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The down operator and expansions of near rectangular k-Schur functions[†]

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Abstract. We prove that the Lam-Shimozono "down operator" on the affine Weyl group induces a derivation of the affine Fomin-Stanley subalgebra of the affine nilCoxeter algebra. We use this to verify a conjecture of Berg, Bergeron, Pon and Zabrocki describing the expansion of k-Schur functions of "near rectangles" in the affine nilCoxeter algebra. Consequently, we obtain a combinatorial interpretation of the corresponding k-Littlewood–Richardson coefficients.

Résumé. Nous montrons que l'opérateur "down", défini par Lam et Shimozono sur le groupe de Weyl affine, induit une dérivation de la sous-algèbre affine de Fomin-Stanley de l'algèbre affine de nilCoxeter. Nous employons cette dérivation pour vérifier une conjecture de Berg, Bergeron, Pon et Zabrocki sur l'expansion des k-fonctions de Schur indexées par les partitions qui sont "presque rectangles". Par conséquent, nous obtenons une interprétation combinatoire des k-coefficients de Littlewood–Richardson correspondants.

Keywords: symmetric functions, k-Schur functions, affine Schubert calculus, dual graded graphs

1 Introduction

k-Schur functions were first introduced by Lapointe, Lascoux and Morse [13] in the study of Macdonald polynomials. Since then, their study has flourished (see for instance [9, 12, 10, 14, 15, 16]). In particular they have been realized as Schubert classes for the homology of the affine Grassmannian. This was done by identifying the algebra of k-Schur functions with the affine Fomin-Stanley subalgebra of the affine nilCoxeter algebra $\mathbb{A}$ [8]. A natural question is to ask for the expansion of a k-Schur function in terms of the standard basis of $\mathbb{A}$, which is indexed by affine permutations.

An important related problem is to describe the multiplicative structure constants of the k-Schur functions, called the *k-Littlewood–Richardson* coefficients due to the similarity with the classical problem of multiplying Schur functions. Lam [7] pointed out that the k-Littlewood–Richardson coefficients are the same coefficients that appear in the expansion of a k-Schur function in the standard basis of $\mathbb{A}$ (see Section 4.1). Hence, results that give such expansions also give information about the k-Littlewood–Richardson coefficients. This paper is one such example; others are [7, 1, 3, 2].

In early 2011, Berg, Bergeron, Pon and Zabrocki conjectured an expansion for k-Schur functions indexed by a k-rectangle R minus its unique removable cell. Their conjecture combined ideas coming from

[†]This manuscript has been shortened to fit the guidelines for submission. All substantial proofs have been omitted. A longer version will appear on the arXiv.

two groups: Pon's [21] description of the generators of the affine Fomin-Stanley subalgebra for arbitrary affine type; and Berg, Bergeron, Thomas and Zabrocki's [3] expansion of $s_R^{(k)}$.

This paper initiates the study of operators on the affine nilCoxeter algebra that stabilize the affine Fomin-Stanley subalgebra. We study one particular family of operators introduced by Lam and Shimozono [12] and prove that they are derivations of the affine Fomin-Stanley subalgebra (Theorem 3.5). As an application, we prove the conjecture of Berg, Bergeron, Pon and Zabrocki and provide a combinatorial interpretation for the corresponding k-Littlewood–Richardson coefficients (Theorem 4.6). Further properties of such operators and their applications to k-Schur functions will be developed in a companion article.

2 k-Combinatorics

In this section, we recall the required terminology associated to the affine type A root system, the affine Weyl group, the connection with bounded partitions and core partitions and the definition of k-Schur functions. We work with the affine type A root system $A_k^{(1)}$. Much of this introduction is borrowed from [2] which in turn was borrowed from [26].

2.1 Affine symmetric group

$I = \{0, 1, \dots, k\}$ will denote the set of nodes of the corresponding Dynkin diagram. We say two nodes $i, j \in I$ are adjacent if $i - j = \pm 1 \mod (k+1)$.

We let W denote the *affine symmetric group* with generators s_i for $i \in I$, and relations $s_i^2 = 1$, $s_i s_j = s_j s_i$, when i and j are not adjacent, and $s_i s_j s_i = s_j s_i s_j$ when i and j are adjacent. An element of the affine symmetric group may be expressed as a word in the generators s_i. Given the relations above, an element of the affine symmetric group may have multiple *reduced words*, words of minimal length which express that element. The *length* of w, denoted $\ell(w)$, is the number of generators in any reduced word of w.

The *Bruhat order* on affine symmetric group elements is a partial order where $v < w$ if there is a reduced word for v that is a subword of a reduced word for w. If $v < w$ and $\ell(v) = \ell(w) - 1$, we write $v \lessdot w$. There is another order on W, called the *left weak order*, which is defined by the covering relation $v \prec\!\!\cdot\, w$ if $w = s_i v$ for some i and $\ell(v) = \ell(w) - 1$.

For $j \in I$, we denote by W_j the subgroup of W generated by the elements s_i with $i \neq j$. We denote by W^j the set of minimal length representatives of the cosets W/W_j.

2.2 Roots and weights

Associated to the affine Dynkin diagram of type $A_k^{(1)}$ we have a root datum, which consists of a free $\mathbb{Z}$-module $\mathfrak{h}$, its dual lattice $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{Z})$, a pairing $\langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{Z}$ given by $\langle \mu, \lambda \rangle = \lambda(\mu)$, and sets of linearly independent elements $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ and $\{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h}$ such that

$$\langle \alpha_i^\vee, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } i \text{ and } j \text{ are adjacent;} \\ 0 & \text{else.} \end{cases} \tag{1}$$

The α_i are known as *simple roots*, and the $\alpha_i^\vee$ are *simple coroots*. The spaces $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h} \otimes \mathbb{R}$ and $\mathfrak{h}_{\mathbb{R}}^* = \mathfrak{h}^* \otimes \mathbb{R}$ are the *coroot* and *root spaces*, respectively.

Given a simple root α_i, we have actions of W on $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}^*_{\mathbb{R}}$ defined by the action of the generators of W as

$$s_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i \quad \text{for } i \in I, \lambda \in \mathfrak{h}^*_{\mathbb{R}}; \tag{2}$$

$$s_i(\mu) = \mu - \langle \mu, \alpha_i \rangle \alpha_i^\vee \quad \text{for } i \in I, \mu \in \mathfrak{h}_{\mathbb{R}}. \tag{3}$$

The action of W satisfies

$$\langle w(\mu), w(\lambda) \rangle = \langle \mu, \lambda \rangle \tag{4}$$

for all $\mu \in \mathfrak{h}_{\mathbb{R}}$, $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ and $w \in W$.

The set of *real roots* is $\Phi_{\text{re}} = W \cdot \{\alpha_i \mid i \in I\}$. Given a real root $\alpha = w(\alpha_i)$, we have an *associated coroot* $\alpha^\vee = w(\alpha_i^\vee)$ and an *associated reflection* $s_\alpha = w s_i w^{-1}$ (these are well-defined, and independent of the choice of w and i). For a Bruhat covering $v \lessdot w$, there exists a unique root $\alpha_{v,w}$ satisfying the equation $v^{-1}w = s_{\alpha_{v,w}}$. We denote by $\alpha_{v,w}^\vee$ the coroot corresponding to the root $\alpha_{v,w}$.

The *fundamental weights* are the elements $\Lambda_i \in \mathfrak{h}^*_{\mathbb{R}}$ satisfying $\langle \alpha_j^\vee, \Lambda_i \rangle = \delta_{ij}$ for $i, j \in I$ for $i, j \in I$. They generate the *weight lattice* $P = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i$. We let $P^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\Lambda_i$ denote the *dominant weights*.

2.3 k-bounded partitions, $(k+1)$-cores and affine Grassmannian elements

Let λ be a partition. To each box (i, j) (row i, column j) of the Young diagram of λ, we associate its *residue* defined by $c_{(i,j)} = (j - i) \mod (k+1)$. We let $\mathcal{P}^{(k)}$ denote the set of *k-bounded partitions*, namely the partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ whose first part λ_1 is at most k.

A *p-core* is a partition that has no removable rim hooks of length p. Lapointe and Morse [15, Theorem 7] showed that the set $\mathcal{P}^{(k)}$ bijects with the set of $(k+1)$-cores. Following their notation, we let $\mathfrak{c}(\lambda)$ denote the $(k+1)$-core corresponding to the partition λ, and $\mathfrak{p}(\mu)$ denote the k-bounded partition corresponding to the $(k+1)$-core μ. We will also use $\mathcal{C}^{(k+1)}$ to represent the set of all $(k+1)$-cores.

W acts on $\mathcal{C}^{(k+1)}$. Specifically, if λ is a $(k+1)$-core then

$$s_i\lambda = \begin{cases} \lambda \cup \{\text{addable residue } i \text{ cells}\} & \text{if } \lambda \text{ has an addable cell of residue } i, \\ \lambda \setminus \{\text{removable residue } i \text{ cells}\} & \text{if } \lambda \text{ has a removable cell of residue } i, \\ \lambda & \text{otherwise.} \end{cases}$$

The *affine Grassmannian elements* are the elements of W^0. These are naturally identified with $(k+1)$-cores in the following way: to a core $\lambda \in \mathcal{C}^{(k+1)}$, we associate the unique element $w \in W^0$ for which $w\emptyset = \lambda$. For a k-bounded partition μ, we let w_μ denote the element of W^0 which satisfies $w_\mu \emptyset = \mathfrak{c}(\mu)$. More details on this can be found in [4].

Example 2.1 *The diagram of the* 4*-core* $\lambda = (5, 2, 1)$ *augmented with its residues, together with the diagrams of the* 4*-cores* $s_1\lambda$ *and* $s_0\lambda$*:*

$$\lambda = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 0 \\ \hline 3 & 0 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array} \qquad s_1\lambda = \begin{array}{|c|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 0 & 1 \\ \hline 3 & 0 & 1 \\ \cline{1-3} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} \qquad s_0\lambda = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} .$$

2.4 k-Schur functions in non-commutative variables

The *nilCoxeter algebra* $\mathbb{A}$ may be defined via generators and relations with generators $\mathbf{u}_i$ for $i \in I$, and relations $\mathbf{u}_i^2 = 0$, $\mathbf{u}_i\mathbf{u}_j = \mathbf{u}_j\mathbf{u}_i$ when i and j are not adjacent and $\mathbf{u}_i\mathbf{u}_j\mathbf{u}_i = \mathbf{u}_j\mathbf{u}_i\mathbf{u}_j$ when i and j are adjacent. Since the braid relations are exactly those of the corresponding affine symmetric group, we may index nilCoxeter elements by elements of the affine symmetric group, e.g., $\mathbf{u}_w = \mathbf{u}_{i_1}\mathbf{u}_{i_2}\cdots\mathbf{u}_{i_k}$, whenever $s_{i_1}s_{i_2}\cdots s_{i_k}$ is a reduced word for w.

Definition 2.2 *For a subset $S \subset I$, one defines a cyclically decreasing word $w_S \in W$ to be the unique element of W for which any (equivalently all) reduced words $s_{i_1}\dots s_{i_m}$ of w_S satisfy:*

1. *each letter from I appears at most once in $\{i_1, \dots, i_m\}$;*
2. *if $j, j+1 \in S$, then $j+1$ appears before j in $i_1, \dots, i_m$ (where the indices are taken modulo $k+1$).*

Furthermore, we let $\mathbf{u}_S = \mathbf{u}_{w_S}$ and

$$\mathbf{h}_i = \sum_{\substack{S \subset I \\ |S|=i}} \mathbf{u}_S \in \mathbb{A}.$$

The elements $\mathbf{h}_i$ are analogues of the i^{th} complete homogeneous symmetric functions.

Example 2.3 *Let $k = 3$. The cyclically decreasing elements of length 2 in the alphabet $\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ are $\mathbf{u}_2\mathbf{u}_1$, $\mathbf{u}_1\mathbf{u}_0$, $\mathbf{u}_0\mathbf{u}_3$, $\mathbf{u}_3\mathbf{u}_2$, $\mathbf{u}_0\mathbf{u}_2$, and $\mathbf{u}_1\mathbf{u}_3$. Thus,*

$$\mathbf{h}_2 = \mathbf{u}_2\mathbf{u}_1 + \mathbf{u}_1\mathbf{u}_0 + \mathbf{u}_0\mathbf{u}_3 + \mathbf{u}_3\mathbf{u}_2 + \mathbf{u}_0\mathbf{u}_2 + \mathbf{u}_1\mathbf{u}_3.$$

Theorem 2.4 (Lam [8]) *The elements $\{\mathbf{h}_i\}_{i \le k}$ commute and freely generate a subalgebra $\mathbb{B}$ of $\mathbb{A}$ called the* affine Fomin-Stanley subalgebra*. Consequently,*

$$\mathbb{B} \cong \Lambda_{(k)} := \mathbb{Q}[h_1, \dots, h_k],$$

where h_i denotes the i^{th} complete homogeneous symmetric function.

The k-Schur functions in non-commutative variables are then the images of the k-Schur functions of Lapointe, Lascoux and Morse [13] under this identification. We take instead the following equivalent definition (see [8, Definition 6.5] and [11, Theorem 4.6]).

Definition 2.5 *The k-*Schur function (in non-commutative variables) *corresponding to a k-bounded partition λ is the unique element $s_\lambda^{(k)} = \sum_w c_w \mathbf{u}_w$ of $\mathbb{B}$ satisfying:*

$$c_{w_\lambda} = 1; \tag{5}$$

$$c_w = 0 \text{ for all other } w \in W^0. \tag{6}$$

3 The Lam-Shimozono up and down operators

In [12], Lam and Shimozono studied two graded graphs whose vertex set is the affine Weyl group W, from which one constructs two closely-related operators defined on the group algebra of W. In this section, we recall the construction of these operators and then develop some properties of the corresponding induced operators on the nilCoxeter algebra $\mathbb{A}$.

3.1 Dual graded graphs

In [5] and [6], Fomin introduced the notion of *dual graded graphs*, generalizing the notion of *differential posets* in [25]. A *graded graph* is a triple $\Gamma = (V, \rho, m, E)$ where V is a set of vertices, ρ is a rank function on V, E is a multiset of edges (x, y) for $x, y \in V$ where $\rho(y) = \rho(x) + 1$, and every edge has multiplicity $m(x, y) \in \mathbb{Z}_{\geq 0}$. The set of vertices of the same rank is called a *level*.

Γ is *locally finite* if every $v \in V$ has finite degree, and we assume this condition for all graphs in this paper. For a graded graph Γ, the linear down and up operators $D, U : \mathbb{Z}V \to \mathbb{Z}V$ are defined as follows.

$$D_\Gamma(v) = \sum_{(u,v)\in E} m(u, v)u \qquad U_\Gamma(v) = \sum_{(v,u)\in E} m(v, u)u$$

In other words, D (respectively U) maps a vertex v to a linear combination of its neighbors in the level immediately below (respectively above) v where the coefficients are the multiplicities of the edges.

A pair of graded graphs (Γ, Γ') is called *dual* if they have the same set of vertices and same rank function, but possibly different edges and multiplicities, and satisfies the following (Heisenberg) commutation relation

$$D_{\Gamma'}U_\Gamma - U_\Gamma D_{\Gamma'} = r\mathrm{Id}$$

for a fixed $r \in \mathbb{Z}_{\geq 0}$, called the *differential coefficient*.

One can find many examples of dual graded graphs in [5] and [6], such as the Young, Fibonacci, and Pascal lattices, the graphs of Ferrers shapes and shifted shapes, and many more.

3.2 The Lam-Shimozono dual graded graphs in affine type A

In [12], Lam and Shimozono introduced pairs of dual graded graphs for arbitrary Kac-Moody algebras. Here, we specialize to the case of affine type $A_k^{(1)}$.

Following [12], we define two graded graph structures on W. The first constructs a graph with an edge from v to w whenever we have a weak cover $v \prec w$. We denote this graph by Γ_w (because its edges depend on weak Bruhat order). The second construction uses strong order. We fix a dominant integral weight $\Lambda \in P^+$ and let $\Gamma_s(\Lambda)$ be the graph that has $\langle \alpha^\vee_{v,w}, \Lambda\rangle$ edges between v and w whenever $v \lessdot w$.

The up and down operators for the dual graded graphs Γ_w and $\Gamma_s(\Lambda)$ induce operators on $\mathbb{A}$. Specifically, define U using the up operator on Γ_w,

$$U(\mathbf{u}_w) = \sum_{v \prec w} \mathbf{u}_v,$$

and define D_Λ using the down operator on $\Gamma_s(\Lambda)$,

$$D_\Lambda(\mathbf{u}_w) = \sum_{v \lessdot w} \langle \alpha^\vee_{v,w}, \Lambda\rangle \mathbf{u}_v.$$

It is clear from the definition and the bilinearity of the pairing $\langle \cdot, \cdot \rangle$ that $D_{\Lambda_i+\Lambda_j} = D_{\Lambda_i} + D_{\Lambda_j}$. With this in mind, we will assume throughout this paper that Λ is a fundamental weight.

Remark 3.1 *Note that the operator U can be realized as left-multiplication by $\mathbf{h}_1$ on $\mathbb{A}$. With this in mind, we define more generally $U_i(\mathbf{u}) := \mathbf{h}_i\mathbf{u}$ for $\mathbf{u} \in \mathbb{A}$.*

Remark 3.2 *Our notation differs slightly from that of [12]. Lam and Shimozono defined the operators U and D as operators on the opposite graphs; D was defined on the weak order graph, and U was defined on the strong order graph.*

Theorem 3.3 (Lam, Shimozono [12], Theorem 2.3) *The graphs Γ_w and $\Gamma_s(\Lambda)$ are dual graded graphs with differential coefficient 1. In other words, $D_\Lambda U - UD_\Lambda = Id$.*

3.3 Properties of the Lam-Shimozono down operator

In this section we further develop properties of the operator D_Λ. Our first observation is a generalization of the Heisenberg relation in Theorem (3.3).

Theorem 3.4 *Let Λ be a fundamental weight. For all $w \in W$,*

$$D_\Lambda(\mathbf{h}_i \mathbf{u}_w) = \mathbf{h}_{i-1}\mathbf{u}_w + \mathbf{h}_i D_\Lambda(\mathbf{u}_w).$$

In particular, $D_\Lambda(\mathbf{h}_i) = \mathbf{h}_{i-1}$ and

$$D_\Lambda \circ U_i - U_i \circ D_\Lambda = U_{i-1}.$$

Next, we study the restrictions of the operators D_Λ to the affine Fomin-Stanley subalgebra $\mathbb{B}$. The following theorem implies that although the operators D_Λ, for distinct fundamental weights Λ, are distinct on $\mathbb{A}$, their restrictions to the affine Fomin-Stanley subalgebra $\mathbb{B}$ coincide. In fact, the action of D_Λ on $\mathbb{B}$ is determined by the conditions that D_Λ is a derivation and $D_\Lambda(\mathbf{h}_i) = \mathbf{h}_{i-1}$.

Theorem 3.5 *Let Λ be a fundamental weight. D_Λ is a derivation on the affine Fomin-Stanley subalgebra $\mathbb{B}$. Explicitly, for $x, y \in \mathbb{B}$,*

$$D_\Lambda(xy) = D_\Lambda(x)y + xD_\Lambda(y).$$

In particular, D_Λ stabilizes $\mathbb{B}$; that is, $D_\Lambda(\mathbb{B}) \subset \mathbb{B}$.

Finally, we describe the coefficients of the operator combinatorially. The next result shows that it suffices to know the value of D_Λ on the elements in W^j. In the case that $j = 0$, this says that it suffices to know the values of D_Λ on the affine Grassmannian elements.

Theorem 3.6 *Suppose $w \in W^j$ and $v \in W_j$. Then*

$$D_{\Lambda_j}(\mathbf{u}_{wv}) = D_{\Lambda_j}(\mathbf{u}_w)\mathbf{u}_v.$$

We now give a combinatorial formula to apply the down operator to the elements of W^j. This generalizes the description of the coefficients given in [10].

Theorem 3.7 *Suppose $w \in W^j$. Then*

$$D_{\Lambda_j}(\mathbf{u}_w) = \sum_{y \lessdot w} c_y^{w,j}\mathbf{u}_y,$$

where $c_y^{w,j}$ is the number of addable $(i_\ell - j)$-cells of the $(k+1)$-core $s_{i_{\ell-1}-j}\cdots s_{i_1-j}\emptyset$, where $s_{i_m}\cdots s_{i_2}s_{i_1}$ is a reduced expression for w and $s_{i_m}\cdots \widehat{s_{i_\ell}}\cdots s_{i_1}$ is a reduced expression for y.

These previous two theorems combine to give a combinatorial method for calculating the down operator on any basis element $\mathbf{u}_w$. We illustrate this in the following example.

Example 3.8 *Fix $k = 3$. We calculate $D_{\Lambda_0}(\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_2)$. By Theorem 3.6,*

$$D_{\Lambda_0}(\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_2) = D_{\Lambda_0}(\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0)\mathbf{u}_2$$

since $s_2s_3s_0s_1s_2s_3s_0 \in W^0$. Hence, it suffices to calculate $D_{\Lambda_0}(\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0)$.

By Theorem 3.7, the coefficient of $\mathbf{u}_2\mathbf{u}_3\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0$ in $D_{\Lambda_0}(\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0)$ is the number of addable 0-cells in the 4-core $s_1s_2s_3s_0 \cdot \emptyset = (2,1,1,1)$, which is 2 (as indicated by the shaded cells in Figure 1).

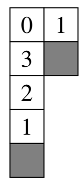


Fig. 1: The addable 0-cells in the 4-core $(2, 1, 1, 1)$.

Similarly, one computes all the other coefficients:

$$\begin{aligned} D_{\Lambda_0}(\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0) &= 3\,\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0 + 2\,\mathbf{u}_2\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_0 + 2\,\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_1\mathbf{u}_2\mathbf{u}_0 \\ &\quad + \mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_3\mathbf{u}_0 + \mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_0 + \mathbf{u}_2\mathbf{u}_3\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3. \end{aligned}$$

4 Expansions of k-Schur functions and k-Littlewood–Richardson coefficients for "near" rectangles

This section describes the connection between expansions of k-Schur functions in the standard basis of $\mathbb{A}$ and the k-Littlewood–Richardson rule. We then recall the expansions of the k-Schur functions for k-rectangles, from which we deduce expansions of the k-Schur functions for the "near" rectangles.

4.1 Expansion of $s_\lambda^{(k)}$ and the k-Littlewood–Richardson coefficients

An important problem in the theory of k-Schur functions is to understand the multiplicative structure coefficients $c_{\lambda,\mu}^{\nu,(k)}$, called the *$k$-Littlewood–Richardson coefficients*:

$$s_\lambda^{(k)} s_\mu^{(k)} = \sum_\nu c_{\lambda,\mu}^{\nu,(k)} s_\nu^{(k)}.$$

Another difficult problem is determining an expansion for $s_\lambda^{(k)}$ in terms of the natural basis $\{\mathbf{u}_w\}_{w\in W}$ of $\mathbb{A}$. In other words, to find the coefficients d_λ^w in the expansion:

$$s_\lambda^{(k)} = \sum_{w\in W} d_\lambda^w \mathbf{u}_w.$$

Lam [7] proved that these two problems are actually equivalent. We reformulate his theorem as follows.

Theorem 4.1 *[7, Proposition 42] The coefficient* $c^{\nu,(k)}_{\lambda,\mu}$ *is nonzero only if* w_μ *is less than* w_ν *in left weak order, and in this case* $c^{\nu,(k)}_{\lambda,\mu} = d_\lambda^{w_\nu w_\mu^{-1}}$.

The main application in this paper of the down operator is to give the coefficients d^w_λ via explicit combinatorics when λ is a "near" rectangle. From this viewpoint our result gives a combinatorial description of the corresponding k-Littlewood–Richardson coefficients. A previous result of [3] will be reviewed in the next section. It contains the combinatorics of the coefficients that appear in the expansion of a k-Schur function corresponding to a rectangle and is needed to prove our main result.

4.2 Expansions of rectangular k-Schur functions

In [3], Berg, Bergeron, Thomas and Zabrocki gave a combinatorial formula for the expansion of the k-Schur function $s^{(k)}_R$ indexed by a k-rectangle R. We recall their result here; it will be a stepping stone for our main result.

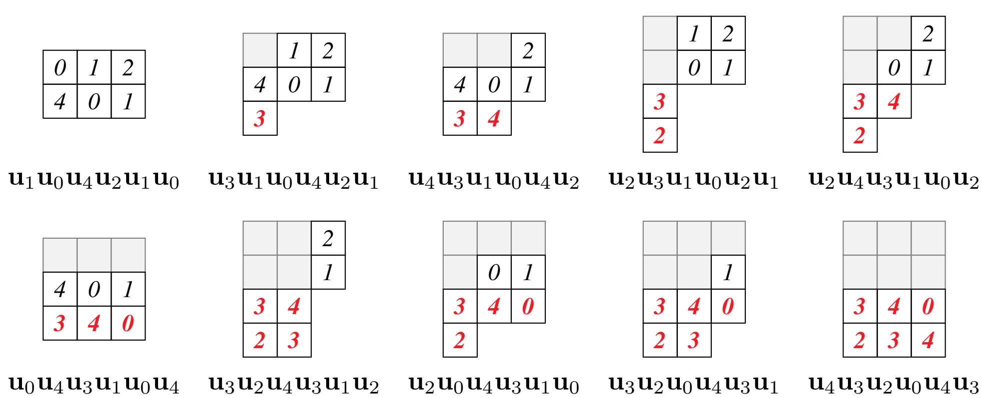
Let ν and μ be k-bounded partitions. For the skew shape ν/μ, let **word**$(\nu/\mu) \in W$ be the word formed by the residues of the cells in ν/μ, reading each row from right to left and taking the rows from bottom to top. See Example 4.3.

Theorem 4.2 (Berg, Bergeron, Thomas, Zabrocki [3]) *Suppose* $R = (c^r)$ *with* $c + r = k + 1$. *The* k*-Schur function* $s^{(k)}_R$ *in non-commutative variables has the expansion:*

$$s^{(k)}_R = \sum_{\lambda \subset R} \mathbf{u}_{\mathbf{word}((R\cup\lambda)/\lambda)},$$

where $\mathbf{u}_{\mathbf{word}((R\cup\lambda)/\lambda)}$ *is the monomial in the generators* $\mathbf{u}_i$ *corresponding to* **word**$((R\cup\lambda)/\lambda)$.

Example 4.3 *Let* $R = (3,3)$ *and* $k = 4$. *Then* $s^{(4)}_R$ *is the sum of all the monomials in* $\mathbf{u}_i$ *corresponding to the reading words of the skew-partitions* $(R\cup\lambda)/\lambda$, *where* λ *is a partition contained inside the rectangle* R, *as shown:*

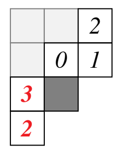


4.3 k-Schur functions for “near” rectangles

Proposition 4.4 *Suppose $R = (c^r)$ with $c+r = k+1$ and let $S = (c^{r-1}, c-1)$ be the partition obtained from R by removing its bottom-right corner. Let Λ be a fundamental weight. Then $D_\Lambda(s_R^{(k)}) = s_S^{(k)}$.*

For $\lambda \subset R$ and a cell $x \in \lambda$, we let $\mathbf{word}(R, \lambda, x)$ denote the word corresponding to the diagram $(R \cup \lambda_x)/\lambda$, where λ_x denotes the diagram with the cell x removed.

Example 4.5 *Let $k = 4$, let $R = (3,3)$, $\lambda = (2,1) \subset R$ and $x = (1,2) \in \lambda$. Then $\mathbf{word}(R, \lambda, x) = s_2 s_3 s_1 s_0 s_2$.*



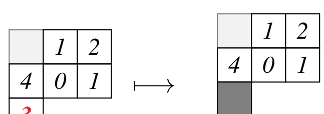
Theorem 4.6

$$s^{(k)}_{(c^{r-1},c-1)} = \sum_{\lambda \subset R} \sum_{x \in \lambda} \mathbf{u}_{\mathbf{word}(R,\lambda,x)}.$$

Proof: This follows from Theorem 3.7 and the application of Proposition 4.4 with the fundamental weight Λ_r. □

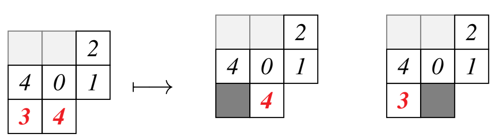
Example 4.7 *Let $k = 4$ and $\lambda = (3,2)$. Using Example 4.3, we can realize $s^{(4)}_{3,2}$ as $D_{\Lambda_3}(s^{(4)}_{3,3})$. D_{Λ_3} acts on the pictures by deleting a bold letter from a term in the expansion of $s^{(4)}_{3,3}$. In particular, the first diagram of $s^{(4)}_{3,3}$ has no bold letters, so it does not contribute any terms to $s^{(4)}_{3,2}$.*

The second diagram gives a term:

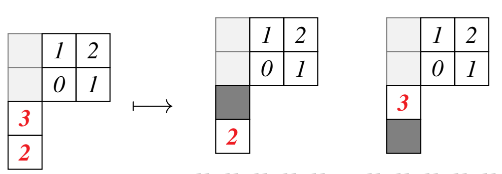


$\mathbf{u}_1\mathbf{u}_0\mathbf{u}_4\mathbf{u}_2\mathbf{u}_1$

The third and fourth diagrams each give two terms:

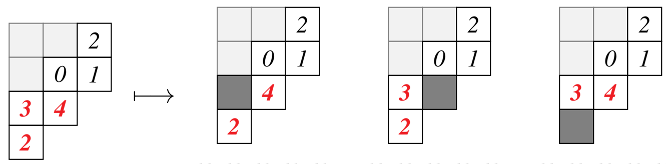


$\mathbf{u}_4\mathbf{u}_1\mathbf{u}_0\mathbf{u}_4\mathbf{u}_2$ $\mathbf{u}_3\mathbf{u}_1\mathbf{u}_0\mathbf{u}_4\mathbf{u}_2$

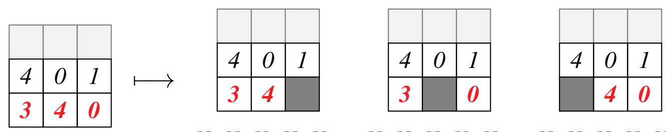


$\mathbf{u}_2\mathbf{u}_1\mathbf{u}_0\mathbf{u}_2\mathbf{u}_1$ $\mathbf{u}_3\mathbf{u}_1\mathbf{u}_0\mathbf{u}_2\mathbf{u}_1$

The fifth and sixth diagrams gives 3 terms each:

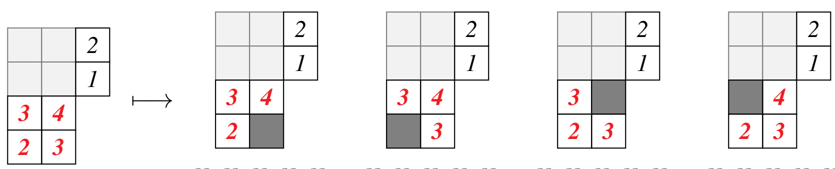


$\mathbf{u}_2\mathbf{u}_4\mathbf{u}_1\mathbf{u}_0\mathbf{u}_2$ $\mathbf{u}_2\mathbf{u}_3\mathbf{u}_1\mathbf{u}_0\mathbf{u}_2$ $\mathbf{u}_4\mathbf{u}_3\mathbf{u}_1\mathbf{u}_0\mathbf{u}_2$

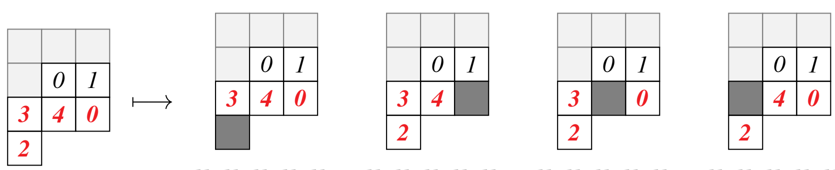


$\mathbf{u}_4\mathbf{u}_3\mathbf{u}_1\mathbf{u}_0\mathbf{u}_4$ $\mathbf{u}_0\mathbf{u}_3\mathbf{u}_1\mathbf{u}_0\mathbf{u}_4$ $\mathbf{u}_0\mathbf{u}_4\mathbf{u}_1\mathbf{u}_0\mathbf{u}_4$

The seventh and eigth diagrams give 4 terms each:

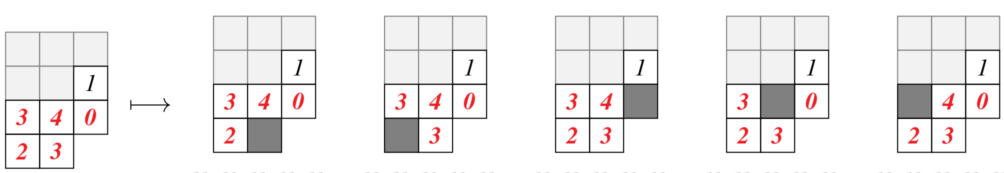


$\mathbf{u}_2\mathbf{u}_4\mathbf{u}_3\mathbf{u}_1\mathbf{u}_2$ $\mathbf{u}_3\mathbf{u}_4\mathbf{u}_3\mathbf{u}_1\mathbf{u}_2$ $\mathbf{u}_3\mathbf{u}_2\mathbf{u}_3\mathbf{u}_1\mathbf{u}_2$ $\mathbf{u}_3\mathbf{u}_2\mathbf{u}_4\mathbf{u}_1\mathbf{u}_2$



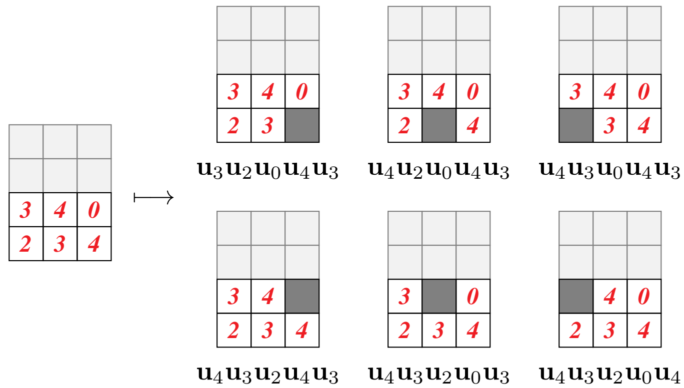
$\mathbf{u}_0\mathbf{u}_4\mathbf{u}_3\mathbf{u}_1\mathbf{u}_0$ $\mathbf{u}_2\mathbf{u}_4\mathbf{u}_3\mathbf{u}_1\mathbf{u}_0$ $\mathbf{u}_2\mathbf{u}_0\mathbf{u}_3\mathbf{u}_1\mathbf{u}_0$ $\mathbf{u}_2\mathbf{u}_0\mathbf{u}_4\mathbf{u}_1\mathbf{u}_0$

The ninth diagram gives 5 terms:



$\mathbf{u}_2\mathbf{u}_0\mathbf{u}_4\mathbf{u}_3\mathbf{u}_1$ $\mathbf{u}_3\mathbf{u}_0\mathbf{u}_4\mathbf{u}_3\mathbf{u}_1$ $\mathbf{u}_3\mathbf{u}_2\mathbf{u}_4\mathbf{u}_3\mathbf{u}_1$ $\mathbf{u}_3\mathbf{u}_2\mathbf{u}_0\mathbf{u}_3\mathbf{u}_1$ $\mathbf{u}_3\mathbf{u}_2\mathbf{u}_0\mathbf{u}_4\mathbf{u}_1$

The tenth and final diagram gives six terms:



Then $s^{(4)}_{3,2}$ is a sum of the 30 words above.

Corollary 4.8 *Let $S = (c^{r-1}, c-1)$ with $c + r = k + 1$. Then the coefficient $c^{\nu,(k)}_{\lambda,S}$ is either 0 or 1.*

Example 4.9 *Continuing the example above, we compute $c^{(3,3,1,1),3}_{(2,1),(3,2)}$. The element $\mathbf{u} = \mathbf{u}_2\mathbf{u}_3\mathbf{u}_1\mathbf{u}_0\mathbf{u}_2$ satisfies $\mathbf{u}(2,1) = (3,3,1,1)$. Therefore the coefficient $c^{(3,3,1,1),3}_{(2,1),(3,2)}$ is the coefficient of $\mathbf{u}$ in the expansion of $s^{(4)}_{3,2}$, which is 1.*

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