On a Unimodality Conjecture in Matroid Theory
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A certain unimodal conjecture in matroid theory states the number of rank-r matroids on a set of size n is unimodal in r and attains its maximum at \( r = \lfloor n/2 \rfloor \). We show that this conjecture holds up to \( r = 3 \) by constructing a map from a class of rank-2 matroids into the class of loopless rank-3 matroids. Similar inequalities are proven for the number of non-isomorphic loopless matroids, loopless matroids and matroids.

Keywords: Matroid Theory, Unimodal Conjecture, Rank-2 matroids, Rank-3 matroids

1 Introduction

Let us first recall some elementary definitions (further details may be found in Oxley (1), Welsh (2) and the excellent appendix of White (3)). Let \( S_n \) be a finite set of size n. A matroid \( M \) on the ground set \( S_n \) is a collection of subsets \( I(M) \) of \( S_n \) satisfying

- \( I(M) \neq \emptyset \),
- if \( X \in I(M) \) and \( Y \subseteq X \) then \( Y \in I(M) \),
- if \( X, Y \in I(M) \) with \( |X| = |Y| + 1 \), then there exists \( x \in X \setminus Y \) such that \( Y \cup \{x\} \in I(M) \).

Sets in \( I(M) \) are called independent sets. The rank of a set \( X \subseteq S_n \), denoted \( r(X) \), is the size of the largest independent set which it contains. The rank of the matroid \( r(M) := r(S_n) \). A set \( X \) is closed (or termed a flat) if \( r(X \cup \{x\}) = r(X) + 1 \) for all \( x \in S_n \setminus X \). We denote by \( F(M) \) the closed sets of \( M \). The loops of \( M \) are the elements of the rank-0 flat. Also note that \( x \in S_n \) is a loop if it is not contained in any of the independent sets of \( M \).

A certain unimodal conjecture in matroid theory states that the sequence of the number of non-isomorphic rank-r matroids on \( S_n \), \( \{f_r(n) : 1 \leq r \leq n\} \), is unimodal in \( r \) and attains its maximum at \( r = \lfloor n/2 \rfloor \) (see Oxley (1) or Welsh (2) p.300). It is easily seen that \( f_1(n) \leq f_2(n) \) holds since \( f_1(n) = n \) and \( f_2(n) = p(1) + \cdots + p(n) - n \), where \( p(n) \) is the number of integer partitions of \( n \). The step between rank-2 and rank-3 is not as clear since the exact value of \( f_3(n) \) remains unknown. We show, through construction of a map between a class of rank-2 matroids and loopless rank-3 matroids and known values of these numbers from the On–line Encyclopedia of Integer Sequences, that this unimodal conjecture holds for
these rank-2 versus rank-3 matroids. Furthermore, we show the corresponding inequalities hold for the number of rank-2, 3 non-isomorphic loopless matroids, \( g_2(n) \leq g_3(n) \), loopless matroids, \( c_2(n) \leq c_3(n) \), and matroids, \( m_2(n) < m_3(n) \).

Let \( b_i(n) \) be the number of partitions of the set \( S_n \) into \( i \) parts and \( b(n) \) be the \( n^\text{th} \) Bell number. Let \( p_i(n) \) the number of partitions of the integer \( n \) into \( i \) parts. The number of rank-2 matroids can be enumerated through considering the points and lines of the associated geometry. We have \( c_2(n) = b(n) - 1 \) and \( m_2(n) = b(n+1) - 2^n \) (for proofs see Dukes [3]). The main results of this paper are given in Theorems 2.3, 2.6, 2.11 and 2.12.

2 Mapping rank-2 to rank-3 matroids

Let \( \mathcal{M}_r(n) \) be the collection of rank-\( r \) matroids on \( S_n \). Let \( \mathcal{A}_r(n) \) be the collection of rank-\( r \) matroids on \( S_n \) with at least one loop and \( \mathcal{B}_r(n) := \mathcal{M}_r(n) \setminus \mathcal{A}_r(n) \). We define the map \( \sigma : \mathcal{A}_2(n) \to \mathcal{B}_3(n) \) as follows: given \( M \in \mathcal{A}_2(n) \) with loops \( F_0 \) and rank-1 flats \( F_j(M) = \{ F_0 \cup F_1, \ldots, F_0 \cup F_m \} \) define \( M' = \sigma(M) \) as:

\[
\begin{align*}
\mathcal{F}_0(M') & := \{ \emptyset \} \\
\mathcal{F}_1(M') & := \{ F_0, F_1, \ldots, F_m \} \\
\mathcal{F}_2(M') & := \{ F_0 \cup F_i | 1 \leq i \leq m \} \cup \{ F_1 \cup \cdots \cup F_m \} \\
\mathcal{F}_3(M') & := \{ S_n \}.
\end{align*}
\]

It is easily checked that these collections of flats satisfy the axioms for a loopless rank-3 matroid. For \( M \in \mathcal{M}_r(n) \), let us write \( d(M) \) for the number of rank-1 flats of \( M \) (which we will refer to as the degree of \( M \)). Let us mention that for any loopless matroid \( M \), the rank-1 flats of \( M \) partition the ground set. Similarly, for any matroid, the rank-1 flats partition the ground set less the set of loops. Also note that in the collection \( \mathcal{F}_2(M') \), there are precisely \( d(M) \) sets containing \( F_0 \), 2 sets containing \( F_i \) (for any \( 1 \leq i \leq d(M) \)) and one set containing \( F_i \cup F_j \) (for all \( 0 \leq i \neq j \leq d(M) \)).

The following lemma shows that to each rank-2 matroid with at least one loop, there corresponds a rank-3 loopless matroid (although not necessarily unique). The following lemma classifies those matroids which map to a unique loopless matroid in \( \mathcal{B}_3(n) \) and those which do not.

**Lemma 2.1** Let \( M_1, M_2 \in \mathcal{A}_2(n) \) be such that \( \mathcal{F}_0(M_1) = \{ F_0^{(1)} \} \), \( \mathcal{F}_0(M_2) = \{ F_0^{(2)} \} \), \( \mathcal{F}_1(M_1) = \{ F_0^{(1)} \cup F_1^{(1)} \cup \cdots \cup F_m^{(1)} \} \), and \( \mathcal{F}_1(M_2) = \{ F_0^{(2)} \cup F_1^{(2)} \cup \cdots \cup F_m^{(2)} \} \). Then \( \sigma(M_1) = \sigma(M_2) \) if and only if \( d(M_1) = d(M_2) = 2 \) and

\[
\begin{align*}
\{ F_0^{(1)}, F_1^{(1)}, F_2^{(1)} \} & = \{ F_0^{(2)}, F_1^{(2)}, F_2^{(2)} \}.
\end{align*}
\]

**Proof:** ONLY IF: Let \( M_1, M_2 \in \mathcal{A}_2(n) \) be such that \( M_1 \neq M_2 \) and \( \sigma(M_1) = \sigma(M_2) \). Let \( M'_1 := \sigma(M_1) \) and \( M'_2 := \sigma(M_2) \). Then we must have \( \mathcal{F}_1(M'_1) = \mathcal{F}_1(M'_2) \) and \( \mathcal{F}_2(M'_1) = \mathcal{F}_2(M'_2) \). Now \( \mathcal{F}_1(M'_1) = \mathcal{F}_1(M'_2) \Rightarrow d(M_1) = d(M_2) \) and \( \mathcal{F}_0^{(1)} \cup \mathcal{F}_0^{(2)} \cup \mathcal{F}_0^{(3)} \Rightarrow d(M_1) > 2 \) then we must have \( \mathcal{F}_0^{(1)} = \mathcal{F}_0^{(2)} \) which would imply \( M_1 = M_2 \). Hence \( d(M_1) = 2 = d(M_2) \). This gives \( \mathcal{F}_2(M'_1) = \{ F_0^{(1)} \cup F_1^{(1)} \cup F_2^{(1)} \} = \mathcal{F}_2(M'_2) \) only if \( \{ F_0^{(1)}, F_1^{(1)}, F_2^{(1)} \} = \{ F_0^{(2)}, F_1^{(2)}, F_2^{(2)} \} \).
IF: This is trivial as \( \{F_0^{(1)}, F_1^{(1)}, F_2^{(1)}\} = \{F_0^{(2)}, F_1^{(2)}, F_2^{(2)}\} \) gives \( \mathcal{F}_1(M'_1) = \mathcal{F}_1(M'_2) \) and

\[
\mathcal{F}_2(M'_1) := \left\{ F_0^{(1)} \cup F_1^{(1)}, F_0^{(1)} \cup F_1^{(1)}, F_0^{(1)} \cup F_1^{(1)} \right\}
= \left\{ F_0^{(2)} \cup F_1^{(2)}, F_0^{(2)} \cup F_1^{(2)}, F_0^{(2)} \cup F_1^{(2)} \right\} =: \mathcal{F}_2(M'_2).
\]

Thus it is seen for each matroid \( M \in \sigma(\mathcal{A}_2(n)) \) such that \( d(M) = 2 \), there are precisely three different matroids \( M_1, M_2, M_3 \in \mathcal{A}_2(n) \) such that \( \sigma(M_1) = \sigma(M_2) = \sigma(M_3) = M \).

**Lemma 2.2** For all \( n \geq 3 \), \( c_3(n) \geq b(n + 1) - b(n) - 3^{n-1} \).

**Proof:** We show that the number of unique matroids in the image of \( \mathcal{A}_2(n) \) under \( \sigma \) is given by \( b(n + 1) - b(n) - 3^{n-1} \), thereby lower-bounding \( c_3(n) \). In the enumeration below, we divide the matroids to be counted into three classes, those matroids \( M \) with \( d(M) = 2 \) and those with \( d(M) > 2 \). The former class projects different matroids to the same matroid in \( \mathcal{B}_3(n) \) and through the use of the previous lemma we take care of this over-counting, hence

\[
\# \{ \sigma(M) | M \in \mathcal{A}_2(n) \} = \# \{ \sigma(M) | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2 \} + \sum_{i=3}^{n} \# \{ \sigma(M) | M \in \mathcal{A}_2(n) \text{ and } d(M) = i \}
= \frac{1}{3} \# \{ M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2 \} + \sum_{i=3}^{n} \# \{ \sigma(M) | M \in \mathcal{A}_2(n) \text{ and } d(M) = i \}
= \sum_{i=2}^{n} \# \{ \sigma(M) | M \in \mathcal{A}_2(n) \text{ and } d(M) = i \} - \frac{2}{3} \# \{ M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2 \}
= \# \mathcal{A}_2(n) - \frac{2}{3} \# \{ M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2 \}
= b(n + 1) - 2^n - (b(n) - 1) - \frac{2}{3} \# \{ M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2 \}.
\]

Note that \( \# \{ M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2 \} = \sum_{i=2}^{n-2} \binom{n}{i} b_2(i) \) and \( b_2(l) = \frac{1}{2} \sum_{j=1}^{l-1} \binom{l}{j} = 2^{l-1} - 1 \), giving:

\[
\# \{ M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2 \} = \sum_{i=2}^{n-2} \binom{n}{i} (2^{i-1} - 1)
= \frac{1}{2} (3^n - 2^n - 2n - 1) - (2^n - n - 2)
= \frac{3}{2} (3^{n-1} - 2^n + 1).
\]

Thus

\[
\# \{ \sigma(M) | M \in \mathcal{A}_2(n) \} = b(n + 1) - 2^n - (b(n) + 1) - \frac{2}{3} \frac{3}{2} (3^{n-1} - 2^n + 1)
= b(n + 1) - b(n) - 3^{n-1}.
\]
The corresponding inequality for the number of non-isomorphic loopless matroids is proved in Lemma 2.3. We do this in a similar manner as before, by showing that each rank-2 matroid (which is not a loopless matroid) of degree greater than 3 corresponds uniquely to a rank-3 loopless matroid.

**Lemma 2.3** For all \( n \geq 4 \), \( g_3(n) \geq \sum_{i=1}^{n-1} p(i) - \frac{1}{12} (2n^2 + 6n + 3) - 1 \).

**Proof:** We show the number of non-isomorphic matroids in the image of \( A_2 \) under \( \sigma \) is given by \( \sum_{i=1}^{n-1} p(i) - \frac{1}{12} (2n^2 + 6n + 3) - 1 \) which lower bounds \( g_3(n) \).

Let us identify \( A \star_2(n) \subseteq A_2(n) \) by placing an ordering on the elements of \( S_n = \{x_1, \ldots, x_n\} \). Given \( M \in A_2(n) \) with \( d(M) = m \), loops \( F_0 \) and rank-1 flats \( \{F_0 \cup F_1, \ldots, F_0 \cup F_m\} \), let \( M \in A \star_2(n) \) if and only if \( F_0 = \{x_1, \ldots, x_{|F_0|}\} \), \( F_1 \) contains the next \( |F_1| \) elements of \( S_n \), i.e. \( \{x_{|F_0|+1}, \ldots, x_{|F_0|+|F_1|}\} \) and so forth. Define

\[
T(n) := \left\{ M \in A_2(n) \mid d(M) = 2 \text{ and } |F_0| \leq |F_1| \leq |F_2| \right\}
\]

and for \( 3 \leq i \leq n-1, \ 3 \leq j \leq i \), define

\[
\Omega_{i,j}(n) := \left\{ M \in A_2(n) \mid d(M) = j, |F_0| = n-i \text{ and } |F_1| \leq \cdots \leq |F_j| \right\}.
\]

Let us now write

\[
A_2^{**}(n) := T(n) \cup \bigcup_{i=3}^{n-1} \bigcup_{j=3}^{i} \Omega_{i,j}(n) \subseteq A_2^*(n).
\]

It is obvious that no two matroids in \( T(n) \) are isomorphic to one-another. Similarly with \( \Omega_{i,j}(n) \). We have simply reduced our class of matroids from \( A_2(n) \) to \( A_2^{**}(n) \) in the same manner as one moves from the set of partitions of a finite set of size \( n \) to the set of integer partitions of \( n \).

The unions in the definition of \( A_2^{**}(n) \) are strictly disjoint and no isomorphisms may occur between matroids in different classes or matroids in the same class. The same is true of the image of \( A_2^{**}(n) \) under the map \( \sigma \). We may directly enumerate the number of non-isomorphic matroids in \( B_3(n) \) in the image of \( A_2^{**}(n) \) under \( \sigma \) as

\[
p_3(n) + \sum_{i=3}^{n-1} \sum_{j=3}^{i} p_j(i).
\]
The rightmost term is bounded below;

\[
\sum_{i=3}^{n-1} \sum_{j=3}^{i} p_j(i) = \sum_{i=3}^{n-1} \{ p(i) - p_1(i) - p_2(i) \} \\
= \sum_{i=3}^{n-1} \{ p(i) - 1 - \lfloor i/2 \rfloor \} \\
= -(n-3) + \sum_{i=3}^{n-1} p(i) - \sum_{i=3}^{n-1} \lfloor i/2 \rfloor \\
\geq \sum_{i=1}^{n-1} p(i) - \frac{(n+1)^2}{4},
\]

for all \( n \geq 2 \). As for \( p_3(n) \), from Hall \((\text{F})\) [p.32], we have

\[
p_3(n) = \begin{cases} 
\lfloor n^2/12 \rfloor, & \text{for } n \not\equiv 3 \pmod{6}, \\
\lceil n^2/12 \rceil, & \text{for } n \equiv 3 \pmod{6}, 
\end{cases}
\]

and so

\[
g_3(n) \geq \sum_{i=1}^{n-1} p(i) + \frac{n^2}{12} - \frac{(n+1)^2}{4} - 1 \\
= \sum_{i=1}^{n-1} p(i) - \frac{1}{12} (2n^2 + 6n + 3) - 1.
\]

2.1 Matroids

The following lemma is needed in order to support the theorem which follows it.

**Lemma 2.4** For all \( n \geq 1 \), \( b(n+1) - 2^n \geq 2^n - (1 + n) \).

**Proof:** We have that \( b(i) \geq 2 \) for all \( i \geq 2 \). Since \( n \geq 2 \), it follows that

\[
b(n+1) - 2^n = \sum_{i=0}^{n} \binom{n}{i} (b(i) - 1) \\
\geq \sum_{i=2}^{n} \binom{n}{i} 1 \\
= 2^n - (1 + n).
\]

\( \square \)
Theorem 2.5 For all \( n > 4 \), \( c_3(n) \geq c_2(n) \).

Proof: From Lemma 2.2 we have that \( c_3(n) \geq b(n + 1) - b(n) - 3^{n-1} \). We know \( c_2(n) = b(n) - 1 \). It suffices to show that

\[ b(n + 1) - b(n) - 3^{n-1} \geq b(n) - 1 \]

for all \( n > 4 \). Let us look at the value \( b(n + 1) - 2b(n) - 3^{n-1} + 2^{n-1} \):

\[
\begin{align*}
b(n + 1) - 2b(n) - 3^{n-1} + 2^{n-1} &= \sum_{i=0}^{n-1} \binom{n}{i} b(i) - \sum_{i=0}^{n-1} \binom{n-1}{i} b(i) - \sum_{i=0}^{n-1} \binom{n-1}{i} 2^i + \sum_{i=0}^{n-1} \binom{n-1}{i} \\
&= \sum_{i=0}^{n-2} \binom{n-1}{i} (b(i + 1) - 2^i)
\end{align*}
\]

and using Lemma 2.4,

\[
\begin{align*}
\geq \sum_{i=1}^{n-2} \binom{n-1}{i} (2^i - (1 + i)) \\
&= 3^{n-1} - 2^n + 1 - \sum_{i=1}^{n-2} \binom{n-1}{i} i \\
&= 3^{n-1} - (n + 3)2^{n-2} + n.
\end{align*}
\]

The problem has been reduced to showing \( 3^{n-1} - (n + 3)2^{n-2} + n \geq 2^{n-1} - 1 \) for all \( n \geq 5 \), which is easily shown by induction.

We now show the number of rank-3 matroids dominates the number of rank-2 matroids by using two things: the first is the result proved previously, that the number of rank-3 loopless matroids is at least as large as the number of rank-2 loopless matroids; the second is the first few known values of the numbers \( c_2(n) \) and \( c_3(n) \). The latter knowledge makes the inequality strict.

Theorem 2.6 For all \( n \geq 5 \), \( m_3(n) \geq m_2(n) \).

Proof: The number of rank-\( r \) matroids on \( S_n \) is related to the number of loopless matroids on \( S_n \) by

\[ m_r(n) = \sum_{i=r}^{n} \binom{n}{i} c_r(i). \]

In Theorem 2.5 we showed that \( c_3(n) \geq c_2(n) \) for all \( n \geq 5 \). Replacing \( r = 3 \) in the above expression and using the first few values of \( c_3(n) \) (taken from row 3, table A058710, of Sloane 186),

\[
\begin{align*}
m_3(n) &= \sum_{i=3}^{n} \binom{n}{i} c_3(i) \\
&= 1 \binom{n}{3} + 11 \binom{n}{4} + 106 \binom{n}{5} + 1232 \binom{n}{6} + \sum_{i=7}^{n} \binom{n}{i} c_3(i) \\
&\geq 1 \binom{n}{3} + 11 \binom{n}{4} + 106 \binom{n}{5} + 1232 \binom{n}{6} + \sum_{i=7}^{n} \binom{n}{i} c_2(i) \\
&= 830 \binom{n}{6} + 75 \binom{n}{5} - 3 \binom{n}{4} - 3 \binom{n}{3} - \binom{n}{2} + \sum_{i=2}^{n} \binom{n}{i} c_2(i)
\end{align*}
\]

It
A simple check shows that 

\[ 830 \binom{n}{6} + 75 \binom{n}{5} - 3 \binom{n}{4} - 3 \binom{n}{3} - \binom{n}{2} \]

is greater than zero and increasing for all \( n \geq 7 \). From Table 1 (see Appendix), the result is also seen to hold for \( n = 5, 6 \). Equality holds only for \( n = 5 \), for all other values of \( n \) the inequality is strict.  

\[ \square \]

### 2.2 Non-isomorphic matroids

Proving the corresponding inequalities for the non-isomorphic numbers is more difficult. We first prove several lemmas related to the numbers \( p(n) \) which we will need in the proofs of the two remaining theorems.

**Lemma 2.7** For all \( n \geq 1 \), \( p(n + 1) \geq p(n) + \left\lfloor \frac{n+1}{2} \right\rfloor \).

**Proof:** The number of partitions of the integer \( n + 1 \) whose first part contains the integer 1 is precisely \( p(n) \). The number beginning with \( i \), for any \( 2 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor \) is at least 1 since we can have the partition \( n + 1 = i + (n + 1 - i) \). Also, the number \( n + 1 \) is a partition by itself, hence,

\[
p(n + 1) \geq p(n) + \left\lfloor \frac{n+1}{2} \right\rfloor - 1 + 1 = p(n) + \left\lfloor \frac{n+1}{2} \right\rfloor.
\]

\[ \square \]

**Lemma 2.8** For all \( n \geq 1 \), \( p(n) \geq 1 + \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \geq \frac{n^2 + 3}{4} \).

**Proof:** From Lemma 2.7 we have

\[
p(n + 1) \geq p(n) + \left\lfloor \frac{n+1}{2} \right\rfloor
\]

for all \( n \geq 1 \). Applying this lemma recursively gives

\[
p(n) \geq p(n - 1) + \left\lfloor \frac{n}{2} \right\rfloor
\]

\[
\geq p(n - 2) + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor
\]

\[
\vdots
\]

\[
\geq p(1) + \left\lfloor \frac{1+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor
\]

\[
\geq 1 + \left\lfloor \frac{1+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor. \quad (1)
\]

Now we wish to evaluate the sum \( \sum_{i=2}^{n} \left\lfloor \frac{i}{2} \right\rfloor \). Let \( n = 2m + 1 \) for some \( m \geq 1 \), then

\[
\sum_{i=2}^{n} \left\lfloor \frac{i}{2} \right\rfloor = \sum_{i=2}^{2m+1} \left\lfloor \frac{i}{2} \right\rfloor
\]

\[
= \sum_{i=1}^{m} \left\lfloor \frac{2i}{2} \right\rfloor + \left\lfloor \frac{2i+1}{2} \right\rfloor
\]

\[
= \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil.
\]
For the \( n = 2m \) case with \( m \geq 1 \), we simply remove the last term in the previous expression, thus

\[
\sum_{i=2}^{n} \left\lfloor \frac{i}{2} \right\rfloor = \sum_{i=2}^{2m+1} \left\lfloor \frac{i}{2} \right\rfloor - \left\lfloor \frac{2m+1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil.
\]

Continuing to the inequality in Equation 1 above,

\[
p(n) \geq 1 + \left\lfloor \frac{1 + 1}{2} \right\rfloor + \cdots + \left\lfloor \frac{n - 1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor
= 1 + \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil,
\]

for all \( n \geq 1 \). If \( n \) is even, then \( \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil = \frac{n^2}{4} \). If \( n \) is odd, then \( \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil = \frac{n-1}{2} \frac{n+1}{2} \). In either case, \( 1 + \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \geq 1 + \frac{n^2-1}{4} \).

\[\square\]

**Lemma 2.9** For all \( n \geq 5 \), \( p(n+1) < 2p(n) - \frac{n+2}{3} \).

**Proof:** Let \( x_1 + x_2 + \ldots + x_k = n + 1 \) be a partition of \( n + 1 \) with \( 1 \leq x_1 < \ldots < x_k \). There are precisely \( p(n) \) partitions with \( x_1 = 1 \), since \( x_2 + \ldots + x_k = n + 1 - 1 \).

For all those partitions with \( x_1 \geq 2 \), we see that reducing \( x_1 \) by 1 will yield a partition of \( n \). Thus an upper bound on the number beginning with \( x_2 \geq 2 \) is \( p(n) \). For all partitions starting with \( x_1 = 2 \), we see that \( x_2 \neq 1 \), thus we may remove all those sequences with \( x_2 = 1 \leq x_3 \leq \cdots \leq x_k \) such that \( 2 + 1 + x_3 + \ldots + x_k = n + 1 \). Reformulated, this means all those partitions with \( x_3 + \ldots + x_k = n - 2 \) and \( 1 \leq x_3 \leq \cdots \leq x_k \) of which there are \( p(n-2) \).

Thus we see that \( p(n + 1) < p(n) + p(n) - p(n - 2) = 2p(n) - p(n - 2) \). From lemma 2.8 we know that for \( n \geq 3 \),

\[
p(n-2) \geq \frac{(n-2)^2 + 3}{4} = \frac{n^2 - 4n + 7}{4}.
\]

Now, we see that the simple inequality \( (3n - 13)(n - 1) \geq 0 \) holds for all \( n \geq \frac{13}{3} \), i.e. \( \frac{(n^2-4n+7)}{4} \geq \frac{(n+2)^2}{3} \). From above, this gives

\[
p(n + 1) < 2p(n) - p(n - 2) \\
\leq 2p(n) - \frac{(n^2 - 4n + 7)}{4} \\
\leq 2p(n) - \frac{(n+2)^2}{3},
\]

for all \( n \geq 5 \) and we are done. A check of the first few values of \( p(n) \) shows the stated inequality to hold for all \( n \geq 2 \).

\[\square\]

**Lemma 2.10** For all \( n \geq 7 \), \( \sum_{i=1}^{n-1} p(i) > p(n) + \frac{1}{12}(2n^2 + 6n + 3) \).
Proof: By simple induction. The result is true for $n = 7$ since $p(1) + p(2) + \cdots + p(6) = 30$ and $p(7) + \frac{1}{12}(2(7)^2 + 6(7) + 3) < 27$. Suppose it to be true for some $n = m \geq 7$, then:

$$\sum_{i=1}^{m} p(i) = p(m) + \sum_{i=1}^{m-1} p(i)$$

$$\geq p(m) + p(m) + \frac{1}{12}(2m^2 + 6m + 3)$$

$$= 2p(m) + \frac{1}{12}(2m^2 + 6m + 3)$$

and using Lemma 2.9,

$$\geq p(m+1) + \frac{m+2}{3} + \frac{1}{12}(2m^2 + 6m + 3)$$

$$= p(m+1) + \frac{1}{12}(2(m+1)^2 + 6(m+1) + 3).$$

Theorem 2.11 For all $n \geq 5$, $g_3(n) \geq g_2(n)$.

Proof: We have that $g_2(n) = p(n) - 1$. Also, we know from Theorem 2.6 that $g_3(n) \geq \sum_{i=1}^{n-1} p(i) - \frac{1}{12}(2n^2 + 6n + 3) - 1$. From Lemma 2.10, we have $\sum_{i=1}^{n-1} p(i) > p(n) + \frac{1}{12}(2n^2 + 6n + 3)$ for all $n \geq 7$. Combining these facts gives

$$g_3(n) \geq \sum_{i=1}^{n-1} p(i) - \frac{1}{12}(2n^2 + 6n + 3) - 1$$

$$\geq p(n) - 1$$

which is $g_2(n)$. From Table 1, the result is also seen to hold for $n = 5, 6$. \qed

Theorem 2.12 For all $n \geq 5$, $f_3(n) \geq f_2(n)$.

Proof: The number of non-isomorphic rank-3 matroids on $S_n$ in terms of loopless non-isomorphic rank-3 matroids is given through the relation $f_3(n) = \sum_{i=3}^{n} g_3(i)$ for all $n \geq 3$. The value $f_2(n) = p(1) + p(2) + \cdots + p(n) - n$ for all $n \geq 2$. From Theorem 2.11 we have $g_3(n) \geq g_2(n)$ for all $n \geq 7$. Applying the above expression for $f_3(n)$, using the known value for $g_3(n)$ (from Sloane [7], row 3 of A058716) and assuming $n \geq 7$,

$$f_3(n) = 38 + \sum_{i=7}^{n} g_3(i)$$

$$> 23 + \sum_{i=7}^{n} g_2(i)$$

$$= \sum_{i=2}^{n} g_2(i)$$

which is precisely $f_2(n)$. From Table 1, the result is seen to hold for $n = 5, 6$. Note that the above inequality is strict for $n \geq 6$ and equality holds only for $n = 5$. \qed
Note that, by duality, an immediate Corollary of Theorems 2.6 and 2.12 is the following.

**Corollary 2.13** For all $n \geq 6$,

$$f_n(n) \leq f_{n-1}(n) \leq f_{n-2}(n) \leq f_{n-3}(n)$$

$$m_n(n) \leq m_{n-1}(n) \leq m_{n-2}(n) \leq m_{n-3}(n).$$

**Acknowledgements**

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**References**


**Appendix**

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Table 1: Known values for the number of rank-2 and rank-3 matroids taken from Sloane (7).