FPSAC 2012, Nagoya, Japan DMTCS proc. **AR**, 2012, 655–660

An algorithm which generates linear extensions for a non-simply-laced d-complete poset with uniform probability

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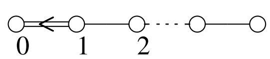
Abstract. The purpose of this paper is to present an algorithm which generates linear extensions for a non-simply-laced d-complete poset with uniform probability.

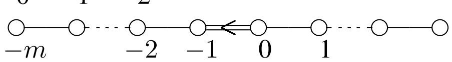
Résumé. Le but de ce papier est présenter un algorithme qui produit des extensions linéaires pour une non-simply-laced d-complete poset avec probabilité constante.

Keywords: d-complete posets, algorithm, linear extension, uniform generation

1 Introduction

In [7](Theorem 4.2), J. Stembridge classified irreducible minuscule elements of Kac-Moody Weyl group over a root system Φ into three classes below:

- Φ is simply-laced,
- Φ has the form 

(namely, of type B) , or
- Φ has the form 

(we say *type* F_m, for simplicity).

In [5][6], the author and S. Okamura constracted an algorithm which generates reduced decompositions for a given minuscule element of simply-laced Weyl group with uniform probability. The algorithm in [6] is described in terms of graphs. Simply-laced minuscule elements are described as certain simple acyclic di-graphs. The transitive-closure of the graph is called a d-complete poset. Then, the reduced decompositions are identified with linear extensions of the graph. This algorithm gives a proof of the hook formula [1] for the number of reduced decompositions of a minuscule element in simply-laced case.

In this paper, we present an algorithm (algorithm A) in terms of graphs (See Section 2 for details). This algorithm is a generalization of an algorithm in [5][6]. We define a certain acyclic multi-di-graph corresponding to a minuscule element of type B (resp. type F_m) in Section 3 (resp. Section 4). Our

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main result (Theorem 5.1) is that the algorithm A generates linear extensions for a minuscule element of type B and F_m with uniform probability. More precisely, the probability the algorithm A generates linear extension L of a graph S is given by:

$$\frac{\prod_{v\in S}(1+\#\mathrm{H}_S\,(v)^+)}{\#S!}, \tag{1.1}$$

where $\mathrm{H}_S\,(v)^+$ is a certain subset of S (See Section 2 for detail). This (1.1) is independent from the choice of L. Hence, we get the hook formula for the number of linear extensions of a given shape S of type B and F_m. Namely, the number of linear extensions of a shape S is given by:

$$\frac{\#S!}{\prod_{v\in S}(1+\#\mathrm{H}_S\,(v)^+)}.$$

In section 6, we give a Lie theoretical description of shape of type B and F_m.

2 An algorithm for a graph Γ

Let $\Gamma = (\Gamma; A, \mathrm{o}, \mathrm{i})$ be a finite acyclic multi-di-graph, where A denotes the set of arrows of Γ, $\mathrm{i}(a)$ the sink of $a \in A$, and $\mathrm{o}(a)$ the source of $a \in A$.

Definition 2.1 *Put $d := \#\Gamma$. A bijection $L : \{1, \cdots, d\} \longrightarrow \Gamma$ is said to be a* linear extension of Γ *if:*

$$L(k) = \mathrm{o}(a) \ \text{ and } \ \mathrm{i}(a) = L(l) \textit{ implies } k > l, \quad k, l \in \{1, \cdots, d\}, \quad a \in A.$$

The set of linear extensions of Γ is denoted by $\mathcal{L}\,(\Gamma)$.

For a given $v \in \Gamma$, we define a set $\mathrm{H}_\Gamma\,(v)^+$ by:

$$\mathrm{H}_\Gamma\,(v)^+ := \left\{\, a \in A(\Gamma) \,\middle|\, v = \mathrm{o}(a) \,\right\}.$$

For a given Γ, we call the following algorithm the *algorithm* A *for* Γ:

GNW1. Set $i := 0$ and set $\Gamma_0 := \Gamma$.

GNW2. (Now Γ_i has $d - i$ nodes.) Set $j := 1$ and pick a node $v_1 \in \Gamma_i$ with the probability $1/(d - i)$.

GNW3. If $\#\mathrm{H}_{\Gamma_i}\,(v_j)^+ \neq 0$, pick an arrow $a_{j+1} \in \mathrm{H}_{\Gamma_i}\,(v_j)^+$ with the probability $1/\#\mathrm{H}_{\Gamma_i}\,(v_j)^+$. If not, go to GNW5.

GNW4. Set $v_{j+1} := \mathrm{i}(a_j)$. Set $j := j + 1$ and return to GNW3.

GNW5. (Now $\#\mathrm{H}_{\Gamma_i}\,(v_j)^+ = 0$.) Set $L(i+1) := v_j$ and set $\Gamma_{i+1} := \Gamma_i \setminus v_j$ (the graph deleted v_j from Γ_i).

GNW6. Set $i := i + 1$. If $i < d$, return to GNW2; if $i = d$, terminate.

We note that the algorithm A stops in finite time since Γ is acyclic. By the definition of the algorithm A for Γ, the map $L : i \mapsto L(i)$ generated above is a linear extension of Γ. We denote by $\mathrm{Prob}_\Gamma(L)$ the probability we get $L \in \mathcal{L}\,(\Gamma)$ by the algorithm A.

3 Shapes of type B

We denote by $\mathbb{N}$ the set of non-negative integers. We define a set $\mathbb{B}$ by:

$$\mathbb{B} := \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} \,\middle|\, i \leq j \right\}.$$

The set $\mathbb{B}$ is depicted in FIGURE 3.1. We equip the $\mathbb{B}$ with the partial order:

$$(i,j) \leq (i',j') \iff i \geq i' \text{ and } j \geq j'.$$

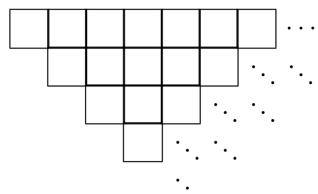


Fig. 3.1: The set $\mathbb{B}$

Definition 3.1 *Let S be a finite order filter of $\mathbb{B}$. We induce to S a graph structure by:*

$$(i,j) \to (i',j') \quad \textit{if and only if} \quad \begin{cases} \text{“}i = j \text{ and } i' = i, j' > j\text{”}, \\ \text{“}i < j \text{ and } i' = i, j' > j\text{”}, \\ \text{“}i < j \text{ and } i' > i, j' = j\text{”}, \\ \text{or “}i < j \text{ and } i' = j, j' > i\text{”}, \end{cases}$$

$$(i,j) \rightrightarrows (i',j') \quad \textit{if and only if} \quad \text{“}i < j \text{ and } i' = j' = j\text{”},$$

and there exists no other adjacency relation. Here, $v \to v'$ means there exists exactly one arrow from v to v', and $v \rightrightarrows v'$ there exists exactly two arrows from v to v'. A graph S is called a shape of type B*. See FIGURE 3.2 for examples of $\mathrm{H}_S(v)^+$.*

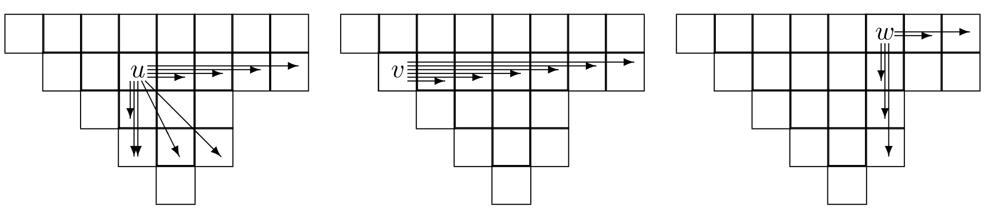


Fig. 3.2: $\mathrm{H}_S(u)^+$, $\mathrm{H}_S(v)^+$, and $\mathrm{H}_S(w)^+$.

Remark 3.2 *A shape of type B as poset is order-isomorphic to a shifted shape. Shifted shapes are also realized as d-complete posets over a root system of type D. The graph-structure of shapes of type D is described in [6] and compatible with notion of hooks (or called bars) of shifted shapes. The algorithm A depends not only on poset-structure but on graph-structure. Hence, we do not consider shapes of type B as shifted shapes.*

4 Shapes of type F_m $(m \geq 2)$.

We denote by $\mathbb{Z}$ the set of integers. Let m be an integer greater than or equal to 2. We define a set $\mathbb{F}_m$ by:

$$\mathbb{F}_m := \left\{ (i,j) \in \mathbb{N} \times \mathbb{Z} \;\middle|\; \begin{array}{l} i = 0 \text{ and } j \geq -m, \\ i = 1 \text{ and } j \geq 0, \text{ or} \\ 2 \leq i \leq m \text{ and } j = 0 \end{array} \right\}$$

For example, the set $\mathbb{F}_3$ is depicted in FIGURE 4.1. We equip the $\mathbb{F}_m$ with the partial order:

$$(i,j) \leq (i',j') \Longleftrightarrow i \geq i' \text{ and } j \geq j'.$$

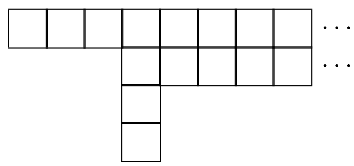


Fig. 4.1: The set $\mathbb{F}_3$

Definition 4.1 *Let S be a finite order filter of $\mathbb{F}_m$. We induce to S a graph structure by:*

$$(i,j) \rightarrow (i',j') \quad \textit{if and only if} \quad \left\{ \begin{array}{l} \text{“}i = 0, j \leq -1 \text{ and } i' \neq -j, j' > j\text{”}, \\ \text{“}i = 0, j = 0, \text{ and } j' > 0\text{”}, \\ \text{“}i = 1, j = 0, \text{ and } i' = 1, j' > 0\text{”}, \\ \text{“}i = 1, j = 0, \text{ and } i' > 1, j' = 0\text{”}, \\ \text{“}i \geq 2, j = 0, \text{ and } i' > i, j' = 0\text{”}, \\ \text{“}j \geq 1 \text{ and } i' = i, j' > j\text{”}, \\ \text{ or “}j \geq 1 \text{ and } i' > i, j' = j\text{”}, \end{array} \right.$$

$$(i,j) \rightrightarrows (i',j') \quad \textit{if and only if} \quad \text{“}i = 0, j = 0, \text{ and } 0 < i', j' = 0\text{”},$$

and there exists no other adjacency relation. A graph S is called a shape of type F_m. *See FIGURE 4.2 for examples of* $\mathrm{H}_S(v)^+$.

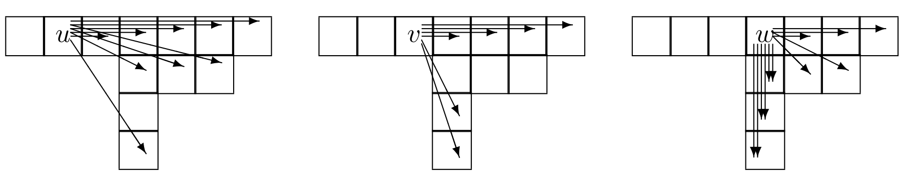


Fig. 4.2: $\mathrm{H}_S(u)^+$, $\mathrm{H}_S(v)^+$, and $\mathrm{H}_S(w)^+$.

5 Main result

Now, we can state the main theorem:

Theorem 5.1 *Let S be a shape of type B or type F_m for some $m \geq 2$. Let $L \in \mathcal{L}(S)$. Then the algorithm* A *for S generates L with the probability*

$$\mathrm{Prob}_S(L) = \frac{\prod_{v \in S}(1 + \#\mathrm{H}_S(v)^+)}{\#S!}. \tag{5.1}$$

Since the right hand side of (5.1) is independent from the choice of $L \in \mathcal{L}(S)$, we have:

Corollary 5.2 *Let S be a shape of type B or type F_m for some $m \geq 2$. Then we have:*

$$\#\mathcal{L}(S) = \frac{\#S!}{\prod_{v \in S}(1 + \#\mathrm{H}_S(v)^+)}.$$

6 Lie theoretical description of main result and Remarks

In this section, we fix a (not necessary simply-laced) Kac-Moody Lie algebra $\mathfrak{g}$ with a simple root system $\Pi = \left\{ \alpha_i \,\middle|\, \in I \right\}$. For all undefined terminology in this section, we refer the reader to [2] [3].

Definition 6.1 *An integral weight λ is said to be* pre-dominant *if:*

$$\langle \lambda, \beta^\vee \rangle \geq -1 \quad \textit{for each } \beta^\vee \in \Phi_+^\vee,$$

where $\Phi_+^\vee$ denotes the set of positive real coroots. The set of pre-dominant integral weights is denoted by $P_{\geq -1}$. For $\lambda \in P_{\geq -1}$, we define the set $\mathrm{D}(\lambda)^\vee$ by:

$$\mathrm{D}(\lambda)^\vee := \left\{ \beta^\vee \in \Phi_+^\vee \,\middle|\, \langle \lambda, \beta^\vee \rangle = -1 \right\}.$$

The set $\mathrm{D}(\lambda)^\vee$ is called the shape of λ. *If $\#\mathrm{D}(\lambda)^\vee < \infty$, then λ is called* finite.

Proposition 6.2 (see [4]) *Let $\lambda \in P_{\geq -1}$ be finite and $\beta^\vee, \gamma^\vee \in \mathrm{D}(\lambda)^\vee$ satisfy $\beta^\vee > \gamma^\vee$ in the ordinary order of coroots. Then we have:*

$$\langle \beta, \gamma^\vee \rangle = 0, 1, \text{ or } 2.$$

By proposition 6.2, we introduce graph-structure into $\mathrm{D}(\lambda)^\vee$ by:

$$\beta^\vee \to \gamma^\vee \Leftrightarrow \beta^\vee > \gamma^\vee \text{ and } \langle \beta, \gamma^\vee \rangle = 1.$$
$$\beta^\vee \rightrightarrows \gamma^\vee \Leftrightarrow \beta^\vee > \gamma^\vee \text{ and } \langle \beta, \gamma^\vee \rangle = 2.$$

If $\beta^\vee \not> \gamma^\vee$, or $\beta^\vee > \gamma^\vee$ and $\langle \beta, \gamma^\vee \rangle = 0$, then no arrows from $\beta^\vee$ to $\gamma^\vee$ exist.

Thus, we get a finite acyclic multi-di-graph $\mathrm{D}(\lambda)^\vee$ for a finite $\lambda \in P_{\geq -1}$.

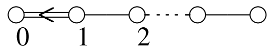
Remark 6.3 *The finite pre-dominant integral weights λ are identified with the minuscule elements w [4]. And, we have $\mathrm{D}(\lambda)^\vee = \left\{ \beta^\vee \in \Phi_+^\vee \,\middle|\, w^{-1}(\beta^\vee) < 0 \right\}$. Furthermore, the linear extensions of $\mathrm{D}(\lambda)^\vee$ are identified with the reduced decompositions of w [4] by the following one-to-one correspondence:*

$$\mathrm{Red}(w) \ni (s_{i_1}, s_{i_2}, \cdots, s_{i_d}) \longleftrightarrow L \in \mathcal{L}(\mathrm{D}(\lambda)^\vee), \quad L(k) = s_{i_1} \cdots s_{i_{k-1}} (\alpha_{i_k})^\vee \in \mathrm{D}(\lambda)^\vee \ (k = 1, \cdots d),$$

where $\mathrm{Red}(w)$ denotes the set of reduced decompositions of w, $d = \ell(w)$ the length of w.

6.1 Case of type B

Suppose that the underlying Dynkin diagram is of type B:



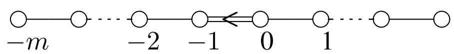
Let $W = \langle s_0, s_1, s_2, \cdots \rangle$ be the Weyl group. Let Λ_0 be the 0-th fundamental weight. Then each $\lambda \in W\Lambda_0$ is a finite pre-dominant integral weight. And, $\mathrm{D}(\lambda)^\vee$ is graph-isomorphic with some shape of type B defined in section 3.

Remark 6.4 *Let $W_0 := \langle s_1, s_2, \cdots \rangle$ be a maximal parabolic subgroup of W, which is the Weyl group of type A. Then a minimal coset representative w in W/W_0 is called a Lagrangian Grassmannian element.*

Let $\lambda \in W\Lambda_0$. Then the corresponding minuscule element w in remark 6.3 is a Lagrangian Grassmannian element. Our result gives the number of reduced decompositions of Lagrangian Grassmannian element w.

6.2 Case of type F_m $(m \geq 2)$

Let $m \in \mathbb{Z}$ be greater than or equal to 2. Suppose that the underlying Dynkin diagram is of type F_m:



Let $W = \langle s_{-m}, \cdots, s_{-2}, s_{-1}, s_0, s_1, \cdots \rangle$ be the Weyl group. Let Λ_{-m} be the $(-m)$-th fundamental weight. Then each $\lambda \in P_{\geq -1} \cap W\Lambda_{-m}$ is a finite pre-dominant integral weight. And, $\mathrm{D}(\lambda)^\vee$ is graph-isomorphic with some shape of type F_m defined in section 4.

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