpha_i^\vee)$ and an associated reflection $s_\alpha = ws_iw^{-1}$ (these are well-defined, and independent of the choice of $w$ and $i$). For a Bruhat covering $v < w$, there exists a unique root $\alpha_{i,v}$ satisfying the equation $v^{-1}w = s_{\alpha_{i,v}}$. We denote by $\alpha_{i,v}^\vee$ the coroot corresponding to the root $\alpha_{i,v}$.

The fundamental weights are the elements $\Lambda_i \in h_\mathbb{R}^+$ satisfying $\langle \alpha_i^\vee, \Lambda_i \rangle = \delta_{ij}$ for $i, j \in I$ for $i, j \in I$. They generate the weight lattice $P = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\Lambda_i$. We let $P^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\Lambda_i$ denote the dominant weights.

2.3 $k$-bounded partitions, $(k+1)$-cores and affine Grassmannian elements

Let $\lambda$ be a partition. To each box $(i, j)$ (row $i$, column $j$) of the Young diagram of $\lambda$, we associate its residue defined by $c(i,j) = (j - i) \mod (k+1)$. We let $\mathcal{P}^{(k)}$ denote the set of $k$-bounded partitions, namely the partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ whose first part $\lambda_1$ is at most $k$.

A $p$-core is a partition that has no removable rim hooks of length $p$. Lapointe and Morse [15] Theorem 7] showed that the set $\mathcal{P}^{(k)}$ bijects with the set of $(k+1)$-cores. Following their notation, we let $c(\lambda)$ denote the $(k+1)$-core corresponding to the partition $\lambda$, and $p(\mu)$ denote the $k$-bounded partition corresponding to the $(k+1)$-core $\mu$. We will also use $\mathcal{C}^{(k+1)}$ to represent the set of all $(k+1)$-cores.

$W$ acts on $\mathcal{C}^{(k+1)}$. Specifically, if $\lambda$ is a $(k+1)$-core then

$$s_i\lambda = \begin{cases} 
\lambda \cup \{\text{addable residue } i \text{ cells}\} & \text{if } \lambda \text{ has an addable cell of residue } i, \\
\lambda \setminus \{\text{removable residue } i \text{ cells}\} & \text{if } \lambda \text{ has a removable cell of residue } i, \\
\lambda & \text{otherwise.}
\end{cases}$$

The affine Grassmannian elements are the elements of $W^0$. These are naturally identified with $(k+1)$-cores in the following way: to a core $\lambda \in \mathcal{C}^{(k+1)}$, we associate the unique element $w \in W^0$ for which $w0 = \lambda$. For a $k$-bounded partition $\mu$, we let $w_\mu$ denote the element of $W^0$ which satisfies $w_\mu0 = c(\mu)$. More details on this can be found in [4].

Example 2.1 The diagram of the 4-core $\lambda = (5, 2, 1)$ augmented with its residues, together with the diagrams of the 4-cores $s_1\lambda$ and $s_0\lambda$:

$$\lambda = \begin{array}{ccc} 
0 & 1 & 2 \\
3 & 0 & 0 \\
2 & & 
\end{array} \quad s_1\lambda = \begin{array}{ccc} 
0 & 1 & 2 \\
3 & 0 & 1 \\
2 & 1 & 
\end{array} \quad s_0\lambda = \begin{array}{ccc} 
0 & 1 & 2 \\
3 & 0 & 0 \\
2 & 1 & 
\end{array}.$$
2.4 \( k \)-Schur functions in non-commutative variables

The nilCoxeter algebra \( A \) may be defined via generators and relations with generators \( u_i \) for \( i \in I \), and relations \( u_i^2 = 0 \), \( u_i u_j = u_j u_i \) when \( i \) and \( j \) are not adjacent and \( u_i u_j u_i = u_j u_i u_j \) when \( i \) and \( j \) are adjacent. Since the braid relations are exactly those of the corresponding affine symmetric group, we may index nilCoxeter elements by elements of the affine symmetric group, e.g., \( u_w = u_{i_1} u_{i_2} \cdots u_{i_k} \), whenever \( s_{i_1} s_{i_2} \cdots s_{i_k} \) is a reduced word for \( w \).

**Definition 2.2** For a subset \( S \subset I \), one defines a cyclically decreasing word \( w_S \in W \) to be the unique element of \( W \) for which any (equivalently all) reduced words \( s_{i_1} \cdots s_{i_m} \) of \( w_S \) satisfy:

1. each letter from \( I \) appears at most once in \( \{i_1, \ldots, i_m\} \);
2. if \( j, j+1 \in S \), then \( j+1 \) appears before \( j \) in \( i_1, \ldots, i_m \) (where the indices are taken modulo \( k+1 \)).

Furthermore, we let \( u_S = u_{w_S} \) and

\[
\mathbf{h}_i = \sum_{|S| = 1} u_S \in A.
\]

The elements \( \mathbf{h}_i \) are analogues of the \( i^{th} \) complete homogeneous symmetric functions.

**Example 2.3** Let \( k = 3 \). The cyclically decreasing elements of length 2 in the alphabet \( \{u_0, u_1, u_2, u_3\} \) are \( u_2 u_1, u_1 u_0, u_0 u_3, u_3 u_2, u_0 u_2, \) and \( u_1 u_3 \). Thus,

\[
\mathbf{h}_2 = u_2 u_1 + u_1 u_0 + u_0 u_3 + u_3 u_2 + u_0 u_2 + u_1 u_3.
\]

**Theorem 2.4 (Lam [8])** The elements \( \{\mathbf{h}_i\}_{1 \leq i \leq k} \) commute and freely generate a subalgebra \( B \) of \( A \) called the affine Fomin-Stanley subalgebra. Consequently,

\[
B \cong \Lambda(k) := \mathbb{Q}[h_1, \ldots, h_k],
\]

where \( h_i \) denotes the \( i^{th} \) complete homogeneous symmetric function.

The \( k \)-Schur functions in non-commutative variables are then the images of the \( k \)-Schur functions of Lapointe, Lascoux and Morse [13] under this identification. We take instead the following equivalent definition (see [8, Definition 6.5] and [11, Theorem 4.6]).

**Definition 2.5** The \( k \)-Schur function (in non-commutative variables) corresponding to a \( k \)-bounded partition \( \lambda \) is the unique element \( s_{\lambda}^{(k)} = \sum_w c_w u_w \) of \( B \) satisfying:

\[
c_{\lambda \lambda} = 1; \quad c_w = 0 \text{ for all other } w \in W^0.
\]

3 The Lam-Shimozono up and down operators

In [12], Lam and Shimozono studied two graded graphs whose vertex set is the affine Weyl group \( W \), from which one constructs two closely-related operators defined on the group algebra of \( W \). In this section, we recall the construction of these operators and then develop some properties of the corresponding induced operators on the nilCoxeter algebra \( A \).
3.1 Dual graded graphs

In [5] and [6], Fomin introduced the notion of dual graded graphs, generalizing the notion of differential posets in [25]. A graded graph is a triple $\Gamma = (V, \rho, m, E)$ where $V$ is a set of vertices, $\rho$ is a rank function on $V$, $E$ is a multiset of edges $(x, y)$ for $x, y \in V$ where $\rho(y) = \rho(x) + 1$, and every edge has multiplicity $m(x, y) \in \mathbb{Z}_{\geq 0}$. The set of vertices of the same rank is called a level.

$\Gamma$ is locally finite if every $v \in V$ has finite degree, and we assume this condition for all graphs in this paper. For a graded graph $\Gamma$, the linear down and up operators $D, U : \mathbb{Z}^V \to \mathbb{Z}^V$ are defined as follows.

$$D_{\Gamma}(v) = \sum_{(u, v) \in E} m(u, v)u$$
$$U_{\Gamma}(v) = \sum_{(v, u) \in E} m(v, u)u$$

In other words, $D$ (respectively $U$) maps a vertex $v$ to a linear combination of its neighbors in the level immediately below (respectively above) $v$ where the coefficients are the multiplicities of the edges.

A pair of graded graphs $(\Gamma, \Gamma')$ is called dual if they have the same set of vertices and same rank function, but possibly different edges and multiplicities, and satisfies the following (Heisenberg) commutation relation

$$D_{\Gamma'}U_{\Gamma} - U_{\Gamma}D_{\Gamma'} = r \text{Id}$$

for a fixed $r \in \mathbb{Z}_{\geq 0}$, called the differential coefficient.

One can find many examples of dual graded graphs in [5] and [6], such as the Young, Fibonacci, and Pascal lattices, the graphs of Ferrers shapes and shifted shapes, and many more.

3.2 The Lam-Shimozono dual graded graphs in affine type A

In [12], Lam and Shimozono introduced pairs of dual graded graphs for arbitrary Kac-Moody algebras. Here, we specialize to the case of affine type $A^{(1)}_k$.

Following [12], we define two graded graph structures on $W$. The first constructs a graph with an edge from $v$ to $w$ whenever we have a weak cover $v \prec w$. We denote this graph by $\Gamma_w$ (because its edges depend on weak Bruhat order). The second construction uses strong order. We fix a dominant integral weight $\Lambda \in P^+$ and let $\Gamma_s(\Lambda)$ be the graph that has $\langle \alpha^\vee_{v, w}, \Lambda \rangle$ edges between $v$ and $w$ whenever $v \preceq w$.

The up and down operators for the dual graded graphs $\Gamma_w$ and $\Gamma_s(\Lambda)$ induce operators on $A$. Specifically, define $U$ using the up operator on $\Gamma_w$,

$$U(u_w) = \sum_{v \prec w} u_v,$$

and define $D_\Lambda$ using the down operator on $\Gamma_s(\Lambda)$,

$$D_\Lambda(u_w) = \sum_{v \preceq w} \langle \alpha^\vee_{v, w}, \Lambda \rangle u_v.$$

It is clear from the definition and the bilinearity of the pairing $\langle \cdot, \cdot \rangle$ that $D_{\Lambda_i + \Lambda_j} = D_{\Lambda_i} + D_{\Lambda_j}$. With this in mind, we will assume throughout this paper that $\Lambda$ is a fundamental weight.

**Remark 3.1** Note that the operator $U$ can be realized as left-multiplication by $h_1$ on $A$. With this in mind, we define more generally $U_i(u) := h_iu$ for $u \in A$. 
Remark 3.2 Our notation differs slightly from that of [12]. Lam and Shimozono defined the operators $U$ and $D$ as operators on the opposite graphs; $D$ was defined on the weak order graph, and $U$ was defined on the strong order graph.

Theorem 3.3 (Lam, Shimozono [12], Theorem 2.3) The graphs $\Gamma_w$ and $\Gamma_s(\Lambda)$ are dual graded graphs with differential coefficient 1. In other words, $D \Lambda U - UD \Lambda = 1d$.

3.3 Properties of the Lam-Shimozono down operator

In this section we further develop properties of the operator $D \Lambda$. Our first observation is a generalization of the Heisenberg relation in Theorem (3.3).

Theorem 3.4 Let $\Lambda$ be a fundamental weight. For all $w \in W$,

$$D \Lambda (h_i u_w) = h_{i-1} u_w + h_i D \Lambda (u_w).$$

In particular, $D \Lambda (h_i) = h_{i-1}$ and

$$D \Lambda \circ U_i - U_i \circ D \Lambda = U_{i-1}.$$

Next, we study the restrictions of the operators $D \Lambda$ to the affine Fomin-Stanley subalgebra $B$. The following theorem implies that although the operators $D \Lambda$, for distinct fundamental weights $\Lambda$, are distinct on $A$, their restrictions to the affine Fomin-Stanley subalgebra $B$ coincide. In fact, the action of $D \Lambda$ on $B$ is determined by the conditions that $D \Lambda$ is a derivation and $D \Lambda (h_i) = h_{i-1}$.

Theorem 3.5 Let $\Lambda$ be a fundamental weight. $D \Lambda$ is a derivation on the affine Fomin-Stanley subalgebra $B$. Explicitly, for $x, y \in B$,

$$D \Lambda (xy) = D \Lambda (x)y + xD \Lambda (y).$$

In particular, $D \Lambda$ stabilizes $B$; that is, $D \Lambda (B) \subset B$.

Finally, we describe the coefficients of the operator combinatorially. The next result shows that it suffices to know the value of $D \Lambda$ on the elements in $W_j$. In the case that $j = 0$, this says that it suffices to know the values of $D \Lambda$ on the affine Grassmannian elements.

Theorem 3.6 Suppose $w \in W^j$ and $v \in W_j$. Then

$$D \Lambda_j (u_w v) = D \Lambda_j (u_w) u_v.$$

We now give a combinatorial formula to apply the down operator to the elements of $W^j$. This generalizes the description of the coefficients given in [10].

Theorem 3.7 Suppose $w \in W^j$. Then

$$D \Lambda_j (u_w) = \sum_{y \leq w} c_{w,j}^{y,j} u_y,$$

where $c_{w,j}^{y,j}$ is the number of addable $(i \ell - j)$-cells of the $(k+1)$-core $s_{i_{k-1} - j} \cdots s_{i_1 - j} \emptyset$, where $s_{i_{m-1}} \cdots s_{i_2} s_{i_1}$ is a reduced expression for $w$ and $s_{i_{m-1}} \cdots s_{i_2} s_{i_1}$ is a reduced expression for $y$. 
These previous two theorems combine to give a combinatorial method for calculating the down operator on any basis element $u_w$. We illustrate this in the following example.

**Example 3.8** Fix $k = 3$. We calculate $D_{\lambda_0}(u_2u_3u_0u_1u_2u_3u_0u_2)$. By Theorem 3.6,

$$D_{\lambda_0}(u_2u_3u_0u_1u_2u_3u_0u_2) = D_{\lambda_0}(u_2u_3u_0u_1u_2u_3u_0)u_2$$

since $s_2s_3s_0s_1s_2s_3s_0 \in W^0$. Hence, it suffices to calculate $D_{\lambda_0}(u_2u_3u_0u_1u_2u_3u_0)$.

By Theorem 3.7 the coefficient of $u_2u_3u_0u_1u_2u_3u_0$ in $D_{\lambda_0}(u_2u_3u_0u_1u_2u_3u_0)$ is the number of addable 0-cells in the 4-core $s_1s_2s_3s_0 \cdot \emptyset = (2, 1, 1, 1)$, which is 2 (as indicated by the shaded cells in Figure 1).

![Fig. 1: The addable 0-cells in the 4-core (2, 1, 1, 1).](image)

Similarly, one computes all the other coefficients:

$$D_{\lambda_0}(u_2u_3u_0u_1u_2u_3u_0) = 3u_3u_0u_1u_2u_3u_0 + 2u_2u_0u_1u_2u_3u_0 + 2u_1u_2u_3u_1u_2u_0 + u_2u_3u_0u_1u_3u_0 + u_2u_3u_0u_1u_2u_0 + u_2u_3u_0u_1u_2u_3.$$

4 Expansions of $k$-Schur functions and $k$-Littlewood–Richardson coefficients for “near” rectangles

This section describes the connection between expansions of $k$-Schur functions in the standard basis of $A$ and the $k$-Littlewood–Richardson rule. We then recall the expansions of the $k$-Schur functions for $k$-rectangles, from which we deduce expansions of the $k$-Schur functions for the “near” rectangles.

4.1 Expansion of $s^{(k)}_\lambda$ and the $k$-Littlewood–Richardson coefficients

An important problem in the theory of $k$-Schur functions is to understand the multiplicative structure coefficients $c^{(k)}_{\lambda, \mu}$, called the $k$-Littlewood–Richardson coefficients:

$$s^{(k)}_\lambda s^{(k)}_\mu = \sum_{\nu} c^{(k)}_{\lambda, \mu} s^{(k)}_\nu.$$

Another difficult problem is determining an expansion for $s^{(k)}_\lambda$ in terms of the natural basis $\{u_w\}_{w \in W}$ of $A$. In other words, to find the coefficients $d^{(k)}_\lambda w$ in the expansion:

$$s^{(k)}_\lambda = \sum_{w \in W} d^{(k)}_\lambda w u_w.$$

Lam [7] proved that these two problems are actually equivalent. We reformulate his theorem as follows.
Theorem 4.1 [7] Proposition 42] The coefficient $c_{\lambda, \mu}^{\nu, (k)}$ is nonzero only if $w_\mu$ is less than $w_\nu$ in left weak order, and in this case $c_{\lambda, \mu}^{\nu, (k)} = d_{\lambda}^{w_\nu w_\mu^{-1}}$.

The main application in this paper of the down operator is to give the coefficients $d_{\lambda}^{w}$ via explicit combinatorics when $\lambda$ is a “near” rectangle. From this viewpoint our result gives a combinatorial description of the corresponding $k$-Littlewood–Richardson coefficients. A previous result of [3] will be reviewed in the next section. It contains the combinatorics of the coefficients that appear in the expansion of a $k$-Schur function corresponding to a rectangle and is needed to prove our main result.

4.2 Expansions of rectangular $k$-Schur functions

In [3], Berg, Bergeron, Thomas and Zabrocki gave a combinatorial formula for the expansion of the $k$-Schur function $s_{(k)}^R$ indexed by a $k$-rectangle $R$. We recall their result here; it will be a stepping stone for our main result.

Let $\nu$ and $\mu$ be $k$-bounded partitions. For the skew shape $\nu / \mu$, let $\text{word}(\nu / \mu) \in W$ be the word formed by the residues of the cells in $\nu / \mu$, reading each row from right to left and taking the rows from bottom to top. See Example 4.3.

Theorem 4.2 (Berg, Bergeron, Thomas, Zabrocki [3]) Suppose $R = (c^r)$ with $c + r = k + 1$. The $k$-Schur function $s_{(k)}^R$ in non-commutative variables has the expansion:

$$s_{(k)}^R = \sum_{\lambda \subseteq R} u_{\text{word}((R \cup \lambda) / \lambda)},$$

where $u_{\text{word}((R \cup \lambda) / \lambda)}$ is the monomial in the generators $u_i$ corresponding to $\text{word}((R \cup \lambda) / \lambda)$.

Example 4.3 Let $R = (3, 3)$ and $k = 4$. Then $s_{(k)}^R$ is the sum of all the monomials in $u_i$ corresponding to the reading words of the skew-partitions $(R \cup \lambda) / \lambda$, where $\lambda$ is a partition contained inside the rectangle $R$, as shown:
4.3 $k$-Schur functions for “near” rectangles

**Proposition 4.4** Suppose $R = (c^r)$ with $c + r = k + 1$ and let $S = (c^{r-1}, c-1)$ be the partition obtained from $R$ by removing its bottom-right corner. Let $\Lambda$ be a fundamental weight. Then $D_\Lambda(s^{(k)}_R) = s^{(k)}_S$.

For $\lambda \subset R$ and a cell $x \in \lambda$, we let $\text{word}(R, \lambda, x)$ denote the word corresponding to the diagram $(R \cup \lambda_x)/\lambda$, where $\lambda_x$ denotes the diagram with the cell $x$ removed.

**Example 4.5** Let $k = 4$, let $R = (3, 3)$, $\lambda = (2, 1) \subset R$ and $x = (1, 2) \in \lambda$. Then $\text{word}(R, \lambda, x) = s_2s_3s_1s_0s_2$.

**Theorem 4.6**

\[ \sum_{(c^{r-1}, c-1)} s^{(k)}_{(c^r)} = \sum_{\lambda \subset R} \sum_{x \in \lambda} u_{\text{word}(R, \lambda, x)}. \]

**Proof:** This follows from Theorem 3.7 and the application of Proposition 4.4 with the fundamental weight $\Lambda_r$. \[ \square \]

**Example 4.7** Let $k = 4$ and $\lambda = (3, 2)$. Using Example 4.3 we can realize $s^{(4)}_{3,3}$ as $D_{\Lambda_3}(s^{(4)}_{3,3})$. $D_{\Lambda_3}$ acts on the pictures by deleting a bold letter from a term in the expansion of $s^{(4)}_{3,3}$. In particular, the first diagram of $s^{(4)}_{3,3}$ has no bold letters, so it does not contribute any terms to $s^{(4)}_{3,2}$.

The second diagram gives a term:

The third and fourth diagrams each give two terms:
The fifth and sixth diagrams give 3 terms each:

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\] \rightarrow

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\] \rightarrow

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

The seventh and eighth diagrams give 4 terms each:

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\] \rightarrow

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\] \rightarrow

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

The ninth diagram gives 5 terms:

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\] \rightarrow

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]
The down operator and expansions of near rectangular $k$-Schur functions

The tenth and final diagram gives six terms:

\[
\begin{array}{ccc}
\begin{array}{cccc}
& 3 & 4 & 0 \\
2 & 3 & 4 & 0 \\
& 2 & 3 & 4
\end{array} & \longrightarrow & \\
\begin{array}{cccc}
3 & 4 & 0 & 2 \\
3 & 4 & 0 & 2 \\
4 & 0 & 2 & 3
\end{array} & \begin{array}{cccc}
3 & 4 & 0 & 2 \\
3 & 4 & 0 & 2 \\
4 & 0 & 2 & 3
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{cccc}
& 3 & 4 & 0 \\
2 & 3 & 2 & 4 \\
& 3 & 4 & 0 & 2 \\
& 3 & 4 & 0 & 2 \\
& 3 & 4 & 2 & 3
\end{array} & \begin{array}{cccc}
3 & 4 & 0 & 2 \\
3 & 4 & 0 & 2 \\
4 & 0 & 2 & 3
\end{array} & \begin{array}{cccc}
3 & 4 & 0 & 2 \\
3 & 4 & 0 & 2 \\
4 & 0 & 2 & 3
\end{array}
\end{array}
\]

Then $s^{(4)}_{3,2}$ is a sum of the 30 words above.

**Corollary 4.8** Let $S = (c^{-1}, c - 1)$ with $c + r = k + 1$. Then the coefficient $c_{X,S}^{(k)}$ is either 0 or 1.

**Example 4.9** Continuing the example above, we compute $c_{(2,1),(3,2)}^{(3,3,1,1),3}$. The element $u = u_2 u_3 u_1 u_0 u_2$ satisfies $u(2,1) = (3,3,1,1)$. Therefore the coefficient $c_{(2,1),(3,2)}^{(3,3,1,1),3}$ is the coefficient of $u$ in the expansion of $s^{(4)}_{3,2}$, which is 1.

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**References**


