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To cite this version:

HAL Id: hal-01283096
https://hal.archives-ouvertes.fr/hal-01283096
Submitted on 5 Mar 2016

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An algorithm which generates linear extensions for a non-simply-laced d-complete poset with uniform probability

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Abstract. The purpose of this paper is to present an algorithm which generates linear extensions for a non-simply-laced d-complete poset with uniform probability.

Résumé. Le but de ce papier est présenter un algorithme qui produit des extensions linéaires pour une non-simply-laced d-complete poset avec probabilité constante.

Keywords: d-complete posets, algorithm, linear extension, uniform generation

1 Introduction

In [7] (Theorem 4.2), J. Stembridge classified irreducible minuscule elements of Kac-Moody Weyl group over a root system $\Phi$ into three classes below:

- $\Phi$ is simply-laced,
- $\Phi$ has the form $\begin{array}{cccc}
0 & 1 & 2 & \\
\end{array}$ (namely, of type $B$), or
- $\Phi$ has the form $\begin{array}{cccc}
-m & \cdots & 0 & 1 & \\
-2 & -1 & 0 & 1 & \\
\end{array}$ (we say type $F_m$, for simplicity).

In [5][6], the author and S. Okamura constructed an algorithm which generates reduced decompositions for a given minuscule element of simply-laced Weyl group with uniform probability. The algorithm in [6] is described in terms of graphs. Simply-laced minuscule elements are described as certain simple acyclic di-graphs. The transitive-closure of the graph is called a d-complete poset. Then, the reduced decompositions are identified with linear extensions of the graph. This algorithm gives a proof of the hook formula [1] for the number of reduced decompositions of a minuscule element in simply-laced case.

In this paper, we present an algorithm (algorithm A) in terms of graphs (See Section 2 for details). This algorithm is a generalization of an algorithm in [5][6]. We define a certain acyclic multi-di-graph corresponding to a minuscule element of type $B$ (resp. type $F_m$) in Section 3 (resp. Section 4). Our

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main result (Theorem 5.1) is that the algorithm $A$ generates linear extensions for a minuscule element of type $B$ and $F_m$ with uniform probability. More precisely, the probability the algorithm $A$ generates linear extension $L$ of a graph $S$ is given by:

$$\frac{\prod_{v \in S} (1 + \#H_S(v)^+)}{\#S!},$$

(1.1)

where $H_S(v)^+$ is a certain subset of $S$ (See Section 2 for detail). This (1.1) is independent from the choice of $L$. Hence, we get the hook formula for the number of linear extensions of a given shape $S$ of type $B$ and $F_m$. Namely, the number of linear extensions of a shape $S$ is given by:

$$\frac{\#S!}{\prod_{v \in S} (1 + \#H_S(v)^+)}.$$

In section 6, we give a Lie theoretical description of shape of type $B$ and $F_m$.

2 An algorithm for a graph $\Gamma$

Let $\Gamma = (\Gamma; A, o, i)$ be a finite acyclic multi-di-graph, where $A$ denotes the set of arrows of $\Gamma$, $i(a)$ the sink of $a \in A$, and $o(a)$ the source of $a \in A$.

**Definition 2.1** Put $d := \#\Gamma$. A bijection $L : \{1, \cdots, d\} \rightarrow \Gamma$ is said to be a linear extension of $\Gamma$ if:

$L(k) = o(a)$ and $i(a) = L(l)$ implies $k > l$, $k, l \in \{1, \cdots, d\}$, $a \in A$.

The set of linear extensions of $\Gamma$ is denoted by $L(\Gamma)$.

For a given $v \in \Gamma$, we define a set $H_\Gamma(v)^+$ by:

$$H_\Gamma(v)^+ := \{ a \in A(\Gamma) \mid v = o(a) \}.$$

For a given $\Gamma$, we call the following algorithm the algorithm $A$ for $\Gamma$:

GNW1. Set $i := 0$ and set $\Gamma_0 := \Gamma$.

GNW2. (Now $\Gamma_i$ has $d - i$ nodes.) Set $j := 1$ and pick a node $v_1 \in \Gamma_i$ with the probability $1/(d - i)$.

GNW3. If $\#H_{\Gamma_i}(v_j)^+ \neq 0$, pick an arrow $a_{j+1} \in H_{\Gamma_i}(v_j)^+$ with the probability $1/\#H_{\Gamma_i}(v_j)^+$. If not, go to GNW5.

GNW4. Set $v_{j+1} := i(a_j)$. Set $j := j + 1$ and return to GNW3.

GNW5. (Now $\#H_{\Gamma_i}(v_j)^+ = 0$.) Set $L(i + 1) := v_j$ and set $\Gamma_{i+1} := \Gamma_i \setminus v_j$ (the graph deleted $v_j$ from $\Gamma_i$).

GNW6. Set $i := i + 1$. If $i < d$, return to GNW2; if $i = d$, terminate.

We note that the algorithm $A$ stops in finite time since $\Gamma$ is acyclic. By the definition of the algorithm $A$ for $\Gamma$, the map $L : i \mapsto L(i)$ generated above is a linear extension of $\Gamma$. We denote by $\text{Prob}_\Gamma(L)$ the probability we get $L \in L(\Gamma)$ by the algorithm $A$. 
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3 Shapes of type $B$

We denote by $\mathbb{N}$ the set of non-negative integers. We define a set $\mathbb{B}$ by:

$$\mathbb{B} := \{ (i, j) \in \mathbb{N} \times \mathbb{N} \mid i \leq j \}.$$ 

The set $\mathbb{B}$ is depicted in FIGURE 3.1. We equip the $\mathbb{B}$ with the partial order:

$$(i, j) \leq (i', j') \iff i \geq i' \text{ and } j \geq j'.$$

![Fig. 3.1: The set $\mathbb{B}$](image)

**Definition 3.1** Let $S$ be a finite order filter of $\mathbb{B}$. We induce to $S$ a graph structure by:

$$\begin{align*}
(i, j) \rightarrow (i', j') & \text{ if and only if } \begin{cases} 
    i = j \text{ and } i' = i, j' > j, \\
    i < j \text{ and } i' = i, j' > j, \\
    i < j \text{ and } i' > i, j' = j, \\
    \text{or } i < j \text{ and } i' = j, j' > i,
\end{cases} \\
(i, j) \Rightarrow (i', j') & \text{ if and only if } \begin{cases} 
    \text{or } i < j \text{ and } i' = j, j' = j,
\end{cases}
\end{align*}$$

and there exists no other adjacency relation. Here, $v \rightarrow v'$ means there exists exactly one arrow from $v$ to $v'$, and $v \Rightarrow v'$ there exists exactly two arrows from $v$ to $v'$. A graph $S$ is called a shape of type $B$. See FIGURE 3.2 for examples of $H_S(v)^+$. 

![Fig. 3.2: $H_S(u)^+$, $H_S(v)^+$, and $H_S(w)^+$](image)

**Remark 3.2** A shape of type $B$ as poset is order-isomorphic to a shifted shape. Shifted shapes are also realized as $d$-complete posets over a root system of type $D$. The graph-structure of shapes of type $D$ is described in [6] and compatible with notion of hooks (or called bars) of shifted shapes. The algorithm A depends not only on poset-structure but on graph-structure. Hence, we do not consider shapes of type $B$ as shifted shapes.
4 Shapes of type $F_m$ ($m \geq 2$).

We denote by $\mathbb{Z}$ the set of integers. Let $m$ be an integer greater than or equal to $2$. We define a set $F_m$ by:

$$F_m := \left\{ (i, j) \in \mathbb{N} \times \mathbb{Z} \middle| \begin{array}{l} i = 0 \text{ and } j \geq -m, \\ i = 1 \text{ and } j \geq 0, \text{ or } \\ 2 \leq i \leq m \text{ and } j = 0 \end{array} \right\}$$

For example, the set $F_3$ is depicted in FIGURE 4.1. We equip the $F_m$ with the partial order:

$$(i, j) \leq (i', j') \iff i \geq i' \text{ and } j \geq j'.$$

Fig. 4.1: The set $F_3$

**Definition 4.1** Let $S$ be a finite order filter of $F_m$. We induce to $S$ a graph structure by:

$$(i, j) \rightarrow (i', j') \text{ if and only if } \begin{cases} "i = 0, j \leq -1 \text{ and } i' \neq -j, j' > j", \\ "i = 0, j = 0, \text{ and } j' > 0", \\ "i = 1, j = 0, \text{ and } i' = 1, j' > 0", \\ "i = 1, j > 0, \text{ and } i' > 1, j' = 0", \\ "i \geq 2, j = 0, \text{ and } i' > i, j' = 0", \\ "j \geq 1 \text{ and } i' = i, j' > j", \\ \text{ or } "j \geq 1 \text{ and } i' > i, j' = j" \\ \text{and there exists no other adjacency relation. A graph } S \text{ is called a shape of type } F_m. \text{ See FIGURE 4.2 for examples of } H_S(v)^+.$$

Fig. 4.2: $H_S(u)^+, H_S(v)^+, \text{ and } H_S(w)^+.$
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5 Main result

Now, we can state the main theorem:

**Theorem 5.1** Let \( S \) be a shape of type \( B \) or type \( F_m \) for some \( m \geq 2 \). Let \( L \in \mathcal{L}(S) \). Then the algorithm \( A \) for \( S \) generates \( L \) with the probability

\[
\text{Prob}_S(L) = \frac{\prod_{v \in S} (1 + \#H_S(v)^\uparrow)}{\#S!}.
\]

Since the right hand side of (5.1) is independent from the choice of \( L \in \mathcal{L}(S) \), we have:

**Corollary 5.2** Let \( S \) be a shape of type \( B \) or type \( F_m \) for some \( m \geq 2 \). Then we have:

\[
\#\mathcal{L}(S) = \frac{\#S!}{\prod_{v \in S} (1 + \#H_S(v)^\uparrow)}.
\]

6 Lie theoretical description of main result and Remarks

In this section, we fix a (not necessary simply-laced) Kac-Moody Lie algebra \( g \) with a simple root system \( \Pi = \{ \alpha_i \mid i \in I \} \). For all undefined terminology in this section, we refer the reader to [2] [3].

**Definition 6.1** An integral weight \( \lambda \) is said to be pre-dominant if:

\[
\langle \lambda, \beta^\vee \rangle \geq -1 \quad \text{for each} \quad \beta^\vee \in \Phi^+_+, \]

where \( \Phi^+_+ \) denotes the set of positive real coroots. The set of pre-dominant integral weights is denoted by \( P_{\geq -1} \). For \( \lambda \in P_{\geq -1} \), we define the set \( D(\lambda)^\vee \) by:

\[
D(\lambda)^\vee := \{ \beta^\vee \in \Phi^+_+ \mid \langle \lambda, \beta^\vee \rangle = -1 \}.
\]

The set \( D(\lambda)^\vee \) is called the shape of \( \lambda \). If \( \#D(\lambda)^\vee < \infty \), then \( \lambda \) is called finite.

**Proposition 6.2** Let \( \lambda \in P_{\geq -1} \) be finite and \( \beta^\vee, \gamma^\vee \in D(\lambda)^\vee \) satisfy \( \beta^\vee > \gamma^\vee \) in the ordinary order of coroots. Then we have:

\[
\langle \beta, \gamma^\vee \rangle = 0, 1, \text{ or } 2.
\]

By proposition 6.2, we introduce graph-structure into \( D(\lambda)^\vee \) by:

\[
\beta^\vee \rightarrow \gamma^\vee \iff \beta^\vee > \gamma^\vee \quad \text{and} \quad \langle \beta, \gamma^\vee \rangle = 1.
\]

\[
\beta^\vee \Rightarrow \gamma^\vee \iff \beta^\vee > \gamma^\vee \quad \text{and} \quad \langle \beta, \gamma^\vee \rangle = 2.
\]

If \( \beta^\vee \neq \gamma^\vee \), or \( \beta^\vee > \gamma^\vee \) and \( \langle \beta, \gamma^\vee \rangle = 0 \), then no arrows from \( \beta^\vee \) to \( \gamma^\vee \) exist.

Thus, we get a finite acyclic multi-di-graph \( D(\lambda)^\vee \) for a finite \( \lambda \in P_{\geq -1} \).

**Remark 6.3** The finite pre-dominant integral weights \( \lambda \) are identified with the minuscule elements \( w \) [4]. And, we have \( D(\lambda)^\vee = \{ \beta^\vee \in \Phi^+_+ \mid \beta^\vee(w) < 0 \} \). Furthermore, the linear extensions of \( D(\lambda)^\vee \) are identified with the reduced decompositions of \( w \) [4] by the following one-to-one correspondence:

\[
\text{Red}(w) \ni (s_{i_1}, s_{i_2}, \cdots, s_{i_d}) \leftrightarrow L \in \mathcal{L}(D(\lambda)^\vee), \quad L(k) = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})^\vee \in D(\lambda)^\vee \quad (k = 1, \cdots d),
\]

where \( \text{Red}(w) \) denotes the set of reduced decompositions of \( w \), \( d = \ell(w) \) the length of \( w \).
6.1 Case of type $B$

Suppose that the underlying Dynkin diagram is of type $B$:

\[
\begin{array}{cccccc}
0 & 1 & 2 & \cdots & \cdots & \cdots \\
\end{array}
\]

Let $W = \langle s_0, s_1, s_2, \cdots \rangle$ be the Weyl group. Let $\Lambda_0$ be the 0-th fundamental weight. Then each $\lambda \in W\Lambda_0$ is a finite pre-dominant integral weight. And, $D(\lambda)^\vee$ is graph-isomorphic with some shape of type $B$ defined in section 3.

**Remark 6.4** Let $W_0 := \langle s_1, s_2, \cdots \rangle$ be a maximal parabolic subgroup of $W$, which is the Weyl group of type $A$. Then a minimal coset representative $w$ in $W/W_0$ is called a Lagrangian Grassmannian element.

Let $\lambda \in W\Lambda_0$. Then the corresponding minuscule element $w$ in remark 6.3 is a Lagrangian Grassmannian element. Our result gives the number of reduced decompositions of Lagrangian Grassmannian element $w$.

6.2 Case of type $F_m$ ($m \geq 2$)

Let $m \in \mathbb{Z}$ be greater than or equal to 2. Suppose that the underlying Dynkin diagram is of type $F_m$:

\[
\begin{array}{cccccc}
-2 & -1 & 0 & 1 & \cdots & \cdots \\
\end{array}
\]

Let $W = \langle s_{-m}, \cdots, s_{-2}, s_{-1}, s_0, s_1, \cdots \rangle$ be the Weyl group. Let $\Lambda_{-m}$ be the $(-m)$-th fundamental weight. Then each $\lambda \in P_{\geq -1} \cap W\Lambda_{-m}$ is a finite pre-dominant integral weight. And, $D(\lambda)^\vee$ is graph-isomorphic with some shape of type $F_m$ defined in section 4.

References


