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Noncommutative symmetric functions with matrix parameters (extended abstract)

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Abstract. We define new families of noncommutative symmetric functions and quasi-symmetric functions depending on two matrices of parameters, and more generally on parameters associated with paths in a binary tree. Appropriate specializations of both matrices then give back the two-vector families of Hivert, Lascoux, and Thibon and the noncommutative Macdonald functions of Bergeron and Zabrocki.

Résumé. Nous définissons de nouvelles familles de fonctions symétriques non-commutatives et de fonctions quasi-symétriques, dépendant de deux matrices de paramètres, et plus généralement, de paramètres associés à des chemins dans un arbre binaire. Pour des spécialisations appropriées, on retrouve les familles à deux vecteurs de Hivert-Lascoux-Thibon et les fonctions de Macdonald non-commutatives de Bergeron-Zabrocki.

Keywords: Noncommutative symmetric functions, Quasi-symmetric functions, Macdonald polynomials

1 Introduction

The theory of Hall-Littlewood, Jack, and Macdonald polynomials is one of the most interesting subjects in the modern theory of symmetric functions. It is well-known that combinatorial properties of symmetric functions can be explained by lifting them to larger algebras (the so-called combinatorial Hopf algebras), the simplest examples being $\mathbf{Sym}$ (Noncommutative symmetric functions [3]) and its dual $QSym$ (Quasi-symmetric functions [5]).

There have been several attempts to lift Hall-Litttlewood and Macdonald polynomials to $\mathbf{Sym}$ and $QSym$ [1, 7, 8, 13, 14]. The analogues defined in [1] were similar to, though different from, those of [8]. These last ones admitted multiple parameters q_i and t_i, which however could not be specialized to recover the version of [1].

The aim of this article is to show that many more parameters can be introduced in the definition of such bases. Actually, one can have a pair of $n \times n$ matrices (Q_n, T_n) for each degree n. The main properties established in [1] and [8] remain true in this general context, and one recovers the BZ and HLT polynomials for appropriate specializations of the matrices.

In the last section, another possibility involving quasideterminants is explored.

2 Notations

Our notations for noncommutative symmetric functions will be as in [3, 10]. The Hopf algebra of noncommutative symmetric functions is denoted by $\mathbf{Sym}$, or by $\mathbf{Sym}(A)$ if we consider the realization in terms of an auxiliary alphabet. Bases of $\mathbf{Sym}_n$ are labelled by compositions I of n. The noncommutative complete and elementary functions are denoted by S_n and Λ_n, and the notation S^I means $S_{i_1}\dots S_{i_r}$. The ribbon basis is denoted by R_I.

The notation $I \vDash n$ means that I is a composition of n. The conjugate composition is denoted by $I^\sim$. The graded dual of $\mathbf{Sym}$ is $QSym$ (quasi-symmetric functions). The dual basis of (S^I) is (M_I) (monomial), and that of (R_I) is (F_I). The *descent set* of $I = (i_1,\dots,i_r)$ is $\mathrm{Des}\,(I) = \{i_1,\ i_1+i_2,\dots,i_1+\cdots+i_{r-1}\}$.

Finally, there are two operations on compositions: if $I=(i_1,\dots,i_r)$ and $J=(j_1,\dots,j_s)$, the composition $I.J$ is $(i_1,\dots,i_r,j_1,\dots,j_s)$ and $I \triangleright J$ is $(i_1,\dots,i_r+j_1,\dots,j_s)$.

3 $\mathbf{Sym}_n$ as a Grassmann algebra

Since for $n>0$, $\mathbf{Sym}_n$ has dimension 2^{n-1}, it can be identified (as a vector space) with a Grassmann algebra on $n-1$ generators $\eta_1,\dots,\eta_{n-1}$ (that is, $\eta_i\eta_j = -\eta_j\eta_i$, so that in particular $\eta_i^2=0$). This identification is meaningful, for example, in the context of the representation theory of the 0-Hecke algebras $H_n(0)$ (see [2]).

If I is a composition of n with descent set $D=\{d_1,\dots,d_k\}$, we make the identification

$$R_I \longleftrightarrow \eta_D := \eta_{d_1}\eta_{d_2}\dots\eta_{d_k}\,. \tag{1}$$

For example, $R_{213} \leftrightarrow \eta_2\eta_3$. We then have

$$S^I \longleftrightarrow (1+\eta_{d_1})(1+\eta_{d_2})\dots(1+\eta_{d_k}) \tag{2}$$

and

$$\Lambda^I \longleftrightarrow \prod_{i=1}^{n-1}\theta_i, \tag{3}$$

where $\theta_i=\eta_i$ if $i\notin D$ and $\theta_i = 1+\eta_i$ otherwise. Other bases have simple expression under this identification, *e.g.*, Ψ_n, Φ_n and Hivert's Hall-Littlewood basis [7].

3.1 Structure on the Grassmann algebra

Let $*$ be the anti-involution given by $\eta_i^* = (-1)^i\eta_i$. The Grassmann integral of any function f is defined by

$$\int d\eta\, f := f_{12\dots n-1}, \qquad \text{where} \quad f=\sum_k\sum_{i_1<\cdots<i_k} f_{i_1\dots i_k}\eta_{i_1}\dots\eta_{i_k}. \tag{4}$$

We define a bilinear form on $\mathbf{Sym}_n$ by

$$(f,g) = \int d\eta\, f^* g. \tag{5}$$

Then,

$$(R_I,R_J) = (-1)^{\ell(I)-1}\delta_{I,\bar{J}^\sim}, \tag{6}$$

so that this is (up to an unessential sign) the Bergeron-Zabrocki scalar product [1, Eq. (4)].

3.2 Factorized elements in the Grassman algebra

Now, for a sequence of parameters $Z = (z_1, \ldots, z_{n-1})$, let

$$K_n(Z) = (1 + z_1\eta_1)(1 + z_2\eta_2)\ldots(1 + z_{n-1}\eta_{n-1})\,. \tag{7}$$

We then have

Lemma 3.1

$$(K_n(X), K_n(Y)) = \prod_{i=1}^{n-1}(y_i - x_i)\,. \tag{8}$$

3.3 Bases of Sym

We shall investigate bases of $\mathbf{Sym}_n$ of the form

$$\tilde{\mathrm{H}}_I = K_n(Z_I) = \sum_J \tilde{\mathbf{k}}_{IJ} R_J\,, \tag{9}$$

where Z_I is a sequence of parameters depending on the composition I of n.

The bases defined in [8] and [1] are of the previous form and for both of them, the determinant of the Kostka matrix $\mathcal{K} = (\tilde{\mathbf{k}}_{IJ})$ is a product of linear factors (as for ordinary Macdonald polynomials). This is explained by the fact that these matrices have the form

$$\begin{pmatrix} A & xA \\ B & yB \end{pmatrix} \tag{10}$$

where A and B have a similar structure, and so on recursively. Indeed, for such matrices,

Lemma 3.2 *Let A, B be two $m \times m$ matrices. Then,*

$$\begin{vmatrix} A & xA \\ B & yB \end{vmatrix} = (y - x)^m \det A \cdot \det B\,. \tag{11}$$

3.4 Duality

Similarly, the dual vector space $QSym_n = \mathbf{Sym}_n^*$ can be identified with a Grassmann algebra on another set of generators $\xi_1, \ldots, \xi_{n-1}$. Encoding the fundamental basis F_I of Gessel [5] by

$$\xi_D := \xi_{d_1}\xi_{d_2}\ldots\xi_{d_k}\,, \tag{12}$$

the usual duality pairing such that the F_I are dual to the R_I is given in this setting by

$$\langle \xi_D, \eta_E \rangle = \delta_{DE}\,. \tag{13}$$

Let

$$L_n(Z) = (z_1 - \xi_1)\ldots(z_{n-1} - \xi_{n-1})\,. \tag{14}$$

Then, as above, we have a factorization identity:

Lemma 3.3

$$\langle L_n(X), K_n(Y) \rangle = \prod_{i=1}^{n-1}(x_i - y_i)\,. \tag{15}$$

4 Bases associated with paths in a binary tree

Let $\mathbf{y} = \{y_u\}$ be a family of indeterminates indexed by all boolean words of length $\leq n-1$. For example, for $n = 3$, we have the six parameters $y_0, y_1, y_{00}, y_{01}, y_{10}, y_{11}$.

We can encode a composition I with descent set D by the boolean word $u = (u_1, \ldots, u_{n-1})$ such that $u_i = 1$ if $i \in D$ and $u_i = 0$ otherwise.

Let us denote by $u_{m\ldots p}$ the sequence $u_m u_{m+1} \ldots u_p$ and define

$$P_I := (1 + y_{u_1}\eta_1)(1 + y_{u_{1\ldots 2}}\eta_2) \ldots (1 + y_{u_{1\ldots n-1}}\eta_{n-1}) \tag{16}$$

or, equivalently,

$$P_I := K_n(Y_I) \qquad \text{with} \quad Y_I = (y_{u_1}, y_{u_{1\ldots 2}}, \ldots, y_u) =: (y_k(I)) \,. \tag{17}$$

Similarly, let

$$Q_I := (y_{w_1} - \xi_1)(y_{w_{1\ldots 2}} - \xi_2) \ldots (y_{w_{1\ldots n-1}} - \xi_{n-1}) =: L_n(Y^I), \tag{18}$$

where $w_{1\ldots k} = u_1 \ldots u_{k-1}\, \overline{u_k}$ where $\overline{u_k} = 1 - u_k$, so that

$$Y^I := (y_{w_1}, y_{w_{1\ldots 2}}, \ldots, y_{w_{1\ldots n-1}}) =: (y^k(I)) \,. \tag{19}$$

4.1 Kostka matrices

The Kostka matrix is defined as the transpose of the transition matrices from P_I to R_J. This matrix is recursively of the form of Eq. (10). Thus, its determinant factors completely. For $n = 4$, it is

$$(y_1 - y_0)^4 (y_{01} - y_{00})^2 (y_{11} - y_{10})^2 (y_{001} - y_{000})(y_{011} - y_{010})(y_{101} - y_{100})(y_{111} - y_{110}). \tag{20}$$

Proposition 4.1 *The bases (P_I) and (Q_I) are adjoint to each other, up to normalization:*

$$\langle Q_I, P_J \rangle = \langle L_n(Y^I), K_n(Y_J) \rangle = \prod_{k=1}^{n-1} \langle y^k(I) - y_k(J) \rangle \,, \tag{21}$$

which is indeed zero unless $I = J$.

From this, it is easy to derive a product formula for the basis P_I.

Proposition 4.2 *Let I and J be two compositions of respective sizes n and m. The product $P_I P_J$ is a sum over an interval of the lattice of compositions*

$$P_I P_J = \sum_{K \in [I \triangleright (m), I \cdot (1^m)]} c_{IJ}^K P_K \tag{22}$$

where

$$c_{IJ}^K = \frac{\langle L_{n+m}(Y^K), K_{n+m}(Y_I \cdot 1 \cdot Y_J) \rangle}{\langle Q_K, P_K \rangle}, \tag{23}$$

where $Y_I \cdot 1 \cdot Y_J$ stands for the sequence $(y_1(I), \ldots, y_n(I), 1, y_1(J), \ldots, y_m(J))$.

4.2 The quasi-symmetric side

As we have seen before, the (Q_I) being dual to the (P_I), the inverse Kostka matrix is given by the simple construction:

Proposition 4.3 *The inverse of the Kostka matrix is given by*

$$(\mathcal{K}_n^{-1})_{IJ} = (-1)^{\ell(I)-1} \prod_{d \in \mathrm{Des}\,(\bar{I}^{\sim})} y^d(J) \prod_{p=1}^{n-1} \frac{1}{y^p(J) - y_p(J)}\,. \tag{24}$$

4.3 Some specializations

Let us now consider the specialization sending all y_w to 1 if w ends with a 1 and denote by $\mathcal{K}'$ the matrix obtained by this specialization. Then, as in [8, p. 10],

Proposition 4.4 *Let n be an integer. Then*

$$S_n = \mathcal{K}_n {\mathcal{K}'_n}^{-1} \tag{25}$$

is lower triangular. More precisely, let Y'_J be the image of Y_J by the previous specialization and define Y'^J in the same way. Then the coefficient s_{IJ} indexed by (I, J) is

$$s_{IJ} = \prod_{k=1}^{n-1} \frac{y_k(I) - y'^k(J)}{y'_k(J) - y'^k(J)}. \tag{26}$$

5 The two-matrix family

5.1 A specialization of the paths in a binary tree

The above bases can now be specialized to bases $\tilde{\mathrm{H}}(A; Q, T)$, depending on two infinite matrices of parameters. Label the cells of the ribbon diagram of a composition I of n with their matrix coordinates as follows:

$$\mathrm{Diagr}\,(4,1,2,1) = \begin{array}{|c|c|c|c|c} \hline (1,1) & (1,2) & (1,3) & (1,4) & \\ \hline & & & (2,4) & \\ \cline{4-5} & & & (3,4) & (3,5) \\ \cline{4-5} & & & & (4,5) \\ \cline{5-5} \end{array} \tag{27}$$

We associate a variable z_{ij} with each cell except $(1,1)$ by setting $z_{ij} := q_{i,j-1}$ if (i,j) has a cell on its left, and $z_{ij} := t_{i-1,j}$ if (i,j) has a cell on its top. The alphabet $Z(I) = (z_j(I))$ is the sequence of the z_{ij} in their natural order.

Next, if J is a composition of the same integer n, form the monomial

$$\tilde{\mathbf{k}}_{IJ}(Q,T) = \prod_{d\in \mathrm{Des}\,(J)} z_d(I)\,. \tag{28}$$

For example, with $I = (4,1,2,1)$ and $J = (2,1,1,2,2)$, we have $\mathrm{Des}\,(J) = \{2,3,4,6\}$ and $\tilde{\mathbf{k}}_{IJ} = q_{12}q_{13}t_{14}q_{34}$.

Definition 5.1 *Let $Q = (q_{ij})$ and $T = (t_{ij})$ $(i,j \geq 1)$ be two infinite matrices of commuting indeterminates. For a composition I of n, the noncommutative (Q,T)-Macdonald polynomial $\tilde{\mathrm{H}}_I(A;Q,T)$ is*

$$\tilde{\mathrm{H}}_I(A;Q,T) = K_n(A;Z(I)) = \sum_{J\vDash n} \tilde{\mathbf{k}}_{IJ}(Q,T)R_J(A)\,. \tag{29}$$

Note that $\tilde{\mathrm{H}}_I$ depends only on the q_{ij} and t_{ij} with $i+j \leq n$.

5.2 (Q,T)-Kostka matrices

The factorization property of the determinant of the (Q,T)-Kostka matrix, which is valid for the usual Macdonald polynomials as well as for the noncommutative analogues of [8] and [1] still holds since the $\tilde{\mathrm{H}}_I$ are specializations of the P_I.

Theorem 5.2 *Let n be an integer. Then*

$$\det \mathcal{K}_n = \prod_{i+j\leq n} (q_{ij} - t_{ij})^{e(i,j)}\,, \tag{30}$$

where $e(i,j) = \binom{i+j-2}{i-1} 2^{n-i-j}$.

5.3 Specializations

For appropriate specializations, we recover (up to indexation) the Bergeron-Zabrocki polynomials $\tilde{\mathrm{H}}_I^{BZ}$ of [1] and the multiparameter Macdonald functions $\tilde{\mathrm{H}}_I^{HLT}$ of [8]:

Proposition 5.3 *Let (q_i), (t_i), $i \geq 1$ be two sequences of indeterminates. For a composition I of n,*

(i) Let ν be the anti-involution of $\mathbf{Sym}$ defined by $\nu(S_n) = S_n$. Under the specialization $q_{ij} = q_{i+j-1}$, $t_{ij} = t_{n+1-i-j}$, $\tilde{\mathrm{H}}_I(Q,T)$ becomes a multiparameter version of $i\nu(\tilde{\mathrm{H}}_I^{BZ})$, to which it reduces under the further specialization $q_i = q^i$ and $t_i = t^i$.

(ii) Under the specialization $q_{ij} = q_j$, $t_{ij} = t_i$, $\tilde{\mathrm{H}}_I(Q,T)$ reduces to $\tilde{\mathrm{H}}_I^{HLT}$.

5.4 The quasi-symmetric side

Families of (Q,T)-quasi-symmetric functions can now be defined by duality by specialization of the (Q_I) defined in the general case. The dual basis of $(\tilde{\mathrm{H}}_J)$ in $QSym$ will be denoted by $(\tilde{\mathrm{G}}_I)$. We have

$$\tilde{\mathrm{G}}_I(X;Q,T) = \sum_J \tilde{\mathbf{g}}_{IJ}(q,t)F_J(X) \tag{31}$$

where the coefficients are given by the transposed inverse of the Kostka matrix: $(\tilde{\mathbf{g}}_{IJ}) = {}^t(\tilde{\mathbf{k}}_{IJ})^{-1}$.

Let $Z'(I)(Q,T) = Z(I)(T,Q) = Z(\bar{I}^{\sim})(Q,T)$. Then, thanks to Proposition 4.3 and to the fact that changing the last bit of a binary word amounts to change a q into a t, we have

Proposition 5.4 *The inverse of the (Q,T)-Kostka matrix is given by*

$$(\mathcal{K}_n^{-1})_{IJ} = (-1)^{\ell(I)-1} \prod_{d\in \mathrm{Des}\,(\bar{I}^{\sim})} z'_d(J) \prod_{p=1}^{n-1} \frac{1}{z_p(J)-z'_p(J)} \,. \tag{32}$$

6 Multivariate BZ polynomials

In this section, we restrict our attention to the multiparameter version of the Bergeron-Zabrocki polynomials, obtained by setting $q_{ij} = q_{i+j-1}$ and $t_{ij} = t_{n+1-i-j}$ in degree n.

6.1 Multivariate BZ polynomials

As in the case of the two matrices of parameters, Q and T, one can deduce the product in the $\tilde{\mathrm{H}}$ basis by some sort of specialization of the general case. However, since t_{ij} specializes to another t where n appears, one has to be a little more cautious to get the correct answer.

Theorem 6.1 *Let I and J be two compositions of respective sizes p and r. Let us denote by $K = I.\bar{J}^{\sim}$ and $n = |K| = p + r$. Then*

$$\tilde{\mathrm{H}}_I\tilde{\mathrm{H}}_J = \frac{(-1)^{\ell(I)+|J|}}{\prod_{k\in \mathrm{Des}\,(K)}(q_k - t_{n-k})} \sum_{K'} \prod_{k\in \mathrm{Des}\,(K)} (-1)^{\ell(K)}(z_k(K') - z'_k(K'))\tilde{\mathrm{H}}_{K'} \tag{33}$$

where the sum is computed as follows. Let I' and J' be the compositions such that $|I'| = |I|$ and either $K' = I' \cdot J'$, or $K' = I' \triangleright J'$. If I' is not coarser than I or if J' is not finer than J, then $\tilde{\mathrm{H}}(K')$ does have coefficient 0. Otherwise, $z_k(K') = q_k$ if k is a descent of K' and t_{n-k} otherwise. Finally, $z'_k(K')$ does not depend on K' and is $(Z(I), 1, Z(J))$.

6.2 The ∇ operator

The ∇ operator of [1] can be extended by

$$\nabla\tilde{\mathrm{H}}_I = \left(\prod_{d=1}^{n-1} z_d(I)\right) \tilde{\mathrm{H}}_I \,. \tag{34}$$

Then,

Proposition 6.2 *The action of ∇ on the ribbon basis is given by*

$$\nabla R_I = (-1)^{|I|+\ell(I)} \prod_{d\in \mathrm{Des}\,(I^{\sim})} q_d \prod_{d\in \mathrm{Des}\,(\bar{I}^{\sim})} t_d \sum_{J\geq \bar{I}^{\sim}} \prod_{i\in \mathrm{Des}\,(I)\cap \mathrm{Des}\,(J)} (t_i + q_{n-i})R_J \,. \tag{35}$$

Note also that if $I = (1^n)$, one has

$$\nabla\Lambda_n = \sum_{J\vDash n} \prod_{j\in \mathrm{Des}\,(J)} (q_j + t_{n-j})R_J = \sum_{J\vDash n} \prod_{j\notin \mathrm{Des}\,(J)} (q_j + t_{n-j} - 1)\Lambda^J \,. \tag{36}$$

As a positive sum of ribbons, this is the multigraded characteristic of a projective module of the 0-Hecke algebra. Its dimension is the number of packed words of length n (called preference functions in

[1]). Let us recall that a packed word is a word w over $\{1, 2, \dots\}$ so that if $i > 1$ appears in w, then $i-1$ also appears in w. The set of all packed words of size n is denoted by PW_n.

Then the multigraded dimension of the previous module is

$$W_n(\mathbf{q}, \mathbf{t}) = \langle \nabla\Lambda_n, F_1^n \rangle = \sum_{w \in \mathrm{PW}_n} \phi(w) \tag{37}$$

where the statistic $\phi(w)$ is obtained as follows.

Let $\sigma_w = \overline{\mathrm{std}(\overline{w})}$, where $\overline{w}$ denotes the mirror image of w. Then

$$\phi(w) = \prod_{i \in \mathrm{Des}\,(\sigma_w^{-1})} x_i \tag{38}$$

where $x_i = q_i$ if $w_i^{\uparrow} = w_{i+1}^{\uparrow}$ and $x_i = t_{n-i}$ otherwise, where $w^{\uparrow}$ is the nondecreasing reordering of w.

For example, with $w = 22135411$, $\sigma_w = 54368721$, $w^{\uparrow} = 11122345$, the recoils of σ_w are 1, 2, 3, 4, 7, and $\phi(w) = q_1 q_2 t_5 q_4 t_1$.

Theorem 6.3 *Denote by d_I the number of permutations σ with descent composition $C(\sigma) = I$. Then, for any composition I of n,*

$$\nabla R_I = (-1)^{|I|+\ell(I)} \theta(\sigma) \sum_{w \in \mathrm{PW}_n;\, \mathrm{ev}(w) \leq I} \frac{R_{C(\sigma_w^{-1})}}{d_{C(\sigma_w^{-1})}}\,, \tag{39}$$

where σ is any permutation such that $C(\sigma^{-1}) = \bar{I}^{\sim}$, and

$$\theta(\sigma) = \prod_{d \in \mathrm{Des}\,(\bar{I}^{\sim})} t_d\,. \tag{40}$$

The behaviour or the multiparameter BZ polynomials with respect to the scalar product

$$[R_I, R_J] := (-1)^{|I|+\ell(I)} \delta_{I, \bar{J}^{\sim}} \tag{41}$$

is the natural generalization of [1, Prop. 1.7]:

$$[\tilde{\mathrm{H}}_I, \tilde{\mathrm{H}}_J] = (-1)^{|I|+\ell(I)} \delta_{I, \bar{J}^{\sim}} \prod_{i=1}^{n-1} (q_i - t_{n-i})\,. \tag{42}$$

7 Quasideterminantal bases

7.1 Quasideterminants of almost triangular matrices

Quasideterminants [4] are noncommutative analogs of the ratio of a determinant by one of its principal minors. Thus, the quasideterminants of a generic matrix are not polynomials, but complicated rational expressions living in the free field generated by the coefficients. However, for almost triangular matrices, *i.e.*, such that $a_{ij} = 0$ for $i > j+1$, all quasideterminants are polynomials, with a simple explicit expression. We shall only need the formula (see [3], Prop.2.6):

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & \boxed{a_{1n}} \\ -1 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & -1 & a_{33} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{n-1n} \\ 0 & \dots & 0 & -1 & a_{nn} \end{vmatrix} = a_{1n} + \sum_{1\le j_1<\cdots<j_k<n} a_{1j_1}a_{j_1+1j_2}a_{j_2+1j_3}\dots a_{j_k+1n}. \tag{43}$$

Recall that the quasideterminant $|A|_{pq}$ is invariant by scaling the rows of index different from p and the columns of index diffrerent from q. It is homogeneous of degree 1 with respect to row p and column q. Also, the quasideterminant is invariant under permutations of rows and columns.

The quasideterminant (43) coincides with the row-ordered expansion of an ordinary determinant

$$\mathrm{rdet}\,(A) := \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma) a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)} \tag{44}$$

which will be denoted as an ordinary determinant in the sequel.

7.2 *Quasideterminantal bases of* $\mathbf{Sym}$

Many interesting families of noncommutative symmetric functions can be expressed as quasi-determinants of the form

$$H(W,G) = \begin{vmatrix} w_{11}G_1 & w_{12}G_2 & \dots & w_{1\,n-1}G_{n-1} & \boxed{w_{1n}G_n} \\ w_{21} & w_{22}G_1 & \dots & w_{2\,n-1}G_{n-2} & w_{2n}G_{n-1} \\ 0 & w_{32} & \dots & w_{3\,n-3}G_{n-3} & w_{3n}G_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & w_{n\,n-1} & w_{nn}G_1 \end{vmatrix} \tag{45}$$

(or of the transposed form), where G_k is some sequence of free generators of $\mathbf{Sym}$, and W an almost-triangular ($w_{ij} = 0$ for $i > j+1$) scalar matrix. For example, S_n over the Λ^I and the Ψ^I (see [3, (37)-(41)]), or over the Θ^I, where $\Theta_n(q) = (1-q)^{-1}S_n((1-q)A)$ (see [10, Eq. (78)]). These examples illustrate relations between sequences of free generators. Quasi-determinantal expressions for some linear bases can be recast in this form as well. For example, the formula for ribbons [3, (50)] can be rewritten as follows. Let U and V be the $n\times n$ almost-triangular matrices

$$U = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ -1 & -1 & \dots & -1 & -1 \\ 0 & -1 & \dots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & -1 \end{bmatrix} \quad V = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 0 \end{bmatrix} \tag{46}$$

Given the pair (U,V), define, for each composition I of n, a matrix $W(I)$ by

$$w_{ij}(I) = \begin{cases} u_{ij} & \text{if } i-1\in \mathrm{Des}\,(I), \\ v_{ij} & \text{otherwise,} \end{cases} \tag{47}$$

and set

$$H_I(U,V;A) := H(W(I),S(A))\,. \tag{48}$$

Then,

$$(-1)^{\ell(I)-1}R_I = H_I(U,V)\,. \tag{49}$$

Indeed, $H_I(U,V;A)$ is obtained by substituting in (43)

$$a_{j_p+1,j_{p+1}} = \begin{cases} -S_{j_{p+1}-j_p} & \text{if } j_p \in \text{Des}\,(I), \\ 0 & \text{otherwise.} \end{cases} \tag{50}$$

This yields

$$\begin{aligned} S_n + \sum_k \sum_{\{j_1<\cdots<j_k\}\subseteq \text{Des}\,(I)} S_{j_1}(-S_{j_2-j_1})\dots(-S_{n-j_k}) \\ = \sum_{\text{Des}\,(K)\subseteq\text{Des}\,(I)} (-1)^{\ell(K)-1} S^K = (-1)^{\ell(I)-1}R_I\,. \end{aligned} \tag{51}$$

For a generic pair of almost-triangular matrices (U,V), the H_I form a basis of $\mathbf{Sym}_n$. Without loss of generality, we may assume that $u_{1j} = v_{1j} = 1$ for all j. Then, the transition matrix M expressing the H_I on the S^J where $J = (j_1,\dots,j_p)$ satisfies:

$$M_{J,I} := x_{1j_1-1}x_{j_1j_2-1}\dots x_{j_pn}. \tag{52}$$

where $x_{ij} = u_{ij}$ if $i-1$ is not a descent of I and v_{ij} otherwise.

As we shall sometimes need different normalizations, we aslo define for arbitrary almost triangular matrices U,V

$$H'(W,G) = \text{rdet} \begin{bmatrix} w_{11}G_1 & w_{12}G_2 & \dots & w_{1\,n-1}G_{n-1} & w_{1n}G_n \\ w_{21} & w_{22}G_1 & \dots & w_{2\,n-1}G_{n-2} & w_{2n}G_{n-1} \\ 0 & w_{32} & \dots & w_{3\,n-3}G_{n-3} & w_{3n}G_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & w_{n\,n-1} & w_{nn}G_1 \end{bmatrix} \tag{53}$$

and

$$H'_I(U,V) = H'(W(I),S(A))\,. \tag{54}$$

7.3 Expansion on the basis (S^I)

For a composition $I = (i_1,\dots,i_r)$ of n, let $I^\sharp$ be the integer vector of length n obtained by replacing each entry k of I by the sequence $(k,0,\dots,0)$ ($k-1$ zeros):

$$I^\sharp = (i_1 0^{i_1-1} i_2 0^{i_2-1} \dots i_r 0^{i_r-1})\,. \tag{55}$$

Proposition 7.1 *The expansion of $H'(W,S)$ on the S-basis is given by*

$$H'(W,S) = \sum_{I\vDash n} \varepsilon(\sigma_I) w_{1\sigma_I(1)} \cdots w_{n\sigma_I(n)} S^I. \tag{56}$$

7.4 Expansion of the basis (R_I)

Proposition 7.2 *For $I = (i_1, \dots, i_r)$ be a composition of n, denote by W_I the product of diagonal minors of the matrix W taken over the first i_1 rows and columns, then the next i_2 ones and so on. Then,*

$$H'(W, S) = \sum_{I \vDash n} W_I R_I \,. \tag{57}$$

7.5 Examples

7.5.1 A family with factoring coefficients

Theorem 7.3 *Let U and V be defined by*

$$u_{ij} = \begin{cases} x^j - y^j & \text{if } i = 1, \\ aq_{i-1}x^{j-i+1} - y^{j-i+1} & \text{if } 1 < i < j+2, \\ 0 & \text{otherwise,} \end{cases} \tag{58}$$

$$v_{ij} = \begin{cases} x^j - y^j & \text{if } i = 1, \\ x^{j-i+1} - bu_{n+1-i}y^{j-i+1} & \text{if } 1 < i < j+2, \\ 0 & \text{otherwise.} \end{cases} \tag{59}$$

Then the coefficients W_J of the expansion of $H'_I(U, V)$ on the ribbon basis all factor as products of binomials.

The formula for the coefficient of R_n is simple enough: if one orders the factors of $\det(U)$ and $\det(V)$ as

$$Z_n = (x - aq_1y, x - aq_2y, \dots, x - aq_{n-1}y) \tag{60}$$

and

$$Z'_n = (y - bu_{n-1}x, y - bu_{n-2}x, \dots, y - bu_1x), \tag{61}$$

then, the coefficient of R_n in $H'_I(U, V)$ is

$$(x - y) \prod_{d \in \mathrm{Des}\,(I)} z'_d \prod_{e \notin \mathrm{Des}\,(I)} z_e \,. \tag{62}$$

A more careful analysis allows one to compute directly the coefficient of R_J in H'_I.

For example,

$$\begin{aligned} \frac{H'_3(U, V)}{(x - y)} =& (x - aq_1y)(x - aq_2y)R_3 + (x - aq_1y)(aq_2x - y)R_{21} \\ &+ a(x - y)(q_1x - q_2y)R_{12} + (aq_1x - y)(aq_2x - y)R_{111}. \end{aligned} \tag{63}$$

$$\begin{aligned} \frac{H'_{21}(U, V)}{(x - y)} =& (x - aq_1y)(bu_1x - y)R_3 + (x - aq_1y)(x - bu_1y)R_{21} \\ &+ (abq_1u_1x - y)(x - y)R_{12} + (aq_1x - y)(x - bu_1y)R_{111}. \end{aligned} \tag{64}$$

7.5.2 An analogue of the $(1-t)/(1-q)$ transform

Recall that for commutative symmetric functions, the $(1-t)/(1-q)$ transform is defined in terms of the power-sums by

$$p_n\left(\frac{1-t}{1-q}X\right) = \frac{1-t^n}{1-q^n}p_n(X)\,. \tag{65}$$

With the specialization $x=1$, $y=t$, $q_i=q^i$, $u_i=1$, $a=b=1$, one obtains a basis such that for a hook composition $I=(n-k,1^k)$, the commutative image of $H'_I(U,V)$ becomes the $(1-t)/(1-q)$ transform of the Schur function $s_{n-k,1^k}$.

7.5.3 An analogue of the Macdonald P-basis

With the specialization $x=1$, $y=t$, $q_i=q^i$, $u_i=t^i$, $a=b=1$, one obtains an analogue of the Macdonald P-basis, in the sense that for hook compositions $I=(n-k,1^k)$, the commutative image of H'_I is proportional to the Macdonald polynomial $P_{n-k,1^k}(q,t;X)$.

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