

Hecke algebra characters and quantum chromatic symmetric functions

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Abstract. We evaluate induced sign characters of $H_n(q)$ at certain elements of $H_n(q)$ and conjecture an interpretation for the resulting polynomials as generating functions for P -tableaux by a certain statistic. Our conjecture relates the quantum chromatic symmetric functions of Shareshian and Wachs to $H_n(q)$ characters.

Résumé. Nous évaluons les caractères de signe induits de $H_n(q)$ à certains éléments de $H_n(q)$ et nous conjecturons une interprétation des polynômes résultants comme fonctions génératrices pour les tableaux- P par une certaine statistique. Cette conjecture établit un lien entre les fonctions chromatiques symétriques de Shareshian et Wachs et les caractères de $H_n(q)$.

Keywords: Hecke algebra character, unit interval order, P -tableau, chromatic symmetric function, quantum analog.

1 Introduction

Let S_n be the symmetric group, let $s_i = (i, i + 1)$, $i = 1, \dots, n - 1$ denote its standard generators, and for all $v \in S_n$, let $\ell(v)$ denote the length of any reduced (short as possible) expression $s_{i_1} \cdots s_{i_\ell}$ for v . (See [12] for more information on this material.) It is well known that each conjugacy class of S_n consists of the permutations whose cycle sizes in weakly decreasing order are equal to some fixed partition $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$. Define the *Young diagram* of λ to be the left-justified array of n boxes having λ_i boxes in row i . The \mathbb{Z} -module of S_n class functions, those functions $f : S_n \rightarrow \mathbb{Z}$ satisfying $f(w^{-1}vw) = f(v)$ for all $v, w \in S_n$, has several well-studied bases:

1. the irreducible characters $\{\chi^\lambda \mid \lambda \vdash n\}$,
2. the induced sign characters $\{\epsilon^\lambda = \text{sgn} \uparrow_{S_\lambda}^{S_n} \mid \lambda \vdash n\}$,
3. the induced trivial characters $\{\eta^\lambda = \text{triv} \uparrow_{S_\lambda}^{S_n} \mid \lambda \vdash n\}$,
4. the monomial class functions $\{\phi^\lambda \mid \lambda \vdash n\}$,

where S_λ is the Young subgroup of S_n of type λ . These bases are related to one another just as are the Schur, elementary, (complete) homogeneous, and monomial bases of the space of homogeneous degree n

symmetric functions. Specifically, we have

$$\begin{aligned} h_\lambda &= \sum_{\mu} K_{\mu, \lambda} s_\mu, & e_\lambda &= \sum_{\mu} K_{\mu^\top, \lambda} s_\mu, & s_\lambda &= \sum_{\mu} K_{\lambda, \mu} m_\mu, \\ \eta^\lambda &= \sum_{\mu} K_{\mu, \lambda} \chi^\mu, & \epsilon^\lambda &= \sum_{\mu} K_{\mu^\top, \lambda} \chi^\mu, & \chi^\lambda &= \sum_{\mu} K_{\lambda, \mu} \phi^\mu, \end{aligned} \tag{1}$$

where $K = (K_{\lambda, \mu})$ is the invertible matrix of *Kostka numbers*, and λ^\top is the partition whose Young diagram is transpose (or *conjugate*) to that of λ . For each of these class functions f which is not an irreducible character, the integer $f(v)$ has a simple combinatorial formula; when $f = \chi^\lambda$, the number $\chi^\lambda(v)$ is computed by a cumbersome algorithm called the Murnaghan-Nakayama rule. This fact is somewhat unfortunate, since the irreducible characters are the most important of the above class functions.

The above class functions extend linearly to the group algebra $\mathbb{Z}[S_n]$. Work of Goulden and Jackson [4] and Greene [5] has led to the study of combinatorially interpreting $f(z)$ for f a class function and z in $\mathbb{Z}[S_n]$. In particular, these authors used S_n subgroups of the form $S_{[i, j]}$, which fix all letters outside of the interval $[i, j] = \{i, \dots, j\}$ of $[n] \stackrel{\text{def}}{=} \{1, \dots, n\}$, to construct elements

$$C'_{[i, j]} \stackrel{\text{def}}{=} \sum_{v \in S_{[i, j]}} v \tag{2}$$

of $\mathbb{Z}[S_n]$, and products of these. It is not difficult to show that we have

$$f(C'_{[i_1, j_1]} \cdots C'_{[i_m, j_m]}) \geq 0 \tag{3}$$

when f is η^λ or ϵ^λ , and to give a combinatorial interpretation of the resulting nonnegative integers in these cases. (See Theorems 2.2-2.3.) Stembridge showed furthermore [19, Cor. 3.3] that the inequality (3) holds for $f = \chi^\lambda$, and conjectured [20, Conj. 2.1] that it holds also for $f = \phi^\lambda$. In neither of these cases is there a proposed combinatorial interpretation, however. Stanley [15] initiated the study of the special case of (3) in which we have $i_1 < \dots < i_m$ and $j_1 < \dots < j_m$, relating this special case of the equation to certain *chromatic symmetric functions*, and to posets and graphs which avoid certain substructures. In this special case Gasharov [2, Thm. 2] found an interpretation when $f = \chi^\lambda$, but when $f = \phi^\lambda$ we still have no proof of positivity and no conjectured combinatorial interpretation.

It is conceivable that quantum analogs of S_n characters may aid in the formulation of the unknown combinatorial interpretations mentioned above. In particular, the (type A) Hecke algebra $H_n(q)$ and certain functions $\chi_q^\lambda, \epsilon_q^\lambda, \eta_q^\lambda, \phi_q^\lambda : H_n(q) \rightarrow \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, known as $H_n(q)$ *traces*, serve as analogs of S_n and its class functions. Furthermore, there are well-studied quantum analogs $C'_{[i, j]}(q) \in H_n(q)$ of the elements (2) in $\mathbb{Z}[S_n]$, and Haiman [6, Lem. 1.1] has proved that we have

$$f_q(C'_{[i_1, j_1]}(q) \cdots C'_{[i_m, j_m]}(q)) \in \mathbb{N}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \tag{4}$$

when f_q is $\eta_q^\lambda, \epsilon_q^\lambda$, or χ_q^λ . He conjectured that the same holds for $f_q = \phi_q^\lambda$. Unfortunately, no simple combinatorial formulas are known or even conjectured for the evaluation of $H_n(q)$ traces at standard basis elements of $H_n(q)$ (in general), or at the special elements in (4). We will state and prove the first such formula for elements of the form (4) in the special case that $i_1 < \dots < i_r, j_1 < \dots < j_r$, and will relate this to the quantum chromatic symmetric functions introduced recently by Shareshian and Wachs [13].

In Section 2 we associate an element $\beta(P)$ of $\mathbb{Z}[S_n]$ to each labeled poset P . We evaluate induced sign characters at $\beta(P)$ and interpret the resulting nonnegative integers in terms of P -tableaux. In Section 3 we restrict our attention to special classes of posets P , especially unit interval orders. We evaluate other class functions at the associated elements $\beta(P)$, again interpreting the resulting nonnegative integers in terms of P -tableaux. We also discuss the appearance of these interpretations in Stanley’s chromatic symmetric functions [15]. In Section 4 we present quantum analogs of much of the earlier material, including an element $\beta_q(P)$ of $H_n(q)$ which we associate to each labeled poset P . In Section 5 we present our main results. For each unit interval order P , we evaluate induced $H_n(q)$ sign characters at $\beta_q(P)$ and interpret the resulting elements of $\mathbb{N}[q]$ as generating functions for P -tableaux by a certain inversion statistic. Finally, we express the Shareshian-Wachs quantum chromatic symmetric functions [13] in terms of induced $H_n(q)$ sign characters.

2 P -tableaux and induced S_n sign characters

Given a poset P , define a P -tableau of shape λ to be a Young diagram of shape λ whose boxes contain the elements of P . A P -tableau is called *column-strict* if whenever elements i_1, \dots, i_r appear from top to bottom in a column, then we have that $i_1 <_P \dots <_P i_r$ is a chain in P . A P -tableau is called *row-semistrict* if whenever i_1, i_2 appear consecutively (from left to right) in a row, then we have $i_1 <_P i_2$ or i_1 incomparable to i_2 in P . A P -tableau is called *semistandard* if it is both column-strict and row-semistrict. When P is simply the chain of integers $1 < \dots < n$, we will refer to a P -tableau as a *Young tableau*.

It is often convenient to label the elements of a poset and to encode the labeled poset as a certain 0-1 matrix. If a labeled poset P has n elements, we will always assume the labels to be $\{1, \dots, n\}$. We call P *naturally labeled* if whenever $i <_P j$, we have $i < j$ as integers. Every poset has at least one natural labeling. Given a labeled poset P , we define the *antiadjacency matrix* $A = A(P) = (a_{i,j})$ of P by

$$a_{i,j} = \begin{cases} 0 & \text{if } i <_P j, \\ 1 & \text{otherwise.} \end{cases} \tag{5}$$

Two different labelings of a poset give antiadjacency matrices which are conjugate to one another by a permutation matrix. If P is naturally labeled, then in its antiadjacency matrix, all entries on or below the diagonal are 1. We may use the antiadjacency matrix of P to count column-strict P -tableaux as follows. Call a sequence (I_1, \dots, I_r) of disjoint subsets of $[n]$ an *ordered set partition of type λ* if we have $|I_k| = \lambda_k$ for $k = 1, \dots, r$ and $I_1 \cup \dots \cup I_r = [n]$. For any $n \times n$ matrix A and any subset $I \subset [n]$, define $A_{I,I}$ to be the $|I| \times |I|$ submatrix $(a_{i,j})_{i,j \in I}$ of A .

Proposition 2.1 *Let P be a labeled poset with antiadjacency matrix A , and let λ be a partition of $n = |P|$. Then the number of column-strict P -tableaux is given by*

$$\sum_{(I_1, \dots, I_r)} \det(A_{I_1, I_1}) \cdots \det(A_{I_r, I_r}), \tag{6}$$

where the sum is over all ordered set partitions of $[n]$ of type λ .

Proof: Suppose that P is naturally labeled. It is straightforward to show that each minor in (6) satisfies

$$\det(A_{I_k, I_k}) = \begin{cases} 1 & \text{if } I_k \text{ is a chain in } P, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

and thus the claim is true in this case. (See, e.g., [16].) Furthermore, it is also straightforward to show that the polynomial function $\text{Mat}_{n \times n}(\mathbb{Z}) \rightarrow \mathbb{Z}$ defined by the expression (6) is invariant under conjugation by a permutation matrix. \square

This relationship between the antiadjacency matrix and P -tableaux allows us to state a relationship between induced sign characters of S_n and P -tableaux. To any labeled n -element poset P we may associate an element $\beta(P)$ of $\mathbb{Z}[S_n]$ by

$$\beta(P) = \sum_v v, \quad (8)$$

where the sum is over all permutations v satisfying $i \not\prec_P v_i$ for all i . Equivalently, $\beta(P)$ is a sum of permutations which correspond to the placement of n nonattacking rooks on entries of $A(P)$ which are equal to 1 [18]. Now we have the following [1, Cor. 4.2]. (See also [18, Eqn. (5.1)].)

Theorem 2.2 *Let P be a labeled poset and let λ be a partition of $|P|$. Then $\epsilon^\lambda(\beta(P))$ equals the number of column-strict P -tableaux of shape λ .*

Proof: Let $n = |P|$ and let $x = (x_{i,j})_{i,j \in [n]}$ be a matrix of n^2 variables. The Littlewood-Merris-Watkins identity for induced sign characters [10, Sec. 6.5], [11, Sec. 1] states that in $\mathbb{Z}[x_{1,1}, \dots, x_{n,n}]$ we have

$$\sum_{(I_1, \dots, I_r)} \det(x_{I_1, I_1}) \cdots \det(x_{I_r, I_r}) = \sum_{v \in S_n} \epsilon^\lambda(v) x_{1, v_1} \cdots x_{n, v_n}. \quad (9)$$

Now let A be the antiadjacency matrix of P and evaluate (9) at $x = A$. Since $a_{1, v_1} \cdots a_{n, v_n} = 1$ precisely when $i \not\prec_P v_i$ for all i , we obtain

$$\epsilon^\lambda \left(\sum_{v \in S_n} a_{1, v_1} \cdots a_{n, v_n} v \right) = \epsilon^\lambda(\beta(P)),$$

and our result follows from Proposition 2.1. \square

A similar argument using permanents of submatrices applies to the induced trivial characters of S_n .

Theorem 2.3 *Let P be a labeled poset and let λ be a partition of $|P|$. Then $\eta^\lambda(\beta(P))$ equals the number of row-semistrict P -tableaux of shape λ .*

Proof: Omitted. \square

3 $(3 + 1)$ -free posets and chromatic symmetric functions

To state more results and conjectures concerning the interpretation of S_n class functions, we restrict our attention to special classes of posets. The following standard definitions concerning posets may be found in [17, Ch. 3] and [21, Sec. 9.2]. Given posets P, Q , let $P \oplus Q$ be the $(|P| + |Q|)$ -element poset whose Hasse diagram is that for P written below that for Q , i.e., with every element of P being less than every element of Q . Letting \mathbf{n} denote an n -element chain $x_1 <_P \cdots <_P x_n$, we have, e.g., $\mathbf{4} \cong \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$. Let $P + Q$ be the $(|P| + |Q|)$ -element poset which is a disjoint union of P and Q , i.e., the poset whose Hasse diagrams is that for P written beside that for Q , with all elements of P being incomparable to all

elements of Q . Thus $\mathbf{a} + \mathbf{b}$ is the disjoint union of an a -element chain and a b -element chain. We say that P is $(\mathbf{a} + \mathbf{b})$ -free if no induced subposet of P is isomorphic to $\mathbf{a} + \mathbf{b}$. Call P an *interval order* if it is $(\mathbf{2} + \mathbf{2})$ -free, and a *unit interval order* if it is $(\mathbf{2} + \mathbf{2})$ -free and $(\mathbf{3} + \mathbf{1})$ -free.

While no true analog of Theorem 2.2 and 2.3 is known for $\chi^\lambda(\beta(P))$ and $\phi^\lambda(\beta(P))$, some results exist for special cases of λ and for P belonging to one of the above classes of posets.

Proposition 3.1 *Let P be a labeled $(\mathbf{3} + \mathbf{1})$ -free poset and let λ be a partition of $|P|$.*

- (1) $\chi^\lambda(\beta(P))$ equals the number of standard P -tableaux of shape λ [2, Thm. 2].
- (2) If P is a unit interval order and $\lambda_1 \leq 2$, then $\phi^\lambda(\beta(P))$ equals zero if there exists a column-strict P -tableaux of shape $\mu \prec \lambda$ in the dominance order; otherwise it equals the number of column-strict P -tableaux of shape λ [1, Thm. 5.7].
- (3) If P is a unit interval order and $\lambda = (k, \dots, k)$, then $\phi^\lambda(\beta(P))$ equals the number of ways to cover P with certain cycles [20, Thm. 2.8].

Furthermore, Stembridge’s [20, Conj. 2.1] has the following special case.

Conjecture 3.2 *Let P be a labeled unit interval order and let λ be a partition of $|P|$. Then we have $\phi^\lambda(\beta(P)) \geq 0$.*

Even with this restriction on P , there is no conjectured combinatorial interpretation for $\phi^\lambda(\beta(P))$ which applies to all λ .

One property of unit interval orders P which facilitates combinatorial interpretation is the fact that certain labelings of P cause the antiadjacency matrix $A(P)$ and the $\mathbb{Z}[S_n]$ element $\beta(P)$ to have particularly nice forms. Define the *altitude* of a poset element i to be $\alpha(i) = \#\{x \mid x <_P i\} - \#\{x \mid x >_P i\}$. Call a labeling an *altitude respecting* or *ar-labeling* if $i < j$ whenever $\alpha(i) < \alpha(j)$. It is known that every ar-labeling of a $(\mathbf{3} + \mathbf{1})$ -free poset is natural. It is also known that in the antiadjacency matrix corresponding to an ar-labeling of an n -element unit interval order, the zero entries form a right justified Young diagram which fits inside the right justified shape $(n - 1, \dots, 1)$. (See [21, Sec. 8.2].) In Proposition 4.1 we will prove that this special form of the antiadjacency matrix implies a nice expression for $\beta(P)$,

$$\beta(P) = \frac{C'_{[i_1, j_1]} \cdots C'_{[i_m, j_m]}}{k}, \quad i_1 < \cdots < i_m, \quad j_1 < \cdots < j_m, \quad k \in \mathbb{N}. \tag{10}$$





S_n class functions may be used to define Stanley’s *chromatic symmetric function* [15] for a poset P ,

$$X_P = \sum_{\lambda \vdash n} c_\lambda m_\lambda, \tag{11}$$

where c_λ is the number of ways to partition P into a sequence of r chains of sizes $(\lambda_1, \dots, \lambda_r)$ respectively, and to assign color k to the k th chain. Equivalently, we have $c_\lambda = \epsilon^\lambda(\beta(P))$. The transition matrices of Kostka numbers in (1) imply furthermore that we have

$$X_P = \sum_{\lambda \vdash n} \epsilon^\lambda(\beta(P)) m_\lambda = \sum_{\lambda \vdash n} \chi^{\lambda^\top}(\beta(P)) s_\lambda = \sum_{\lambda \vdash n} \phi^\lambda(\beta(P)) e_\lambda. \tag{12}$$

Monomial nonnegative by definition, the chromatic symmetric functions are sometimes Schur nonnegative and even elementary nonnegative as well. For example, the four posets $P = \mathbf{3}, \mathbf{2} + \mathbf{1}, (\mathbf{1} + \mathbf{1}) \oplus \mathbf{1}, \mathbf{1} + \mathbf{1} + \mathbf{1}$ and their chromatic symmetric functions X_P are

				(13)
$\begin{aligned} X_P &= 6m_{111} + 3m_{21} + m_3 \\ &= s_{111} + 2s_{21} + s_3 \\ &= e_{111}, \end{aligned}$	$\begin{aligned} X_P &= 6m_{111} + m_{21} \\ &= 4s_{111} + s_{21} \\ &= 3e_3 + e_{21}, \end{aligned}$	$\begin{aligned} X_P &= 6m_{111} + 2m_{21} \\ &= 2s_{111} + 2s_{21} \\ &= 2e_{21}, \end{aligned}$	$\begin{aligned} X_P &= 6m_{111} \\ &= 6s_{111} \\ &= 6e_3. \end{aligned}$	

These nonnegativity phenomena are addressed in the following result of Gasharov [2, Thm. 2] and conjecture of Stanley and Stembridge [15, Conj. 5.1], [18, Conj. 5.5].

Theorem 3.3 *If P is $(\mathbf{3} + \mathbf{1})$ -free, then X_P is Schur nonnegative.*

Conjecture 3.4 *If P is $(\mathbf{3} + \mathbf{1})$ -free, then X_P is elementary nonnegative.*

By (12), Theorem 3.3 is equivalent to Proposition 3.1 (1), Conjecture 3.4 is stronger than Conjecture 3.2, and two special cases of Conjecture 3.4 have been proved in Proposition 3.1 (2)-(3).

4 The Hecke algebra and quantum polynomial ring in n^2 variables

Let $H_n(q)$ be the (type A) Hecke algebra, the noncommutative $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -algebra with multiplicative identity $T_e = 1$, generated by $T_{s_1}, \dots, T_{s_{n-1}}$, subject to relations

$$\begin{aligned} T_{s_i}^2 &= (q - 1)T_{s_i} + qT_e, & \text{for } i = 1, \dots, n - 1, \\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j}, & \text{if } |i - j| = 1, \\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i}, & \text{if } |i - j| \geq 2. \end{aligned} \tag{14}$$

If $s_{i_1} \dots s_{i_\ell}$ is (any) reduced expression for $v \in S_n$ we define $T_v = T_{s_{i_1}} \dots T_{s_{i_\ell}}$. The $n!$ elements $\{T_v \mid v \in S_n\}$ are a basis of $H_n(q)$ as a $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -module.

Let χ_q^λ be the irreducible $H_n(q)$ character indexed by λ , and let ϵ_q^λ and η_q^λ be the sign and trivial characters induced from Young subalgebras of type λ . All of these $H_n(q)$ characters are $H_n(q)$ traces, i.e., they satisfy $f_q(gh) = f_q(hg)$ for $g, h \in H_n(q)$. Like a class function, a trace f_q satisfies $f_q(g^{-1}hg) = f_q(h)$ whenever g is invertible. Define $\phi_q^\lambda = \sum_{\mu} K_{\lambda, \mu}^{-1} \chi_q^\mu$ to be the monomial trace indexed by λ . These sets of traces form four bases for the space of $H_n(q)$ traces and are related to one another by the same transition matrices of Kostka numbers (1) that relate the S_n class function bases used in Sections 1-3:

$$\eta_q^\lambda = \sum_{\mu \vdash n} K_{\mu, \lambda} \chi_q^\mu, \quad \epsilon_q^\lambda = \sum_{\mu \vdash n} K_{\mu, \lambda} \chi_q^\mu, \quad \chi_q^\lambda = \sum_{\mu \vdash n} K_{\lambda, \mu} \phi_q^\mu. \tag{15}$$

(See, e.g., [3].) For none of these traces f_q do we have a known combinatorial interpretation of $f_q(T_v)$ for all $v \in S_n$. Neither do we have a known combinatorial interpretation of

$$f_q(C'_{[i_1, j_1]}(q) \cdots C'_{[i_m, j_m]}(q)), \tag{16}$$

where

$$C'_{[i,j]}(q) \stackrel{\text{def}}{=} q^{-\frac{\ell(w)}{2}} \sum_{v \in S_{[i,j]}} T_v,$$

and w is the longest element in $S_{[i,j]}$. We therefore consider special cases of the elements (16) which are quantum analogs of the elements $\beta(P)$ in $\mathbb{Z}[S_n]$. To any labeled n -element poset P we may associate an element $\beta_q(P)$ of $H_n(q)$ by

$$\beta_q(P) = \sum_{v \in S_n} T_v, \tag{17}$$

where the sum is over all permutations $v \in S_n$ satisfying $i \not\prec_P v_i$ for all i . The ar -labeling of a unit interval order P gives a nice expression for $\beta_q(P)$.

Proposition 4.1 *Let P be an ar -labeled n -element unit interval order. Then there exist subintervals $[i_1, j_1], \dots, [i_m, j_m]$ of $[n]$ with $i_1 < \dots < i_m, j_1 < \dots < j_m$, a polynomial $p \in \mathbb{N}[q]$, a 3412-, 4231-avoiding permutation $w = w(P)$ in S_n , and a nonnegative integer c such that*

$$\beta_q(P) = \frac{q^{\frac{\ell(w)+c}{2}} C'_{[i_1, j_1]}(q) \cdots C'_{[i_m, j_m]}(q)}{p(q)} = \sum_{v \leq w} T_v. \tag{18}$$

Proof: (Idea.) The n -element unit interval orders correspond bijectively to certain planar networks G having n source and n sink vertices, with each such network being composed of smaller networks indexed by intervals $[i_1, j_1], \dots, [i_m, j_m]$. For each such planar network G , one defines a matrix $B = (b_{i,j})$ by setting $b_{i,j} = 1$ if there exists a path from source i to sink j , and $b_{i,j} = 0$ otherwise. One also defines an element $\beta_q(G) = \sum_v b_{1,v_1} \cdots b_{n,v_n} T_v$ of $H_n(q)$. If in the above bijection G corresponds to P , then one can show that the antiadjacency matrix A of P satisfies $a_{1,v_1} \cdots a_{n,v_n} = b_{1,v_1} \cdots b_{n,v_n}$ for all $v \in S_n$ (although A and B are not in general equal). Thus we have $\beta_q(P) = \beta_q(G)$.

By [14, Thm. 4.3], there exist a certain nonnegative integer c and a 3412-, 4231-avoiding permutation $w \in S_n$ associated to G such that $q^{-\frac{\ell(w)}{2}} \beta_q(G)$ is equal to $q^{\frac{c}{2}} C'_{[i_1, j_1]}(q) \cdots C'_{[i_m, j_m]}(q)$ divided by a polynomial in $\mathbb{N}[q]$. By [14, Lem. 5.3], w also satisfies

$$b_{1,v_1} \cdots b_{n,v_n} = \begin{cases} 1 & \text{if } v \leq w \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Thus we have $\beta_q(P) = \beta_q(G) = \sum_{v \leq w} T_v$. □

For P an ar -labeled unit interval order, let $w(P)$ denote the permutation appearing in the above proposition. We remark that by results in [9], $\beta_q(P)$ is equal to $q^{\frac{\ell(w)}{2}}$ times the Kazhdan-Lusztig basis element $C'_{w(P)}(q)$ of $H_n(q)$ [7], and the factors $C'_{[i,j]}(q)$ also are Kazhdan-Lusztig basis elements.

Even for the special case (18) of (16), there is no published combinatorial interpretation for $f_q(\beta_q(P))$ when $f_q \in \{\chi_q^\lambda, \eta_q^\lambda, \epsilon_q^\lambda, \phi_q^\lambda\}$. On the other hand, Haiman's [6, Lem. 1.1, Conj. 2.1] have the following special cases which suggest that combinatorial interpretations for some of these expressions should exist.

Proposition 4.2 *Let P be an ar -labeled unit interval order and let λ be a partition of $|P|$. Then we have $\chi_q^\lambda(\beta_q(P)) \in \mathbb{N}[q]$, and therefore $\epsilon_q^\lambda(\beta_q(P)) \in \mathbb{N}[q]$ and $\eta_q^\lambda(\beta_q(P)) \in \mathbb{N}[q]$.*

Conjecture 4.3 *Let P be an ar-labeled unit interval order and let λ be a partition of $|P|$. Then we have $\phi_q^\lambda(\beta_q(P)) \in \mathbb{N}[q]$.*

In Section 5 we give a combinatorial interpretation for the polynomial $\epsilon_q^\lambda(q_{e,w}\beta_q(P))$ for P an ar-labeled unit interval order. To do this, we employ a quantum analog of the identity (9) which holds in the quantum polynomial ring $\mathcal{A}_n(q)$. Let $\mathcal{A}_n(q)$ be the noncommutative $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -algebra generated by n^2 variables $x = (x_{1,1}, \dots, x_{n,n})$, subject to the relations

$$\begin{aligned} x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{i,\ell}, & x_{j,k}x_{i,\ell} &= x_{i,\ell}x_{j,k}, \\ x_{j,k}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{j,k}, & x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})x_{i,\ell}x_{j,k}, \end{aligned} \tag{20}$$

for all indices $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$.

$\mathcal{A}_n(q)$ is a multigraded $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -module whose components $\mathcal{A}_{L,M}(q)$ are spanned by monomials having a fixed multiset L of row indices and a fixed multiset M of column indices. In particular, the component $\mathcal{A}_{[n],[n]}(q)$ contains monomials having total degree n , with all row indices and all column indices appearing exactly once. Defining $x^{u,v} = x_{u_1,v_1} \cdots x_{u_n,v_n}$ for $u, v \in S_n$, we thus have

$$\mathcal{A}_{[n],[n]}(q) = \text{span}\{x^{u,v} \mid u, v \in S_n\}.$$

By the relations (20), each multigraded component has a $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -basis of monomials in which variables appear in lexicographic order. Thus the component $\mathcal{A}_{[n],[n]}(q)$ has the natural basis $\{x^{e,v} \mid v \in S_n\}$, and contains the quantum determinant, defined by

$$\det_q(x) \stackrel{\text{def}}{=} \sum_{v \in S_n} (-q^{-\frac{1}{2}})^{\ell(v)} x^{e,v}.$$

The quantized Littlewood-Merris-Watkins identity, due to Konvalinka and the second author [8, Thm. 5.4], gives generating functions for the induced sign characters of $H_n(q)$.

Theorem 4.4 *Fix any partition $\lambda \vdash n$. Then in $\mathcal{A}_{[n],[n]}(q)$ we have the identity*

$$\sum_{(I_1, \dots, I_r)} \det_q(x_{I_1, I_1}) \cdots \det_q(x_{I_r, I_r}) = \sum_{v \in S_n} \epsilon_q^\lambda(q^{-\frac{\ell(v)}{2}} T_v) x^{e,v}, \tag{21}$$

where the first sum is over all ordered set partitions of $[n]$ of type λ .

5 Main results

Recall that for P a unit interval order with antiadjacency matrix A , the combinatorial interpretation of $\beta(P)$ in Theorem 2.2 follows immediately from the evaluation of both sides of the classical Littlewood-Merris-Watkins identity (9) at $x = A$. However, the quantum analog (21) of this identity does not allow us to compute $\epsilon^\lambda(\beta_q(P))$ as the evaluation of the left-hand side at $x = A$, since the noncommuting variables $(x_{i,j})$ satisfy nontrivial relations. If we are to use the left-hand side of (21) to understand $\epsilon^\lambda(\beta_q(P))$, we must first express it in terms of the natural basis $\{x^{e,v} \mid v \in S_n\}$. We therefore define a family of maps as follows. For each matrix $A \in \text{Mat}_{n \times n}(\mathbb{Z})$, let $\sigma_A : \mathcal{A}_{[n],[n]}(q) \rightarrow \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ be the $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -linear map defined by

$$\sigma_A(x^{e,v}) = q^{\frac{\ell(v)}{2}} a_{1,v_1} \cdots a_{n,v_n}.$$

This definition has the following immediate consequence.

Proposition 5.1 *Let P be an ar-labeled poset with antiadjacency matrix A , let $w = w(P)$, and let λ be a partition of $n = |P|$. Then we have*

$$\sum_{(I_1, \dots, I_r)} \sigma_A(\det_q(x_{I_1, I_1}) \cdots \det_q(x_{I_r, I_r})) = \epsilon_q^\lambda(\beta_q(P)), \tag{22}$$

where the sum is over all ordered set partitions of $[n]$ of type λ .

Proof: The left-hand side of (22) equals

$$\sigma_A\left(\sum_{v \in S_n} \epsilon_q^\lambda(q^{-\frac{\ell(v)}{2}} T_v) x^{e,v}\right) = \sum_{v \in S_n} \epsilon_q^\lambda(T_v) q^{-\frac{\ell(v)}{2}} \sigma_A(x^{e,v}) = \epsilon_q^\lambda\left(\sum_{v \leq w} T_v\right) = \epsilon_q^\lambda(\beta_q(P)),$$

since the proof of Proposition 4.1 implies that $a_{1,v_1} \cdots a_{n,v_n}$ is 1 if $v \leq w$ and is 0 otherwise. □

To combinatorially interpret each term on the left-hand side of (22), we introduce the following statistic on P -tableaux. Given a P -tableau U , define an *inversion* in U to be a pair (i, j) of incomparable elements in P satisfying $i < j$, with i appearing in a column of U to the right of the column containing j . Let $\text{INV}(U)$ denote the number of inversions in U .

Lemma 5.2 *Let P be a labeled unit interval order with antiadjacency matrix A , let λ be a partition of $n = |P|$, let (I_1, \dots, I_r) be an ordered set partition of $[n]$ of type λ , and let U be the P -tableau whose j th column contains the poset elements labeled by I_j , with labels increasing upward in that column. If U is a column-strict P -tableau, then we have*

$$\sigma_A(\det_q(x_{I_1, I_1}) \cdots \det_q(x_{I_r, I_r})) = q^{\text{INV}(U)}. \tag{23}$$

Proof: Omitted. □

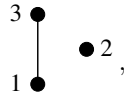
The authors believe that if we keep the hypotheses of Lemma 5.2, but assume that the P -tableau U is *not* column-strict, then $\sigma_A(\det_q(x_{I_1, I_1}) \cdots \det_q(x_{I_r, I_r}))$ vanishes. In light of Proposition 5.1, we thus have the following conjecture.

Conjecture 5.3 *Let P be an ar-labeled unit interval order, and let λ be a partition of $|P|$. Then we have*

$$\epsilon_q^\lambda(\beta_q(P)) = \sum_U q^{\text{INV}(U)}, \tag{24}$$

where the sum is over all column-strict P -tableaux of shape λ^\top .

For example, let P be the poset $2 + 1$ with its unique ar-labeling



and its two pairs of incomparable elements $(1, 2)$ and $(2, 3)$. Then there are six column-strict P -tableaux of shape $111^\top = 3$, one of shape $21^\top = 21$, and none of shape $3^\top = 111$. Enumerating these and indicating inversions by arrows, we have

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & 1 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & 3 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 3 & 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 3 & 2 \\ \hline 1 & 2 \\ \hline \end{array}. \tag{25}$$

Comparing these tableaux with the known evaluations

$$\epsilon_q^{111}(\beta_q(P)) = 1 + 4q + q^2, \quad \epsilon_q^{21}(\beta_q(P)) = q, \quad \epsilon_q^3(\beta_q(P)) = 0, \tag{26}$$

we see that Conjecture 5.3 holds in this case.

Shareshian and Wachs [13] defined a quantum analog of Stanley’s chromatic symmetric function. Given a labeled poset P , their *quantum chromatic quasisymmetric function* for P is the sum

$$X_{P,q} = \sum_{\kappa} q^{\#\{(i,j) \mid i \not\leq_P j, j \not\leq_P i, i < j, \kappa(i) > \kappa(j)\}} \prod_{k \in \mathbb{P}} x_k^{\#\{i \in P \mid \kappa(i) = k\}} \tag{27}$$

over all colorings $\kappa : P \rightarrow \mathbb{N}$ of P , where a function κ is termed a *coloring* if it satisfies $\kappa(i) \neq \kappa(j)$ whenever i and j are incomparable in P . By [13, Prop. 4.4], when P is an n -element *ar*-labeled unit interval order then $X_{P,q}$ is in fact symmetric,

$$X_{P,q} = \sum_{\lambda \vdash n} c_{\lambda,q} m_{\lambda}, \tag{28}$$

and it follows that $c_{\lambda,q} = \sum_U q^{\text{INV}(U)}$, where the sum is over all column-strict P -tableaux of shape λ^{\top} . For example, when the four unit interval orders in (13) are *ar*-labeled, their quantum chromatic symmetric functions are

$$\begin{aligned} X_{\mathbf{3},q} &= 6m_{111} + 3m_{21} + m_3 & X_{\mathbf{2+1},q} &= (1 + 4q + q^2)m_{111} + qm_{21} \\ &= s_{111} + 2s_{21} + s_3 & &= (1 + 2q + q^2)s_{111} + qs_{21} \\ &= e_{111}, & &= (1 + q + q^2)e_3 + qe_{21}, \end{aligned}$$

$$\begin{aligned} X_{(\mathbf{1+1}) \oplus \mathbf{1},q} &= (3 + 3q)m_{111} + (1 + q)m_{21} & X_{\mathbf{1+1+1},q} &= (1 + 2q + 2q^2 + q^3)m_{111} \\ &= (1 + q)s_{111} + (1 + q)s_{21} & &= (1 + 2q + 2q^2 + q^3)s_{111} \\ &= (1 + q)e_{21}, & &= (1 + 2q + 2q^2 + q^3)e_3. \end{aligned}$$

Now Conjecture 5.3 and Equation (15) imply the following.

Conjecture 5.4 *If P is an n -element ar-labeled unit interval order then the Shareshian-Wachs quantum chromatic symmetric function satisfies*

$$X_{P,q} = \sum_{\lambda \vdash n} \epsilon_q^{\lambda}(\beta_q(P)) m_{\lambda} = \sum_{\lambda \vdash n} \chi_q^{\lambda^{\top}}(\beta_q(P)) s_{\lambda} = \sum_{\lambda \vdash n} \phi_q^{\lambda}(\beta_q(P)) e_{\lambda}.$$

Shareshian and Wachs have formulated the following quantum analog [13, Conj. 4.8] of Stanley and Stembridge’s [15, Conj. 5.1] ([18, Conj. 5.5]).

Conjecture 5.5 *If P is an ar-labeled unit interval order, then the coefficients arising in the elementary expansion of $X_{P,q}$ belong to $\mathbb{N}[q]$.*

By Conjecture 5.4, the authors believe this to be equivalent to the special case of Haiman’s [6, Conj. 2.1] which we have stated in Conjecture 4.3. Furthermore, Shareshian and Wachs have proved the following weaker result [13, Thm. 4.11], which is a quantum analog of the special case of Gasharov’s [2, Thm. 2] stated in Theorem 3.3.

Theorem 5.6 *If P is an ar-labeled unit interval order, then the coefficient of s_λ in the Schur expansion of $X_{P,q}$ is $\sum_U q^{\text{INV}(U)}$, where the sum is over all semistandard P -tableaux of shape λ^\top .*

This coincides with the authors' conjectured interpretation of the irreducible $H_n(q)$ characters.

Conjecture 5.7 *If P is an ar-labeled unit interval order and $w = w(P)$, then $\chi_q^\lambda(\beta_q(P)) = \sum_U q^{\text{INV}(U)}$, where the sum is over all semistandard P -tableaux of shape λ^\top .*

Since Theorem 3.3 and Conjecture 3.4 apply to $(\mathbf{3} + \mathbf{1})$ -free posets, it would be interesting to see if the symmetry of $X_{P,q}$ holds more generally for ar-labeled $(\mathbf{3} + \mathbf{1})$ -free posets, and to interpret $\epsilon_q^\lambda(\beta_q(P))$, $\eta_q^\lambda(\beta_q(P))$, and $\chi_q^\lambda(\beta_q(P))$ in these cases.

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