Cumulants of the q-semicircular law, Tutte polynomials, and heaps

Matthieu Josuat-Vergès[1†]

[1] *CNRS and Institut Gaspard Monge, Université de Marne-la-Vallée, 5 Boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée CEDEX 2, FRANCE*

Abstract. The q-semicircular law as introduced by Bożejko and Speicher interpolates between the Gaussian law and the semicircular law, and its moments have a combinatorial interpretation in terms of matchings and crossings. We prove that the cumulants of this law are, up to some factor, polynomials in q with nonnegative coefficients. This is done by showing that they are obtained by an enumeration of connected matchings, weighted by the evaluation at $(1, q)$ of a Tutte polynomial. The two particular cases $q = 0$ and $q = 2$ have also alternative proofs, related with the fact that these particular evaluation of the Tutte polynomials count some orientations on graphs. Our methods also give a combinatorial model for the cumulants of the free Poisson law.

Résumé. La loi q-semicirculaire introduite par Bożejko et Speicher interpole entre la loi gaussienne et la loi semi-circulaire, et ses moments ont une interprétation combinatoire en termes de couplages et croisements. Nous prouvons que les cumulants de cette loi sont, à un facteur près, des polynômes en q à coefficients positifs. La méthode consiste à obtenir ces cumulants par une énumération de couplages connexes, pondérés par l'évaluation en $(1, q)$ d'un polynôme de Tutte. Les cas particuliers $q = 0$ et $q = 2$ ont une preuve alternative, relié au fait que des évaluations particulières du polynôme de Tutte comptent certaines orientations de graphes. Nos méthodes donnent aussi un modèle combinatoire aux cumulants de la loi de Poisson libre.

Keywords: moments, cumulants, matchings, Tutte polynomials, heaps

1 Introduction

Let us consider the sequence $m_n(q)$ defined by the generating function

$$\sum_{n\geq 0} m_n(q) z^n = \cfrac{1}{1 - \cfrac{[1]_q z^2}{1 - \cfrac{[2]_q z^2}{1 - \ddots}}}$$

where $[i]_q = \frac{1-q^i}{1-q}$. For example, $m_0(q) = m_2(q) = 1$, $m_4(q) = 2 + q$, and the odd values are 0. The generating function being a Stieltjes continued fraction, $m_n(q)$ is the nth moment of a symmetric

[†] Supported by the Austrian Research Foundation (FWF) via the grant Y463.

probability measure on $\mathbb{R}$. An explicit formula for the density $w(x)$ such that $m_n(q) = \int x^n w(x) \mathrm{d}x$ can be found as a theta function :

$$w(x) = \begin{cases} \frac{1}{\pi}\sqrt{1-q}\sin\theta \prod_{n=1}^{\infty}(1-q^n)|1-q^n e^{2i\theta}|^2 & \text{if } -2 \leq x\sqrt{1-q} \leq 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta \in [0,\pi]$ is such that $2\cos\theta = x\sqrt{1-q}$. At $q=0$, it is the semicircular distribution with density $(2\pi)^{-1}\sqrt{4-x^2}$ supported on $[-2,2]$, whereas at the limit $q \to 1$ it becomes the Gaussian distribution with density $(2\pi)^{-1/2}e^{-x^2/2}$. This law is therefore known either as the q-Gaussian or the q-semicircular law. It can be conviently characterized by its orthogonal polynomials, defined by the relation $xH_n(x|q) = H_{n+1}(x|q) + [n]_q H_{n-1}(x|q)$ together with $H_1(x|q) = x$ and $H_0(x|q) = 1$, and called the continuous q-Hermite polynomials (but we do not insist on this point of view since the notion of cumulant is not particularly relevant for orthogonal polynomials).

The semicircular law is the analogue in free probability of the Gaussian law ([Hiai and Petz(2000), Nica and Speicher(2006)]). More generally, the q-semicircular measure plays an important role in non-commutative probability theories, see [Anshelevich et al.(2010) , Bożejko et al.(1997)Bożejko, Kümmerer, and Speicher, Bożejko and Speicher(1992)]. In particular, this law appeared in the work of Bożejko and Speicher who used creation and annihilation operators in a twisted Fock space to build generalized Brownian motions.

Combinatorially, the moments $m_n(q)$ count matchings, and the free cumulants $c_n(q)$ count connected matchings, see the next section. The goal of this work is to examine the combinatorial meaning of the classical cumulants $k_n(q)$. The first values lead to the observation that

$$\tilde{k}_{2n}(q) = \frac{k_{2n}(q)}{(q-1)^{n-1}}$$

is a polynomial in q with nonnegative coefficients. For example:

$$\tilde{k}_2(q) = \tilde{k}_4(q) = 1, \qquad \tilde{k}_6(q) = q+5, \qquad \tilde{k}_8(q) = q^3 + 7q^2 + 28q + 56.$$

We actually show in Theorem 1 that this $\tilde{k}_{2n}(q)$ can be given a meaning as a generating function of connected matchings. However, the weight function on connected matching is not as simple as in the case of free cumulants, it is given by the value at $(1,q)$ of the Tutte polynomial of a graph attached to each connected matching.

There are various points where the evalutation of a Tutte polynomials has combinatorial meaning, in particular $(1,0)$, $(1,1)$ and $(1,2)$. In the first and third case ($q=0$ and $q=2$), they can be used to give an alternative proof of Theorem 1. The integers $\tilde{k}_{2n}(0)$ were recently considered by Lassalle ([Lassalle(2012)]) who defines them as a sequence simply related with Catalan numbers. Being the (classical) cumulants of the semicircular law, it might seem unnatural to consider this quantity since this law belongs to the world of free probability, but on the other side the free cumulants of the Gaussian have numerous properties (see [Belinschi et al.(2011)]). The interesting feature is that this particular case $q=0$ can be proved via the theory of heaps ([Cartier and Foata(1969), Viennot(1989)]). As for the case $q=2$, even though the q-semicircular law is only defined when $|q|<1$, its moments and cumulants and the link between still exist because (1) can be seen as an identity between formal power series in z. The particular proof for $q=2$ is an application of the exponential formula.

Eventually, it would be interesting to explain why the same combinatorial objects appear both for $c_{2n}(q)$ and $k_{2n}(q)$. This suggests that there exists some quantity that interpolates between the classical and free cumulants of the q-semicircular law, however, building a noncommutative probability theory that encompasses the classical and free ones appear to be elusive. More precisely, it means that building such an interpolation would rely not only on the q-semicircular law and its moments, but on its realization as a noncommutative random variable.

2 Preliminaries

Let us first precise some terms used in the introduction. Besides the moments $\{m_n(q)\}_{n\geq 0}$, the q-semicircular law can be characterized by its *cumulants* $\{k_n(q)\}_{n\geq 1}$ formally defined by

$$\sum_{n\geq 1} k_n(q)\frac{z^n}{n!} = \log\left(\sum_{n\geq 0} m_n(q)\frac{z^n}{n!}\right), \tag{1}$$

or by its *free cumulants* $\{c_n(q)\}_{n\geq 1}$ ([Nica and Speicher(2006)]) formally defined by

$$1 + C(zM(z)) = M(z) \quad \text{ where } M(z) = \sum_{n\geq 0} m_n(q)z^n, \quad C(z) = \sum_{n\geq 1} c_n(q)z^n.$$

The first non-zero values are

$$\begin{aligned} &c_2(q) = 1, && c_4(q) = q, && c_6(q) = q^3 + 3q^2, \\ &k_2(q) = 1, && k_4(q) = q - 1, && k_6(q) = q^3 + 3q^2 - 9q + 5. \end{aligned}$$

For any finite set V, let $\mathcal{P}(V)$ denote the lattice of set partitions of V, and let $\mathcal{P}(n) = \mathcal{P}(\{1,\dots,n\})$. We will denote by $\hat{1}$ the maximal element and by μ the Möbius function of these lattices, without mentioning V explicitly. See [Stanley(1999), Chapter 3] for details. When we have some sequence $(u_n)_{n\geq 0}$, for any $\pi \in \mathcal{P}_n$ we will use the notation:

$$u_\pi = \prod_{b\in\pi} u_{\#b}.$$

Then the relations between moments and cumulants read:

$$m_n(q) = \sum_{\pi\in\mathcal{P}(n)} k_\pi(q), \qquad k_n(q) = \sum_{\pi\in\mathcal{P}(n)} m_\pi(q)\mu(\pi,\hat{1}). \tag{2}$$

These are equivalent via the Möbius inversion formula and both can be obtained from (1) using Faà di Bruno's formula. Similarly, there is a relation between $m_n(q)$ and $c_n(q)$ using the lattice of noncrossing partitions. We refer to [Hiai and Petz(2000)] for more on this subject.

Let $\mathcal{M}(V) \subset \mathcal{P}(V)$ denote the set of *matchings*, i.e. set partitions whose all blocks have size 2. As is customary, a block of $\sigma \in \mathcal{M}(V)$ will be called an *arch*. When $V \subset \mathbb{N}$, a *crossing* ([Ismail et al.(1987)]) of $\sigma \in \mathcal{M}(V)$ is a pair of arches $\{i,j\}$ and $\{k,\ell\}$ such that $i < k < j < \ell$. Let $\operatorname{cr}(\sigma)$ denote the number of crossings of $\sigma \in \mathcal{M}(V)$. Let $\mathcal{N}(V) \subset \mathcal{M}(V)$ denote the set of *noncrossing* matchings, i.e.

those such that $\mathrm{cr}(\sigma) = 0$, let also $\mathcal{M}(2n) = \mathcal{M}(\{1,\dots,2n\})$ and $\mathcal{N}(2n) = \mathcal{N}(\{1,\dots,2n\})$. Let $\mathcal{M}^c(2n)$ be the set of *connected* matchings (i.e. such that no proper interval is a union of arches), see ([Belinschi et al.(2011)]) for various properties of these objects in the context of free probability.

It is known that :

$$m_{2n}(q) = \sum_{\sigma\in\mathcal{M}(2n)} q^{\mathrm{cr}(\sigma)}, \qquad c_{2n}(q) = \sum_{\sigma\in\mathcal{M}^c(2n)} q^{\mathrm{cr}(\sigma)}. \tag{3}$$

The first equality was proved in ([Ismail et al.(1987)]). The second equality was obtained in ([Lehner(2002)]) using the Möbius inversion relating moments and free cumulants.

3 A combinatorial formula for $k_n(q)$A combinatorial formula for kn(q)

We will use the Möbius inversion formula in Equation (2), but we first need to consider the combinatorial meaning of the products $m_\pi(q)$.

Lemma 1 *For any $\sigma \in \mathcal{M}(2n)$ and $\pi \in \mathcal{P}(2n)$ such that $\sigma \leq \pi$, let $\mathrm{cr}(\sigma,\pi)$ be the number of crossings $\{\{i,j\},\{k,\ell\}\}$ of σ such that $\{i,j,k,\ell\} \subset b$ for some $b \in \pi$. Then we have:*

$$m_\pi(q) = \sum_{\substack{\sigma\in\mathcal{M}(2n)\\ \sigma\leq\pi}} q^{\mathrm{cr}(\sigma,\pi)}. \tag{4}$$

Proof: Denoting $\sigma|_b = \{x \in \sigma \ : \ x \subset b\}$, the map $\sigma \mapsto (\sigma|_b)_{b\in\pi}$ is a natural bijection between the set $\{\sigma \in \mathcal{M}(2n) \ : \ \sigma \leq \pi\}$ and the product $\Pi_{b\in\pi}\mathcal{M}(b)$, in such a way that $\mathrm{cr}(\sigma,\pi) = \sum_{b\in\pi} \mathrm{cr}(\sigma|_b)$. This allows to factorize the right-hand side in (4) and obtain $m_\pi(q)$. □

From Equation (2) and the previous lemma, we have:

$$\begin{aligned} k_{2n}(q) &= \sum_{\pi\in\mathcal{P}(2n)} m_\pi(q)\mu(\pi,\hat{1}) = \sum_{\pi\in\mathcal{P}(2n)} \sum_{\substack{\sigma\in\mathcal{M}(2n)\\ \sigma\leq\pi}} q^{\mathrm{cr}(\sigma,\pi)}\mu(\pi,\hat{1}) \\ &= \sum_{\sigma\in\mathcal{M}(2n)} \sum_{\substack{\pi\in\mathcal{P}(2n)\\ \pi\geq\sigma}} q^{\mathrm{cr}(\sigma,\pi)}\mu(\pi,\hat{1}) = \sum_{\sigma\in\mathcal{M}(2n)} W(\sigma), \end{aligned} \tag{5}$$

where for each $\sigma \in \mathcal{M}(2n)$ we have introduced:

$$W(\sigma) = \sum_{\substack{\pi\in\mathcal{P}(2n)\\ \pi\geq\sigma}} q^{\mathrm{cr}(\sigma,\pi)}\mu(\pi,\hat{1}). \tag{6}$$

A key point is to note that $W(\sigma)$ only depends on how the arches of σ cross with respect to each other, which can be encoded in a graph. This leads to the following:

Definition 1 *Let $\sigma \in \mathcal{M}(2n)$. The crossing graph $G(\sigma)$ is as follows: vertices are the arches of σ, and there is an edge between the vertices $\{i,j\}$ and $\{k,\ell\}$ if and only if $i < k < j < \ell$, i.e. there is a crossing between $\{i,j\}$ and $\{k,\ell\}$.*

See Figure 1 for an example. Note that the graph $G(\sigma)$ is connected if and only if σ is a connected matching in the sense of the previous section.

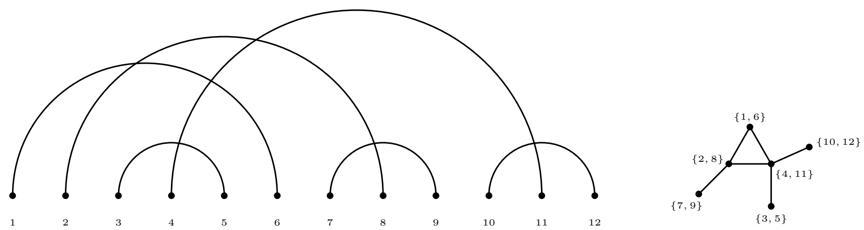

Fig. 1: A matching σ and its crossing graph $G(\sigma)$.

Lemma 2 *Let $\sigma \in \mathcal{M}(2n)$ and $G(\sigma) = (V, E)$ be its crossing graph. If $\pi \in \mathcal{P}(V)$, let $i(E, \pi)$ be the number of elements in the edge set E such that both endpoints are in the same block of π. Then we have:*

$$W(\sigma) = \sum_{\pi \in \mathcal{P}(V)} q^{i(E,\pi)} \mu(\pi, \hat{1}). \tag{7}$$

Proof: There is a natural bijection between the interval $[\sigma, \hat{1}]$ in $\mathcal{P}(2n)$ and the set $\mathcal{P}(V)$, in such a way that $\mathrm{cr}(\sigma, \pi) = i(E, \pi)$. Hence Equation (7) is just a rewriting of (6) in terms of the graph $G(\sigma)$. $\square$

Now we can use the following proposition, which allows to recognize $(q-1)^{-n+1}W(\sigma)$ as an evaluation of the Tutte polynomial $T_{G(\sigma)}(x, y)$, except that it is 0 when the graph is not connected.

Proposition 1 *Let $G = (V, E)$ be a graph (possibly with multiple edge and loops), $T_G(x, y)$ its Tutte polynomial. Let $n = \#V$. With $i(E, \pi)$ defined as in Lemma 2, we have:*

$$\frac{1}{(q-1)^{n-1}} \sum_{\pi \in \mathcal{P}(V)} q^{i(E,\pi)} \mu(\pi, \hat{1}) = \begin{cases} T_G(1, q) & \textit{if } G \textit{ is connected,} \\ 0 & \textit{otherwise.} \end{cases} \tag{8}$$

This is actually a particular case of [Burman and Shapiro(2006), Theorem 9]. It is elementary to check that both sides satisfy a common recurrence relation.

Gathering Equations (5), (7), and (8), we have proved:

Theorem 1 *For any $n \geq 1$,*

$$\tilde{k}_{2n}(q) = \sum_{\sigma \in \mathcal{M}^c(2n)} T_{G(\sigma)}(1, q).$$

In particular $\tilde{k}_{2n}(q)$ is a polynomial in q with nonnegative coefficients.

4 The case $q = 0$, Lasalle's sequence and heapsThe case q=0, Lasalle's sequence and heaps

In the case $q = 0$, the substitution $z \to iz$ recasts Equation (1) as

$$-\log\left(\sum_{n\geq 0}(-1)^n C_n \frac{z^{2n}}{(2n)!}\right) = \sum_{n\geq 1}\tilde{k}_{2n}(0)\frac{z^{2n}}{(2n)!}, \tag{9}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number, known to be the cardinal of $\mathcal{N}(2n)$, see [Stanley(1999)]. The integer sequence $\{\tilde{k}_{2n}(0)\}_{n\geq 1} = (1, 1, 5, 56, \dots)$ was previously defined by Lassalle ([Lassalle(2012)]) via an equation equivalent to (9), and Theorem 1 from ([Lassalle(2012)]) states that the integers $\tilde{k}_{2n}(0)$ are positive and increasing.

The goal of this section is to give a meaning to (9) in the context of the theory of heaps ([Viennot(1989)], [Cartier and Foata(1969), Appendix 3]). This will give an alternative proof of Theorem 1 for the case $q = 0$, based on a classical result on the evaluation $T_G(1, 0)$ of a Tutte polynomial in terms of some orientations of the graph G.

Definition 2 *A graph $G = (V, E)$ is rooted when it has a distinguished vertex $r \in V$, called the root. An orientation of G is root-connected, if for any vertex $v \in V$ there exists a directed path from the root to v.*

Proposition 2 ([Greene and Zaslavsky(1983)]) *If G is a rooted and connected graph, $T_G(1, 0)$ is the number of its root-connected acyclic orientations.*

The notion of heap was introduced in ([Viennot(1989)]) as a geometric interpretation of elements in the Cartier-Foata monoid (see [Cartier and Foata(1969)]), and has various applications in enumeration. We refer to [Cartier and Foata(1969), Appendix 3] for a modern presentation of this subject (and comprehensive bibliography).

Let M be the monoid built on the generators $(x_{ij})_{1\leq i<j}$ subject to the relations $x_{ij}x_{k\ell} = x_{k\ell}x_{ij}$ if $i < j < k < \ell$ or $i < k < \ell < j$. We call it the Cartier-Foata monoid (but in other contexts it could be called a partially commutative free monoid or a trace monoid as well). Following [Viennot(1989)], we call an element of M a *heap*.

Any heap can be represented as a "pile" of segments, as in the left part of Figure 2 (this is remindful of [Bousquet-Mélou and Viennot(1992)]). This pile is described inductively: the generator x_{ij} correspond to a single segment whose extremities have abscissas i and j, and multiplication m_1m_2 is obtained by placing the pile of segments corresponding to m_2 above the one corresponding to m_1. In terms of segments, the relation $x_{ij}x_{k\ell} = x_{k\ell}x_{ij}$ if $i < j < k < \ell$ has a geometric interpretation: segments are allowed to move vertically as long as they do not intersect (this is the case of x_{34} and x_{67} in Figure 2). Similarly, the other relation $x_{ij}x_{k\ell} = x_{k\ell}x_{ij}$ if $i < k < \ell < j$ can be treated by thinking of each segment as the projection of an arch as in the central part of Figure 2. In this three-dimensional representation, all the commutation relations are translated in terms of arches that are allowed to move along the dotted lines as long as they do not intersect.

A heap can also be represented as a poset. Consider two segments s_1 and s_2 in a pile of segments, then the relation is defined by saying that $s_1 < s_2$ if s_1 is always below s_2, after any movement of the arches (along the dotted lines and as long as they do not intersect, as above). See the right part of Figure 2 for an example and [Cartier and Foata(1969), Appendice 3] for details.

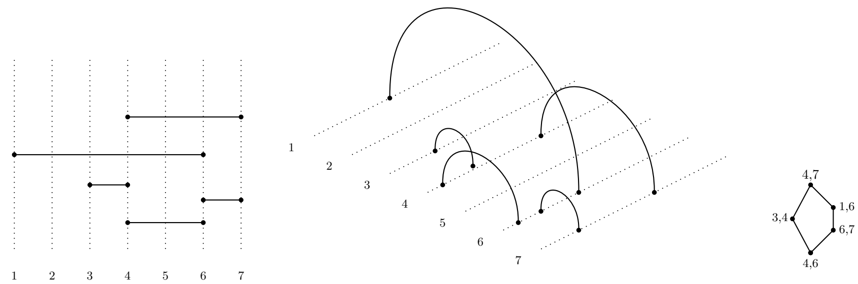

Fig. 2: The heap $m = x_{46}x_{67}x_{34}x_{16}x_{47}$ as a pile of segments and the Hasse diagram of the associated poset.

Definition 3 *For any heap $m \in M$, let $|m|$ denote its length as a product of generators. Moreover, $m \in M$ is called a trivial heap if it is a product of pairwise commuting generators. Let $M^\circ \subset M$ denote the set of trivial heaps.*

Let $\mathbb{Z}[[M]]$ denote the ring of formal power series in M, i.e. all formal sums $\sum_{m \in M} \alpha_m m$ with multiplication induced by the one of M. A fundamental result of from [Cartier and Foata(1969)] is the identity in $\mathbb{Z}[[M]]$ as follows:

$$\left(\sum_{m \in M^\circ} (-1)^{|m|} m \right)^{-1} = \sum_{m \in M} m. \tag{10}$$

Note that M° contains the neutral element of M so that the sum in the left-hand side is invertible, being a formal power series with constant term equal to 1.

Definition 4 *An element $m \in M$ is called a pyramid if the associated poset has a unique maximal element. Let $P \subset M$ denote the subset of pyramids.*

A fundamental result of the theory of heaps links the generating function of pyramids with the one of all heaps [Cartier and Foata(1969), Viennot(1989)]. It essentially relies on the exponential formula for labelled combinatorial objects, and reads:

$$\log \left(\sum_{m \in M} m \right) =_{\text{comm}} \sum_{p \in P} \frac{1}{|p|} p, \tag{11}$$

where the sign $=_{\text{comm}}$ means that the equality holds in any commutative quotient of $\mathbb{Z}[[M]]$. Combining (10) and (11), we obtain:

$$- \log \left(\sum_{m \in M^\circ} (-1)^{|m|} m \right) =_{\text{comm}} \sum_{p \in P} \frac{1}{|p|} p. \tag{12}$$

Now, let us examine how to apply this general equality to the present case.

The following lemma is a direct consequence of the definitions, and permits to identify trivial heaps with noncrossing matchings.

Lemma 3 *The map*

$$\Phi : x_{i_1 j_1} \cdots x_{i_n j_n} \mapsto \{\{i_1, j_1\}, \dots, \{i_n, j_n\}\} \tag{13}$$

defines a bijection between the set of trivial heaps M° and the disjoint union of $\mathcal{N}(V)$ where V runs through the finite subsets (of even cardinal) of $\mathbb{N}_{>0}$.

For a general heap $m \in M$, we can still define $\Phi(m)$ via (13) but it may not be a matching, for example $\Phi(x_{1,2}x_{2,3}) = \{\{1,2\},\{2,3\}\}$. Let us first consider the case of $m \in M$ such that $\Phi(m)$ is really a matching.

Lemma 4 *Let $\sigma \in \mathcal{M}(V)$ for some $V \subset \mathbb{N}_{>0}$. Then the heaps $m \in M$ such that $\Phi(m) = \sigma$ are in bijection with acyclic orientations of $G(\sigma)$. Thus, such a heap $m \in M$ can be identified with a pair (σ, r) where r is an acyclic orientation of $G(\sigma)$.*

Proof: An acyclic orientation r on $G(\sigma)$ defines a partial order on σ by saying that two arches x and y satisfy $x < y$ if there is a directed path from y to x. In this partial order, two crossing arches are always comparable since they are adjacent in $G(\sigma)$. We recover the description of heaps in terms of posets, as described above, so each pair (σ, r) corresponds to a heap $m \in M$ with $\Phi(m) = \sigma$. □

To treat the case of $m \in M$ such that $\Phi(m)$ is not a matching, such as $x_{12}x_{23}$, we are led to introduce a set of commuting variables $(a_i)_{i\geq 1}$ such that $a_i^2 = 0$, and consider the specialization $x_{ij} \mapsto a_i a_j$ which defines a morphism of algebras $\omega : \mathbb{Z}[[M]] \to \mathbb{Z}[[a_1, a_2, \dots]]$. This way, for any $m \in M$ we have either $\omega(m) = 0$, or $\Phi(m) \in \mathcal{M}(V)$ for some $V \subset \mathbb{N}_{>0}$.

Let $m \in M$ such that $\omega(m) \neq 0$. As seen in Lemma 4, it can be identified with the pair (σ, r) where $\sigma = \Phi(m)$, and r is an acyclic orientation of $G(\sigma)$. Then the condition defining pyramids is easily translated in terms of (σ, r), indeed we have $m \in P$ if and only if the acyclic orientation r has a unique source (where a *source* is a vertex having no ingoing arrows).

Under the specialization ω, the generating function of trivial heaps is:

$$\omega\Bigg(\sum_{m \in M^\circ} (-1)^{|m|} m \Bigg) = \sum_{n \geq 0} (-1)^n C_n e_{2n}, \tag{14}$$

where e_{2n} is the $2n$th elementary symmetric functions in the a_i's. Indeed, let $V \subset \mathbb{N}_{>0}$ with $\#V = 2n$, then the coefficient of $\prod_{i\in V} a_i$ in the left-hand side of (14) is $(-1)^n \#\mathcal{N}(V) = (-1)^n C_n$, as can be seen using Lemma 3. In particular, it only depends on n so that this generating function can be expressed in terms of the e_{2n}. Moreover, since the variables a_i have vanishing squares their elementary symmetric functions satisfy

$$e_{2n} = \frac{1}{(2n)!} e_1^{2n},$$

so that the right-hand side of (14) is actually the exponential generating of the Catalan numbers (evaluated at e_1). It remains to understand the meaning of taking the logarithm of the left-hand side of (14) using pyramids and Equation (12).

Note that the relation $=_{\text{comm}}$ becomes a true equality after the specialization $x_{ij} \mapsto a_i a_j$. So taking the image of (12) under ω and using (14), this gives

$$-\log \Bigg(\sum_{n \geq 0} (-1)^n C_n e_{2n} \Bigg) = \sum_{p \in P} \frac{1}{|p|} \omega(p).$$

The argument used to obtain (14) shows as well that the right-hand side of the previous equation is $\sum \frac{x_n}{n} e_{2n}$ where $x_n = \#\{p \in P \ : \ \omega(p) = a_1 \cdots a_{2n}\}$. So we have

$$-\log\left(\sum_{n\geq 0}(-1)^n C_n e_{2n}\right) = \sum_{n\geq 0} \frac{x_n}{n} e_{2n},$$

and comparing this with (9), we obtain $\tilde{k}_{2n}(0) = \frac{x_n}{n}$.

Clearly, a graph with an acyclic orientation always has a source, and it has a unique source only when it is root-connected (for an appropriate root, namely the source). So a pyramid p such that $\omega(p) \neq 0$ can be indentified with a pair (σ, r) where r is a root-connected acyclic orientation of $G(\sigma)$. Then using Proposition 2, it follows that

$$x_n = n \sum_{\sigma\in\mathcal{M}^c(2n)} T_{G(\sigma)}(1,0).$$

Here, the factor n in the right-hand side accounts for the n possible choices of the source in each graph $G(\sigma)$. Eventually, we obtain

$$\tilde{k}_{2n}(0) = \sum_{\sigma\in\mathcal{M}^c(2n)} T_{G(\sigma)}(1,0), \tag{15}$$

i.e. we have proved the particular case $q = 0$ of Theorem 1.

Let us state again the result in an equivalent form. We can consider that if $\sigma \in \mathcal{M}(2n)$, the graph $G(\sigma)$ has a canonical root which the arch containing 1. Then, Equation (15) gives a combinatorial model for the integers $\tilde{k}_{2n}(0)$:

Theorem 2 *The integer $\tilde{k}_{2n}(0)$ counts pairs (σ, r) where $\sigma \in \mathcal{M}^c(2n)$, and r is an acyclic orientation of $G(\sigma)$ whose unique source is the arch of σ containing 1.*

From this, it is possible to give a combinatorial proof that the integers $\tilde{k}_{2n}(0)$ are increasing, as suggested in ([Lassalle(2012)]) who gave an algebraic proof. Indeed, we can check that pairs (σ, r) where $\{1, 3\}$ is an arch of σ are in bijection with the same objects but of size one less, hence $\tilde{k}_{2n}(q) \leq \tilde{k}_{2n+2}(q)$.

Before ending this section, note that the left-hand side of (9) is $-\log(\frac{1}{z}J_1(2z))$ where J_1 is the Bessel function of order 1. There are quite a few other cases where the combinatorics of Bessel functions is related with the theory of heaps, see ([Fédou(1994), Fédou(1995), Bousquet-Mélou and Viennot(1992)]).

5 The case $q = 2$, the exponential formulaThe case q=2, the exponential formula

The specialization at $(1, 2)$ of a Tutte polynomial has combinatorial significance in terms of connected spanning subgraphs (see [Aigner(2007), Chapter 9]), so it is natural to consider the case $q = 2$ of Theorem 1. This case is particular because the factor $(q-1)^{n-1}$ disappears, so that $\tilde{k}_{2n}(2) = k_{2n}(2)$. We can then interpret the logarithm in the sense of combinatorial species, by showing that $\tilde{k}_{2n}(2)$ counts some *primitive* objects and $m_{2n}(2)$ counts *assemblies* of those, just like permutations that are formed by assembling cycles (this is the exponential formula for labelled combinatorial objects, see [Aigner(2007), Chapter 3]). What we obtain is another more direct proof of Theorem 1, based on an interpretation of $T_G(1, 2)$ as follows.

Proposition 3 ([Gioan(2007)]) *If G is a rooted and connected graph, $T_G(1,2)$ is the number of its root-connected orientations.*

This differs from the more traditionnal interpretation of $T_G(1,2)$ in terms of connected spanning subgraphs mentionned above, but it is what naturally appears in this context.

Definition 5 *Let $\mathcal{M}^+(2n)$ be the set of pairs (σ, r) where $\sigma \in \mathcal{M}(2n)$ and r is an orientation of the graph $G(\sigma)$. Such a pair is called an augmented matching, and is depicted with the convention that the arch $\{i,j\}$ lies above the arch $\{k,\ell\}$ if there is an oriented edge $\{i,j\} \to \{k,\ell\}$, and behind it if there is an oriented edge $\{k,\ell\} \to \{i,j\}$.*

See Figure 3 for example. Clearly, $\#\mathcal{M}^+(2n) = m_{2n}(2)$. Indeed, each graph $G(\sigma) = (V,E)$ has $2^{\#E}$ orientations, and $\#E = \mathrm{cr}(\sigma)$, so this follows from (3).

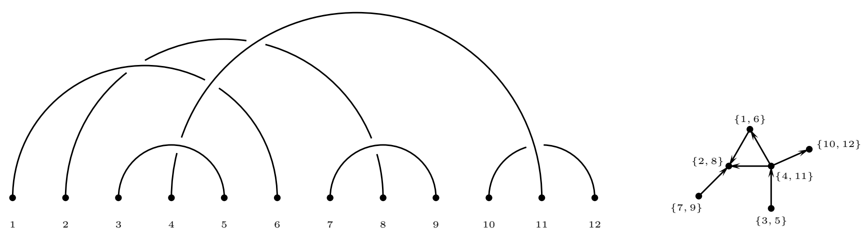

Fig. 3: An augmented matching (σ, r) and the corresponding orientation of $G(\sigma)$.

Notice that if there is no directed cycle in the oriented graph $(G(\sigma), r)$, the augmented matching (σ, r) can be identified with a heap $m \in M$ as defined in the previous section. The one in Figure 3 would be $x_{3,5}x_{4,11}x_{10,12}x_{1,6}x_{7,9}x_{2,8}$. Actually, the application of the exponential formula in the present section is quite reminiscent of the link between heaps and pyramids as seen in the previous section.

Definition 6 *Recall that each graph $G(\sigma)$ is rooted with the convention that the root is the arch containing 1. Let $\mathcal{I}(2n) \subset \mathcal{M}^+(2n)$ be the set of augmented matchings (σ, r) such that σ is connected and r is a root-connected orientation of $G(\sigma)$. The elements of $\mathcal{I}(2n)$ are called primitive augmented matchings. For any $V \subset \mathbb{N}_{>0}$ with $\#V = 2n$, we also define the set $\mathcal{I}(V)$, with the same combinatorial description as $\mathcal{I}(2n)$ except that matchings are based on the set V instead of $\{1,\dots,2n\}$.*

Using Proposition 3, we have

$$\#\mathcal{I}(2n) = \sum_{\sigma \in \mathcal{M}^c(2n)} T_{G(\sigma)}(1,2),$$

so that the particular case $q = 2$ of Theorem 1 is the equality $\#\mathcal{I}(2n) = k_{2n}(2)$. To prove this from (1) and using the exponential formula, we have to see how an augmented matching can be decomposed into an assembly of primitive ones, as stated in Proposition 4 below. This decomposition thus proves the case $q = 2$ of Theorem 1. Note also that the bijection given below is equivalent to the first identity in (2).

Proposition 4 *There is a bijection*

$$\mathcal{M}^+(2n) \longrightarrow \biguplus_{\pi \in \mathcal{P}(n)} \prod_{V \in \pi} \mathcal{I}(V).$$

Proof: Let $(\sigma, r) \in \mathcal{M}^+(2n)$, the bijection is defined as follows. Consider the vertices of $G(\sigma)$ which are accessible from the root. This set of vertices defines a matching on a subset $V_1 \subset \{1, \dots, 2n\}$. For example, in the case in Figure 3, the root is $\{1, 6\}$ and the only other accessible vertex is $\{2, 8\}$, so $V_1 = \{1, 2, 6, 8\}$. Together with the restriction of the orientation r on this subset of vertices, this defines an augmented matching $(\sigma_1, r_1) \in \mathcal{M}^+(V_1)$ which by construction is primitive. By repeating this operation on the set $\{1, \dots, 2n\} \backslash V_1$, we find $V_2 \subset \{1, \dots, 2n\} \backslash V_1$ and $(\sigma_2, r_2) \in \mathcal{I}(V_2)$, and so on. See Figure 4 for the result, in the case of the augmented matching in Figure 3.

It is easy to describe explicitly the inverse bijection. □

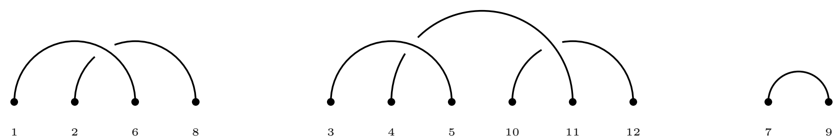

Fig. 4: Decomposition of an augmented matching.

6 Cumulants of the free Poisson law

The free Poisson law appears in free probability and random matrices theory, and can be characterized by the fact that all free cumulants are equal to some $\lambda > 0$. It follows that the moments count noncrossing partitions (which form a subset $\mathcal{NC}(n) \subset \mathcal{P}(n)$), i.e. the coefficients are given by the Narayana numbers ([Stanley(1999)]):

$$m_n(\lambda) = \sum_{\pi \in \mathcal{NC}(n)} \lambda^{\#\pi} = \sum_{k=1}^{n} \frac{\lambda^k}{n} \binom{n}{k} \binom{n}{k-1}.$$

The corresponding cumulants are as before defined by

$$\sum_{n \geq 1} k_n(\lambda) \frac{z^n}{n!} = \log \left(\sum_{n \geq 0} m_n(\lambda) \frac{z^n}{n!} \right).$$

For any set partition π, we can define a crossing graph $G(\pi)$, whose vertices are blocks of π, and there is an edge between $b_1, b_2 \in \pi$ if $\{b_1, b_2\}$ is not a noncrossing partition. Denoting $\mathcal{P}^c(n) \subset \mathcal{P}(n)$ the set of connected set partitions, i.e. those whose crossing graph is connected, the two different proofs for the semicircular cumulants show as well the following:

Theorem 3 *For any* $n \geq 1$, *we have:*

$$k_n(\lambda) = - \sum_{\pi \in \mathcal{P}^c(n)} (-\lambda)^{\#\pi} T_{G(\pi)}(1, 0).$$

Acknowledgements

This work was initiated during the trimester “Bialgebras in Free Probability” at the Erwin Schrödinger Institute in Vienna. I thank Natasha Blitvić and Franz Lehner for enlightening conversations.

References

[Aigner(2007)] M. Aigner. *A course in enumeration*, volume 238 of *Graduate Texts in Mathematics*. Springer, Berlin, 2007.

[Anshelevich et al.(2010)] M. Anshelevich, S. T. Belinschi, M. Bożejko, and F. Lehner. Free infinite divisibility for q-Gaussians. *Math. Res. Lett.*, 17(5):905–916, 2010.

[Belinschi et al.(2011)] S. T. Belinschi, M. Bożejko, F. Lehner, and R. Speicher. The normal distribution is +-infinitely divisible. *Adv. Math.*, 226(4):3677–3698, 2011.

[Bousquet-Mélou and Viennot(1992)] M. Bousquet-Mélou and X. G. Viennot. Empilements de segments et q-énumération de polyominos convexes dirigés. *J. Combin. Theory Ser. A*, 60(2):196–224, 1992.

[Bożejko and Speicher(1992)] M. Bożejko and R. Speicher. An example of a generalized Brownian motion. II. In *Quantum probability & related topics*, QP-PQ, VII, pages 67–77. World Sci. Publ., River Edge, NJ, 1992.

[Bożejko et al.(1997)Bożejko, Kümmerer, and Speicher] M. Bożejko, B. Kümmerer, and R. Speicher. q-Gaussian processes: non-commutative and classical aspects. *Comm. Math. Phys.*, 185(1):129–154, 1997.

[Burman and Shapiro(2006)] Y. Burman and B. Shapiro. Around matrix-tree theorem. *Math. Res. Lett.*, 13(5-6):761–774, 2006.

[Cartier and Foata(1969)] P. Cartier and D. Foata. *Problèmes combinatoires de commutation et réarrangements*. Lecture Notes in Mathematics, No. 85. Springer-Verlag, Berlin, 1969.

[Fédou(1994)] J.-M. Fédou. Combinatorial objects enumerated by q-Bessel functions. *Rep. Math. Phys.*, 34(1):57–70, 1994. ISSN 0034-4877. Algebraic topology and anyon superconductivity (Poznań, 1993).

[Fédou(1995)] J.-M. Fédou. Sur les fonctions de Bessel. *Discrete Math.*, 139(1-3):473–480, 1995. ISSN 0012-365X. Formal power series and algebraic combinatorics (Montreal, PQ, 1992).

[Gioan(2007)] E. Gioan. Enumerating degree sequences in digraphs and a cycle-cocycle reversing system. *European J. Combin.*, 28(4):1351–1366, 2007. ISSN 0195-6698.

[Greene and Zaslavsky(1983)] C. Greene and T. Zaslavsky. On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs. *Trans. Amer. Math. Soc.*, 280(1):97–126, 1983.

[Hiai and Petz(2000)] F. Hiai and D. Petz. *The semicircle law, free random variables and entropy*, volume 77 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.

[Ismail et al.(1987)] M. E. H. Ismail, D. Stanton, and G. Viennot. The combinatorics of q-Hermite polynomials and the Askey-Wilson integral. *European J. Combin.*, 8(4):379–392, 1987. ISSN 0195-6698.

[Lassalle(2012)] M. Lassalle. Two integer sequences related to catalan numbers. *J. Comb. Theory, Ser. A*, 119(4):923–935, 2012.

[Lehner(2002)] F. Lehner. Free cumulants and enumeration of connected partitions. *European J. Combin.*, 23(8):1025–1031, 2002.

[Nica and Speicher(2006)] A. Nica and R. Speicher. *Lectures on the combinatorics of free probability*, volume 335 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.

[Stanley(1999)] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. ISBN 0-521-56069-1; 0-521-78987-7. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[Viennot(1989)] G. X. Viennot. Heaps of pieces. I. Basic definitions and combinatorial lemmas. In *Graph theory and its applications: East and West (Jinan, 1986)*, volume 576 of *Ann. New York Acad. Sci.*, pages 542–570. New York Acad. Sci., New York, 1989.