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Star^1-convex functions on tropical linear spaces of complete graphs

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Abstract. Given a fan Δ and a cone $\sigma \in \Delta$, let star$^1(\sigma)$ be the set of cones that contain σ and are one dimension bigger than σ. In this paper we study two cones of piecewise linear functions defined on Δ: the cone of functions which are convex on $\mathrm{star}^1(\sigma)$ for all cones, and the cone of functions which are convex on $\mathrm{star}^1(\sigma)$ for all cones of codimension 1. We give nice combinatorial descriptions for these two cones given two different fan structures on the tropical linear space of complete graphs. For the complete graph K_5, we prove that with the finer fan subdivision the two cones are not equal, but with the coarser subdivision they are the same. This gives a negative answer to a question of Gibney-Maclagan that for the finer subdivision the two cones are the same.

Résumé. Soit Δ un fan, pour $\sigma \in \Delta$ nous définissons star$^1(\sigma)$ comme l'ensemble de cônes qui contiennent σ dont la dimension est un de plus que la dimension de σ. Nous étudions deux cônes d'applications linéaires par morceaux définis sur Δ: le cône de fonctions convexes sur star$^1(\sigma)$, où $\sigma \in \Delta$ est un cône quelconque, et le cône de fonctions convexes sur star$^1(\sigma)$ où σ est un cône de codimension 1. Étant donnés deux structures sur l'espace tropical linéaire de graphes complets, nous donnons de beaux descriptions combinatoires des cônes décrits en haut. Pour le graphe complet K_5, on démontre que avec la subdivision en fans plus fine, les deux cônes sont différentes, mais avec la subdivision plus gros ils sont cônes sont les mêmes. Ce résultant réponde négativement une question de Gibney-Maclagan.

Keywords: Tropical linear space, functions convex on a fan, matroid, nef cone

1 Introduction

Consider piecewise linear functions over a fan Δ which are linear on each of its cones. Given a cone $\sigma \in \Delta$ we wish to study such functions that are convex on $\mathrm{star}^1(\sigma)$, the set of cones containing σ and one dimension bigger than σ. Gibney and Maclagan [5] defined the cones $\mathfrak{L}(\Delta)$, consisting of piecewise linear functions convex on $\mathrm{star}^1(\sigma)$ for all cones of Δ, and $\mathfrak{U}(\Delta)$, consisting of piecewise linear functions convex on $\mathrm{star}^1(\sigma)$ for all cones of codimension 1.

In this paper we will investigate the cones $\mathfrak{L}(\Delta)$ and $\mathfrak{U}(\Delta)$ when Δ is the tropical variety of the complete graph K_n with two different fan structures: $\mathcal{T}_f(K_n)$ and $\mathcal{T}_c(K_n)$ which we define in section 2. The tropical variety of K_n is also called the Bergman fan of K_n. One main idea we will study is whether:

Question 1.1 (Gibney-Maclagan) *Is $\mathfrak{L}(\Delta) = \mathfrak{U}(\Delta)$ when Δ is a fan subdivision of the tropical variety of the complete graph?*

This question was proposed by Gibney and Maclagan as a method to prove that the nef cone of the moduli space $\overline{M}_{0,n}$ is polyhedral. In general nef$(\overline{M}_{0,n})$ is hard to compute, but useful in describing maps from $\overline{M}_{0,n}$ to a projective space. In [5] they propose a new candidate for this nef cone which is closely related to $\mathfrak{L}(\Delta)$ and $\mathfrak{U}(\Delta)$. By proving the question above it would follow that the candidate they propose is indeed the nef cone of $\overline{M}_{0,n}$. We explain more of this motivation below.

Given an arbitrary matroid M the fine subdivision $\mathcal{T}_f(M)$ is easier to describe, however the coarse subdivision is in general more useful. In order to prove the equality $\mathfrak{L}(\Delta) = \mathfrak{U}(\Delta)$ for the coarse subdivision, Gibney and Maclagan suggested the following approach:

Question 1.2 *Does* $\mathfrak{L}(\mathcal{T}_f(K_n)) = \mathfrak{U}(\mathcal{T}_f(K_n))$*?*

Question 1.3 *Does* $\mathfrak{L}(\mathcal{T}_f(K_n)) = \mathfrak{U}(\mathcal{T}_f(K_n))$ *imply that* $\mathfrak{L}(\mathcal{T}_c(K_n)) = \mathfrak{U}(\mathcal{T}_c(K_n))$*?*

In Section 5 we verify that $\mathfrak{L}(\mathcal{T}_c(K_5)) = \mathfrak{U}(\mathcal{T}_c(K_5))$. However, in Section 4 we prove that $\mathfrak{L}(\mathcal{T}_f(K_5)) \subsetneq \mathfrak{U}(\mathcal{T}_f(K_5))$. Thus we show this approach does not work.

The tropical variety of the complete graph can be described using rooted trees. In Theorem 5.2 and Corollary 5.5 we prove that the cones $\mathfrak{L}(\mathcal{T}_c(K_n))$ and $\mathfrak{U}(\mathcal{T}_c(K_n))$ have a nice combinatorial structure which depends on non-binary vertices of rooted trees. In the case of the fine subdivision, in Theorem 4.2 we construct a combinatorial structure of $\mathfrak{U}(\mathcal{T}_f(K_n))$ using intervals of flats.

We now describe the motivation to the problem studied which is due to Gibney and Maclagan [5]. This motivation is independent from the rest of the paper. We start by talking about divisors, for more information see [7]. A divisor D of a variety X is a generalization of a codimension one subvariety. A curve C in X generally intersects a codimension one subvariety D in a finite number of points; let $D \cdot C$ count the number of times they intersect. The operation $D \cdot C$ is defined as a generalization of product just described. Weil divisors are finite sums of codimension one subvarieties with real coefficients. Extend the operation $D \cdot C$ to Weil divisors linearly. A Weil divisor is effective if all its coefficients are nonnegative.

We are actually interested in Cartier divisors, a generalization of codimension one subvarieties which is better behaved. The two notions of divisors of X are related as there is a map from Cartier divisors to Weil divisors, which is sometimes an isomorphism. For these divisors we can define the number $D \cdot C$ where D is a Cartier divisor and C a curve. A Cartier divisor D of X is numerically effective (abbreviated as *nef*) if $D \cdot C \geq 0$ for every curve C in X.

Define an equivalence class of Cartier divisors as $D_1 \sim D_2$ if for every curve C in X we have that $D_1 \cdot C = D_2 \cdot C$. The nef cone of X, nef(X), is the cone generated by classes of nef divisors of X. The interior of the nef cone of X consists of the ample divisors, which are central to describe the maps from X to a projective space. The nef cone is generally very hard to compute and there is a lot of interest in giving simpler descriptions for particular X.

We are interested in computing the nef cone of moduli spaces $\overline{M}_{0,n}$. We give some idea of these spaces: for more information see [6]. Moduli spaces are spaces whose points correspond to geometric objects. One of the most studied moduli spaces is $M_{0,n}$, whose points correspond to isomorphism classes of smooth curves of genus 0 with n distinct marked points. The moduli space $\overline{M}_{0,n}$ is a compactification of $M_{0,n}$ and it is in general hard to compute. There are a lot of questions about the nef cone of $\overline{M}_{0,n}$, for example it is not known whether it is polyhedral. The F-conjecture states that nef$(\overline{M}_{0,n})$ is equal to the polyhedral cone $F_{0,n}$ generated by the divisors of $\overline{M}_{0,n}$ that nonnegatively intersect certain family of curves called the F-curves. In general $\overline{M}_{0,n} \subseteq F_{0,n}$ and the conjecture is known to be true for $n \leq 7$.

Recent work of Angela Gibney and Diane Maclagan [5] has been the proposal of a new candidate for nef$(\overline{M}_{0,n})$ arising from a connection between $M_{0,n}$ and the Grassmannian $G(2, n)$, the space of two dimensional linear subspaces of $\mathbb{C}^n$. They embed $\overline{M_{0,n}}$ into a toric variety X_Δ where Δ is the quotient of tropical Grassmanian trop$G(2,n)$ by its lineality space. This fan is the same as the tropical variety of the complete graph K_{n-1}. The new candidate for nef$(\overline{M}_{0,n})$ they propose is the cone $\mathfrak{L}(\Delta)$. Gibney and Maclagan describe the upper bound $F_{0,n}$ in terms of $\mathfrak{U}(\Delta)$ and find a lower bound for nef$(\overline{M}_{0,n})$ which depends on $\mathfrak{L}(\Delta)$. This bounds imply that if $\mathfrak{L}(\Delta) = \mathfrak{U}(\Delta)$ then the F-conjecture follows. Moreover we will see that $\mathfrak{L}(\Delta)$ and $\mathfrak{U}(\Delta)$ have a really nice combinatorial structure, so we would also have a nice description for nef$(\overline{M}_{0,n})$.

2 Tropical Linear Spaces

The tropical semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ is defined by

$$x \oplus y = \min(x, y) \qquad\qquad x \odot y = x + y.$$

Given a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ we obtain its tropicalization $p = \text{trop}(f)$ by replacing $+$ by $\oplus$ and $\cdot$ by $\odot$. Notice that a tropical polynomial p is the minimum of a finite collection of linear functions $\mathcal{L}$. Define a tropical hypersurface as

$$V(p) := \{w \in \mathbb{R}^n : \min_{f \in \mathcal{L}}(f(w)) \text{ is attained at least twice}\}.$$

Given an ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ define the tropical variety of I as

$$\mathcal{T}(I) := \bigcap_{f \in I} V(\text{trop}(f)).$$

If I is a linear ideal then we will call $\mathcal{T}(I)$ a **tropicalized linear space**.

We can generalize tropical linear spaces using matroids, a combinatorial object that generalizes various notions of independence.

Definition 2.1 *A* ***matroid*** *M is a pair $(E, \mathcal{C})$ where E is a finite set and a collection $\mathcal{C}$ of subsets of E, referred to as the circuits of M, such that $\emptyset \notin \mathcal{C}$, no proper subset of a circuit is a circuit, and if C_1 and C_2 are distinct circuits and $e \in C_1 \cap C_2$, then there is a circuit contained in $(C_1 \cup C_2) \setminus \{e\}$.*

Example 2.2 *The complete graph on n vertices determines a matroid with E the set of edges and $\mathcal{C}$ to be the set of cycles. We denote this matroid K_n.*

One can associate a matroid $M(I)$ to I by defining the circuits to be the inclusion minimal sets of the form $C_f := \{i : \text{ the coefficient of } x_i \text{ in } f \text{ is nonzero}\}$ where $f \in I$ is linear. It turns out that $\mathcal{T}(I)$ only depends on $M(I)$. The following construction, due to Sturmfels, is a generalization of the above construction to any matroid M.

Definition 2.3 *Given a matroid M with ground set E, where $|E| = n$, the* ***tropical linear space*** *associated to M is*

$$\mathcal{T}(M) := \{w \in \mathbb{R}^n : \min_{i \in C}(w_i) \text{ is attained twice for all circuits } C \text{ of } M\}$$

This linear space can be subdivided into a fan in many different ways. We now describe two possible ways which arise from combinatorics.

2.1 Fine subdivision of $\mathcal{T}(M)$

Given a matroid $M = (E, \mathcal{C})$ a subset F of E is a **flat** of M if whenever $|C \setminus F| \geq 1$ for a circuit C, then $|C \setminus F| \geq 2$. The **rank** of a flat F, denoted by $r(F)$, is the largest $|A|$ such that $A \subset F$ and A contains no circuits. The lattice of flats of M is ordered by inclusion and graded by the rank function.

Theorem 2.4 (Ardila-Klivans [[1] Sect. 2 Thm. 1]) *The cones of the* ***fine subdivision****, $\mathcal{T}_f(M)$, of $\mathcal{T}(M)$ are in one to one correspondence with the flags of flats of the matroid containing $\emptyset$ and E. The correspondence is given by*

$$\mathcal{F} = \{\emptyset = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{k-1} \subsetneq F_k = E\}$$

$$\updownarrow$$

$$pos(e_{F_1}, \ldots, e_{F_{k-1}}, e_E, -e_E) := \left\{\lambda_1 e_{F_1} + \cdots \lambda_{k-1} k e_{F_{k-1}} + \mu e_E : \lambda_i \geq 0\right\}$$

where $e_F := \sum_{i \in F} e_i$ and $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$.

Note that this fan is simplicial and it has a lineality space generated by e_E.

2.2 Coarse subdivision of $\mathcal{T}(K_n)$

We now discuss the coarsest fan structure that $\mathcal{T}(M)$ can have when $M = K_n$. The rays of this fan are vectors of the form e_F where $F \subseteq K_n$ is also a complete graph. A collection $\mathcal{C}$ of sets is **compatible** if for every $S, T \in \mathcal{C}$ either $S \subseteq T$, $T \subseteq S$ or $S \cap T = \emptyset$.

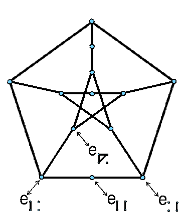
Theorem 2.5 (Feichtner-Sturmfels [[3] Thm. 4.1]) *Given $F \subseteq K_n$ a complete graph let V_F be the set of vertices of F. The cones of the* ***coarse subdivision****, $\mathcal{T}_c(K_n)$, of $\mathcal{T}(K_n)$ are*

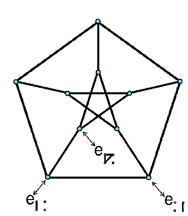
$$pos(e_{F_1}, \ldots, e_{F_k}, e_E, -e_E)$$

where each F_i is a complete subgraph of K_n and the collection $\mathcal{C} = \{V_{F_1}, \ldots, V_{F_k}, V_E\}$ of the vertices of each flat is compatible.

This fan is refined by the fine subdivision $\mathcal{T}_f(K_n)$ of $\mathcal{T}(K_n)$. The following example illustrates the fine and coarse subdivisions of $\mathcal{T}(K_4)$. It is explained in detail in [1].

Example 2.6 ([1]) *The following drawings are obtained by intersecting each fan with the hyperplane $\{w \in \mathbb{R}^6 : w_1 + \ldots + w_6 = 0\}$ and the sphere $\{w \in \mathbb{R}^6 : w_1^2 + \ldots + w_6^2 = 1\}$. The outcomes are the simplicial complexes below, which illustrate that $\mathcal{T}_c(K_4)$, on the right, coarsens $\mathcal{T}_f(K_4)$, on the left.*





3 Star^1-convex functions defined on a fan

Let $\Delta \subseteq \mathbb{R}^n$ be a fan. Given a cone $\sigma \in \Delta$ let

$$\mathrm{star}^1(\sigma) := \{\tau \in \Delta : \sigma \subseteq \tau \text{ and } \dim(\tau) = \dim(\sigma) + 1\}.$$

Define $N(\Delta)$ to be the set of piecewise linear functions $\varphi : \bigcup_{\sigma\in\Delta} \Delta \to \mathbb{R}$ that are linear on each cone.

Definition 3.1 *Let $\sigma \in \Delta$, we say that $\varphi \in N(\Delta)$ is* star^1**-convex** *on σ if it satisfies that*

$$\varphi\,(u_1 + \cdots + u_k) \leq \varphi(u_1) + \cdots \varphi(u_k)$$

for each $u_1, \ldots, u_k$ such that $u_1 + \cdots + u_k \in \sigma$ and each $u_i \in \tau_i$ for some $\tau_i \in \mathrm{star}^1(\sigma)$.

We will write $u \in \mathrm{star}^1(\sigma)$ to mean that there exists $\tau \in \mathrm{star}^1(\sigma)$ such that $u \in \tau$. Consider the following cones in $N(\Delta)$:

$$\begin{aligned}
\mathfrak{C}(\sigma) &:= \left\{\varphi \in N(\Delta) : \varphi \text{ is } \mathrm{star}^1\text{-convex on } \sigma\right\},\\
\mathfrak{L}(\Delta) &:= \bigcap_{\sigma\in\Delta} \mathfrak{C}(\sigma),\\
\mathfrak{U}(\Delta) &:= \bigcap_{\sigma\in\Delta,\ \mathrm{codim}(\sigma)=1} \mathfrak{C}(\sigma).
\end{aligned}$$

The following theorem gives a containment for the cones $\mathfrak{L}$ and $\mathfrak{U}$ for two fan subdivisions, one coarser than the other. We state it in terms of $\mathcal{T}_f(K_n)$ and $\mathcal{T}_c(K_n)$, but the result holds pair of comparable fans.

Proposition 3.2 *We have that $\mathfrak{L}(\mathcal{T}_c(K_n)) \subseteq \mathfrak{L}(\mathcal{T}_f(K_n))$ and $\mathfrak{U}(\mathcal{T}_c(K_n)) \subseteq \mathfrak{U}(\mathcal{T}_f(K_n))$.*

4 Fine Subdivision (flats and flags)

In this section we restrict our attention to the fan $\mathcal{T}_f(M)$. Notice that every function $\varphi \in N(\mathcal{T}_f(M))$ is completely characterized by its values on the rays e_F for a flat $F \neq \emptyset$, therefore $N(\mathcal{T}_f(M)) \cong \mathbb{R}^{\#\text{ of flats}-1}$. To ease notation we will let $\varphi(F) := \varphi(e_F)$; we will also write $\mathfrak{L}_f(M)$ and $\mathfrak{U}_f(M)$ instead of $\mathfrak{L}(\mathcal{T}_f(M))$ and $\mathfrak{U}(\mathcal{T}_f(M))$, respectively. Given a flag of flats $\mathcal{F}$, we will use this notation to mean both the cone it defines in $\mathcal{T}_f(M)$ and the flag of flats.

Notice that cones of codimension one in $\mathcal{T}_f(M)$ correspond to flags of the form

$$\mathcal{F} = \{\emptyset = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{k-1} = F \subsetneq F_{k+1} = G \subsetneq \cdots \subsetneq F_n = E\} \tag{1}$$

with $rank(F_i) = i$. The following theorem gives a description of the cone $\mathfrak{C}(\mathcal{F})$ of functions star^1-convex on $\mathcal{F}$ as a halfspace that only depends on the interval $[F, G]$ in the lattice of flats.

Theorem 4.1 *Suppose we have a flag $\mathcal{F}$ corresponding to a cone of codimension 1, that is, a flag of the form (1). Then the function φ is* star^1*-convex on $\mathcal{F}$ if and only if*

$$(|(F,G)| - 1)\varphi(F) + \varphi(G) \leq \sum_{H\in(F,G)} \varphi(H),$$

where F, G are the flats in the flag such that $(F,G) := \{H : F \subsetneq H \subsetneq G\}$ is a nonempty interval in the lattice of flats.

Proof: Let $(F, G) = \{H_1, \dots, H_m\}$. Since these sets are flats of a matroid then $H_1 \setminus F, \dots, H_m \setminus F$ partition $G \setminus F$. Suppose $u_1, \dots, u_k$ satisfy that $\sum_i u_i \in \mathcal{F}$ and each $u_i \in \mathcal{F}_i$ where $\mathcal{F}_i \in \mathrm{star}^1(\mathcal{F})$. Then by writing each u_i as a nonnegative linear combination of the rays generating the cone corresponding to $\mathcal{F}_i$ in and combining terms we obtain the following equation:

$$\mu_1 e_{F_1} + \dots + \mu_{k-1} e_F + \mu_{k+1} e_G + \dots + \mu_n e_{F_n} = \lambda_1 e_{H_1} + \dots + \lambda_m e_{H_m}, \tag{2}$$

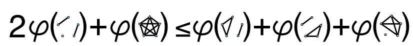
where the coefficients are nonnegative. By restricting this equation to different values $i \in E$ we can show that $\mu_j = 0$, $\lambda_j = \mu_{k+1}$, $\mu_j = 0$ if $j < k-1$, and $\mu_{k-1} = (m-1)\mu_{k+1}$, if $k - 1 \neq 0$. So really (2) looks like

$$(m-1)\mu_{k+1} e_F + \mu_{k+1} e_G = \mu_{k+1} e_{H_1} + \dots + \mu_{k+1} e_{H_m}.$$

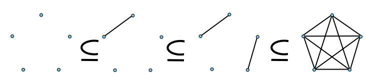
For $\varphi \in N(\mathcal{T}_f(M))$ to be star^1-convex on $\mathcal{F}$ we must have that when we apply φ to (2) the $=$ sign should become a $\leq$ and this characterizes $\mathfrak{C}(\mathcal{F})$. □

Theorem 4.2 *$\mathfrak{U}_f(M)$ is the intersection of the halfspaces described above, where the intersection runs over all flats F, G with $F \subseteq G$ and $rank(F) + 2 = rank(G)$.*

Example 4.3 *Let $M = K_5$. It has 51 nonempty flats and $\mathcal{T}(K_5)$ is 3-dimensional. Applying the theorems above, we have that $\mathfrak{U}_f(K_5)$ is the intersection of 115 halfspaces in $\mathbb{R}^{51}$. By using the software polymake [4] we find that $\mathfrak{U}_f(K_5)$ has dimension 51. As an illustration of the theorem, the inequality*



helps define $\mathfrak{U}_f(K_5)$ and it is associated to the flag



If $\mathcal{F}$ has codimension bigger than one, the cone $\mathfrak{C}(\mathcal{F})$ can be the intersection of many halfspaces. However, the theorem below and its following corollary tell us that given a flag $\mathcal{F}$, we can focus our attention to the equations that depend only on the nonempty intervals (F_i, F_{i+1}), and thus simplifying the computation.

Theorem 4.4 *Let $\mathcal{F} = \{\emptyset = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_n = E\}$ be a flag corresponding to a cone of codimension ≥ 1. Suppose that $F_{i_1} \subsetneq F_{i_2} \subsetneq \dots \subsetneq F_{i_k}$ are all the flats in $\mathcal{F}$ such that the interval (F_i, F_{i+1}) is nonempty in the lattice of flats. We then have that λ is a nonnegative solution to*

$$\sum_{i=1}^{n} \lambda_{F_i} e_{F_i} = \sum_{\substack{F \text{ flat} \\ \{F\} \cup \mathcal{F} \text{ flag}}} \lambda_F e_F, \tag{3}$$

if and only if $\lambda = \lambda^1 + \dots + \lambda^k$ where each λ^j is a nonnegative solution to

$$\lambda^j_{F_{i_j}} e_{F_{i_j}} + \lambda^j_{F_{i_j+1}} e_{F_{i_j+1}} = \sum_{F \in (F_j, F_{j+1})} \lambda^j_F e_F, \tag{4}$$

and $\lambda^j_F = 0$ if it does not appear in the above equation.

Proof: It suffices to prove that λ is a nonnegative solution to equation (3) if and only if $\lambda = \lambda^1 + \mu$ where λ^1 is a nonnegative solution to equation (4) for $j = 1$, and μ is a nonnegative solution to

$$\sum_{i=i_1+1}^{n} \mu_{F_i} e_{F_i} = \sum_{\substack{F \supseteq F_{i_1+1} \\ \{F\} \cup \mathcal{F} \text{ flag}}} \mu_F e_F, \tag{5}$$

where $\mu_F = 0$ if it does not appear in the above equation.

We can only extend $\mathcal{F}$ into a bigger flag by adding one flat in an interval (F_i, F_{i+1}). By restricting equation (3) to $x \in F_1$ and to $y \in F_{i_1} \setminus F_{i_1-1}$, we can show that $\lambda_i = 0$ for all $i < i_1$ if λ is a nonnegative solution to (3). Throughout this proof we will denote $F_{i_1} = G$ and $F_{i_1+1} = H$.

Assume λ^1 and μ are as desired, then $\lambda = \lambda^1 + \mu$ satisfies equation (3) because:

$$\sum_{i=1}^{n} \lambda_{F_i} e_{F_i} = \sum_{i=1}^{n} \lambda^1_{F_i} e_{F_i} + \sum_{i=1}^{n} \mu_{F_i} e_{F_i} = \sum_{F \in (G,H)} \lambda_F e_F + \sum_{\substack{F \supseteq H \\ \{F\} \cup \mathcal{F} \text{ flag}}} \lambda_F e_F = \sum_{\substack{F \text{ flat} \\ \{F\} \cup \mathcal{F} \text{ flag}}} \lambda_F e_F.$$

Now assume λ is a solution to (3). If the interval (F_{i_1+1}, F_{i_1+2}) is empty, define:

$$\lambda^1_F = \begin{cases} \lambda_F, & \text{if } F \in [G, H] \\ 0, & \text{otherwise} \end{cases}$$

and $\mu = \lambda - \lambda^1$. If we restrict equation λ to any $x \in F_{i_1+2} \setminus H$, we can show that μ satisfies the desired equation. If we now restrict equation λ to some $x \in G$ we can show that λ^1 satisfies the desired equation.

If (F_{i_1+1}, F_{i_1+2}) is nonempty, let $x \in H \setminus G$ and restrict the equation for λ to x. We then get that

$$\lambda_H + \sum_{i=i_1+2}^{n} \lambda_{F_i} = \sum_{\substack{x \in F \\ F \in (G,H)}} \lambda_F + \sum_{\substack{F \supseteq H \\ \{F\} \cup \mathcal{F} \text{ flag}}} \lambda_F. \tag{6}$$

Notice that if we choose a different $x \in H \setminus G$ we get that the sum $\sum_{\substack{x \in F \\ F \in (G,H)}} \lambda_F$ does not vary. Define

$$\lambda^1_I = \begin{cases} \lambda_I, & \text{if } I \in [G, H) \\ \sum_{\substack{t \in F \\ F \in (G,H)}} \lambda_F, & \text{if } I = H \\ 0, & \text{otherwise,} \end{cases}$$

and $\mu = \lambda - \lambda^1$. If we restrict the equation that λ satisfies to some $y \in F_{i_1+2} \setminus H$, we can prove that μ is nonnegative. We can similarly restrict equation λ for some $z \in G$ and combine the equations we get by restricting to x, y, z we can show that λ^1 and μ are solutions for equations (4) and (5), respectively. □

Finding the nonnegative solutions to equation (3) allows us to describe $\mathfrak{C}(\mathcal{F})$ for a flag $\mathcal{F}$ as the intersection of cones, each depending on the nonempty intervals (F_i, F_{i+1}).

Corollary 4.5 *Given a flag $\mathcal{F} = \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = E\}$, then*

$$\mathfrak{C}(\mathcal{F}) = \bigcap_{i=1}^{n-1} \mathfrak{C}\left(F_i, F_{i+1}\right),$$

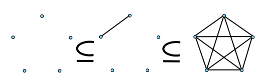
where

$$\mathfrak{C}(F_i, F_{i+1}) = \{\varphi \in N(\mathcal{T}_f(K_n)) : \ \lambda_{F_i}\varphi(F_i) + \lambda_{F_{i+1}}\varphi(F_{i+1}) \le \sum_{F \in (F_i, F_{i+1})} \lambda_F \varphi(F)$$
$$\textit{for each } \lambda \ge 0 \textit{ solution for } 4\}.$$

We now focus our attention to whether $\mathfrak{L} = \mathfrak{U}$ for the fine subdivision of $\mathcal{T}(M)$.

Theorem 4.6 *It is not true that $\mathfrak{L}_f(K_n) = \mathfrak{U}_f(K_n)$, in particular for $n = 5$, $\mathfrak{L}_f(K_n) \subsetneq \mathfrak{U}_f(K_n)$.*

Proof: We show that $\mathfrak{L}_f(K_5) \subsetneq \mathfrak{U}_f(K_5)$ by finding a function $\varphi \in \mathfrak{U}_f(K_5) \setminus \mathfrak{C}(\mathcal{F})$ if $\mathcal{F}$ is



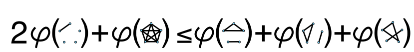
Let $F \subseteq G$ be the two nonempty flats in the above flag. Let $H_1, \ldots, H_{13}$ be all the flats in (F, G). Notice that $\mathfrak{C}(\mathcal{F})$ is the set of all functions $\varphi \in N(\mathcal{T}_f(K_n))$ such that

$$a\varphi(F) + b\varphi(G) \le c_1\varphi(H_1) + \cdots + c_{13}\varphi(H_{13}),$$

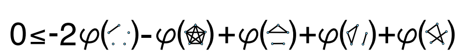
where $a, b, c_1, \ldots, c_{13}$ are all the nonegative solutions to the equation

$$ae_F + be_G = c_1 e_{H_1} + \cdots + c_{13} e_{H_{13}}.$$

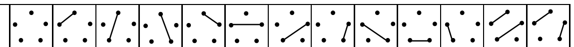
We use the software polymake to find its generating rays of the cone these solutions generate; these rays give rise to the 47 inequalities that define $\mathfrak{C}(\mathcal{F})$. Of these inequalities it turns out that


(7)

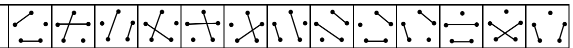
makes the intersection smaller and this is the only one dimensional flag, up to symmetry, that makes $\mathfrak{L}_f(K_5)$ smaller. To verify this, we change this inequality into


(8)

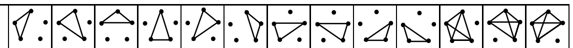
then minimize the equation on the right over $\mathfrak{U}_f(K_5)$, using the software Mathematica [8]. The outcome is a function φ described by the table below. This function is not in $\mathfrak{L}_f(K_5)$, but it does satisfy the 120 inequalities defining $\mathfrak{U}_f(K_5)$.



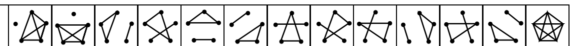
flat													
φ (flat)	0	4	0	0	0	2	2	0	2	2	1	4	4



flat													
φ (flat)	4	2	0	2	2	-2	1	2	-4	1	0	4	-2



flat													
φ (flat)	4	4	4	0	2	-2	4	4	4	4	4	4	4



flat													
φ (flat)	-4	4	-4	-4	-4	4	-4	4	4	-4	0	-2	-4

$\square$

5 Coarse subdivision (partitions and collections)

In this section look at the case $M = K_n$ with the coarse fan structure. In Theorems 5.1 and 5.5 describe for $\mathfrak{U}(\mathcal{T}_c(K_n))$ and $\mathfrak{L}(\mathcal{T}_c(K_n))$ combinatorially. In Section 5 we prove that $\mathfrak{L}(\mathcal{T}_c(K_5)) = \mathfrak{U}(\mathcal{T}_c(K_5))$. To ease notation, write $\mathfrak{L}_c(K_n)$ and $\mathfrak{U}_c(K_n)$ instead of $\mathfrak{L}(\mathcal{T}_c(K_n))$ and $\mathfrak{U}(\mathcal{T}_c(K_n))$, respectively.

A **rooted tree** is a tree that has a vertex of degree at least 2 labelled as the **root** of the tree. The **leaves** of the tree are all the vertices of degree 1. We will work with trees whose only vertex of degree 2 is the root and whose leaves are labelled from 1 to n. Given a vertex v there is a unique path connecting it with the root; we define the **parent** of v to be the vertex connected to it in this path. The **children** of v are all vertices having v as a parent and the **descendants** of v are all the vertices for which v appears in the unique path to the root. We will denote the set of children of v as $\operatorname{child}(v)$ and the set of vertices of a tree T as V_T. A rooted tree is **binary** if every vertex that is not a leaf has exactly 2 children.

The bijection $T \longleftrightarrow \{\text{Set of leaves descending from } v \text{ for each vertex } v \text{ of } T\}$ gives a correspondence between compatible collections of subsets of $[n] := \{1, \ldots, n\}$ containing the set $[n]$, and rooted trees with n labelled leaves. This allows us to label the vertices of a tree T with n leaves in terms of subsets of $[n]$, thus we identify each vertex of T with the set of leaves under it. This bijection also gives us a new description for the fan $\mathcal{T}_c(K_n)$; its cones are in one-to-one correspondence with rooted trees on n leaves.

Let us introduce some more notation. Given a tree T we will write expressions like $\mathfrak{C}(T)$ and $\operatorname{star}^1(T)$, where we are letting T refer to the cone it defines. Recall that the compatible sets defining the cones of $\mathcal{T}_c(K_n)$ come from the vertices of complete graphs contained in K_n. Given $S \subseteq [n]$ we let $K(S)$ be the complete graph whose vertices are the elements of S. It is easy to see that $R \subseteq S$ implies $K(R) \subseteq K(S)$, that if R and S are disjoint then $K(R)$ and $K(S)$ are disjoint as well. Notice that a function $\varphi \in N(\mathcal{T}_c(M))$ is determined by its values on the rays $e_{K(S)}$ where $S \subseteq [n]$ and $|S| > 1$ since $\varphi(e_{K(S)}) = 0$ if $|S| \leq 1$. For convenience we will write $\varphi(S) := \varphi(e_{K(S)})$.

A full-dimensional cone $\sigma \in \mathcal{T}_c(K_n)$ corresponds to a binary rooted tree T and every cone inside σ corresponds to a tree obtained by contractions of edges of T that preserve the number of leaves; each contraction drops the dimension of the cone by one. As we contract edges, some vertices of the new tree become **non-binary**, i.e. they have more than two children. Cones of codimension one are precisely the rooted trees that have only one non-binary vertex and this vertex has exactly 3 children.

Theorem 5.1 *Let T be a tree corresponding to a cone in $\mathcal{T}_c(K_n)$ of codimension 1. Then $\mathfrak{C}(T)$ depends only on its non-binary vertex ands its children. The cone $\mathfrak{C}(T)$ is the halfspace given by the inequality*

$$\varphi(a)+\varphi(b)+\varphi(c)+\varphi(r)\leq\varphi(a\cup b)+\varphi(a\cup c)+\varphi(b\cup c)$$

where r is the non-binary vertex and a, b, c are its children.

Proof: Describing the vectors $u_1,\dots,u_l$ such that $u_1+\cdots+u_l\in T$ and each $u_i\in\mathrm{star}^1(T)$ is the same as describing the cone of nonnegative solutions of the equation

$$\sum_{X\in V_T}\lambda_X e_{K(X)}=\sum_{i=1}^{k}\mu_{N_i}e_{K(N_i)}. \tag{9}$$

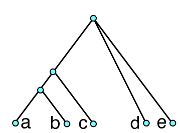
We then study the possible ways we can extend the tree formed by r and its children a , b, and c, where r, a, b, c are the sets corresponding to the vertices of T. We use this extensions to rewrite equation (9) as

$$\sum_{X\in V_T}\lambda_X e_{K(X)}=\mu_{a\cup b}e_{K(a\cup b)}+\mu_{a\cup c}e_{K(a\cup c)}+\mu_{b\cup c}e_{K(b\cup c)}. \tag{10}$$

Then, we restrict equation (10) to different values of $i\in E$, where E is the set of edges of K_n to find the nonnegative solutions. The functions in $\mathfrak{C}(T)$ are precisely those that change the $=$ to a $\leq$ in equation (10) when applied to each side of this equation. Taking into account the solution above we have that $\mathfrak{C}(T)$ is the halfspace given by the inequality described in the statement of the theorem. $\square$

Theorem 5.2 *The cone $\mathfrak{U}_c(K_n)$ is the intersection of the halfspaces defined by the inequalities in the last theorem, where the intersection runs through all trees with just one vertex having exactly 3 children.*

Example 5.3 *For K_5, each tree of codimension 1 gives us an inequality for $\mathfrak{U}_c(K_5)$, however some trees give rise to the same inequality. This is because the non-binary vertex and its children for each tree can form the same subtree. Therefore, $\mathfrak{U}_c(K_5)$ is the intersection of 65 halfspaces. Using polymake we verify that it is a 26 dimensional cone living in $\mathbb{R}^{26}$. As an illustration of the theorem, the tree*



gives rise to the the inequality

$$\varphi(abc)+\varphi(d)+\varphi(e)+\varphi(abcde)\leq\varphi(abcd)+\varphi(abce)+\varphi(de),\ \textit{where } \varphi(d)=\varphi(e)=0.$$

In our method to describe $\mathfrak{C}(T)$ for a tree T a fundamental step is finding the nonnegative solutions to

$$\sum_{v\in V_T}\lambda_v e_{K(v)}=\sum_{u\in P}\lambda_u e_{K(u)}, \tag{11}$$

where $P=\bigcup_{T'\in\mathrm{star}^1(T)}(V_{T'}\setminus V_T)$. The children of non-binary vertices determine the elements of P so they play a fundamental role for solving this equation. In Theorem 5.4 and Corollary 5.5 we prove that finding nonnegative solutions to (11) can be reduced to solving k smaller equations, each given by a tree consisting of a non-binary vertex and its children. With these results we can describe $\mathfrak{L}_c(K_n)$ in a much simpler way. We omit the proof of the following theorem, which is a generalization of the proof of Theorem 4.4.

Theorem 5.4 *Let T be a rooted tree of codimension ≥ 1, $r_1, \dots, r_k$ be all its non-binary vertices, P_i be the set all unions of at least two children of r_i that do not give r_i, and $P = \bigcup_i P_i$. Then λ is a nonnegative solution to the equation*

$$\sum_{v \in V_T} \lambda_v e_{K(v)} = \sum_{u \in P} \lambda_u e_{K(u)} \tag{12}$$

if and only if $\lambda = \lambda^1 + \cdots + \lambda^k$ where each λ^i is a nonnegative solution to

$$\lambda^i_{r_i} e_{K(r_i)} + \sum_{v \in child(r_i)} \lambda^i_v e_{K(v)} = \sum_{u \in P_i} \lambda^i_u e_{K(u)} \tag{13}$$

and $\lambda^i_u = 0$ if it does not appear in the above equation.

The following corollary simplifies the computation of $\mathfrak{C}(T)$ to the computation the cones of nonnegative solutions to equations $eq(r)$ where r is a non-binary vertex of T; this equation is written explicitly in (13). Recall that a function $\varphi \in N(\mathcal{T}_c(K_n))$ is determined by its values on the subsets $S \subseteq [n]$ with $|S| > 1$, therefore $N(\mathcal{T}_c(K_n)) \cong \mathbb{R}^{2^n-n-1}$. These solutions can be first found in the space $\mathbb{R}^{1+|\mathrm{child}(r)|+|P_r|}$, where P_r is the set of all unions of at least two children of r that do not give r. Once we have the solutions in the smaller space, we can easily extend them to solutions inside $\mathbb{R}^{2^n-n-1}$ by allowing the coordinates not corresponding to elements of $\{r\} \cup \mathrm{child}(r) \cup P_r$ to have any value in $\mathbb{R}$.

Corollary 5.5 *Suppose T is a tree of codimension ≥ 1, then*

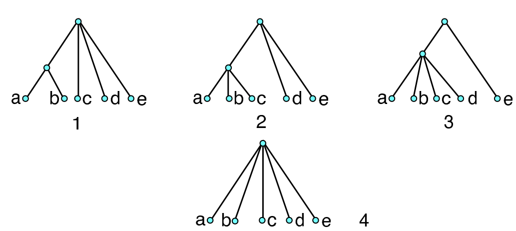
$$\mathfrak{C}(T) = \bigcap_{r \in \textit{non-bin}(T)} \mathfrak{C}^{\upharpoonright}(r),$$

where non-bin(T) is the set of all non-binary vertices of T and

$$\mathfrak{C}^{\upharpoonright}(r) = \{\varphi \in N(\mathcal{T}_c(K_n)) : \lambda_r \varphi(r) + \sum_{v \in child(r)} \lambda_v \varphi(v) \leq \sum_{u \in P_r} \lambda_u \varphi(u)$$
$$\text{where } \lambda \geq 0 \text{ is a solution for } eq(r)\ \},$$

where P_r is the set of all unions of at least two children of r that do not give r.

Example 5.6 *Recall that $\mathfrak{L}_c(K_5)$ is the intersection of the cones $\mathfrak{C}(T)$ where T is a rooted tree with 5 labelled leaves. Up to symmetry there are 3 trees of codimension 2 and one tree of codimension 3.*



In the drawing above, the top trees have codimension 2 and the lower one has codimension 3. We first use polymake to find generating rays of the cone of nonnegative solutions to

$$\sum_{v \in V_T} \lambda_v e_{K(v)} = \sum_{u \in P} \lambda_u e_{K(u)} \,.$$

Each of these solutions λ gives us a defining halfspace for $\mathfrak{C}(T)$, the halfspace associated to the inequality

$$\sum_{v \in V_T} \lambda_v \varphi(K(v)) \leq \sum_{u \in P} \lambda_u \varphi(K(u)) \,.$$

This method gives us that $\mathfrak{C}$ for tree 1 is the intersection of 10 halfspaces, $\mathfrak{C}$ for tree 2 is the intersection of 2 halfspaces, $\mathfrak{C}$ for tree 3 is the intersection of 10 halfspaces, and $\mathfrak{C}$ for tree 4 is the intersection of 2750 halfspaces. The other inequalities are those defining $\mathfrak{U}_c(K_5)$.

Thus we have the inequalities that define $\mathfrak{L}_c(K_5)$ and we now use linear programming to verify that every element of $\mathfrak{U}_c(K_5)$ satisfies them. If H is the hyperplane associated to an inequality defining $\mathfrak{L}_c(K_5)$, we use the software Mathematica to find the minimum of H inside $\mathfrak{U}_c(K_5)$ and verify that the minimum is consistent with the associated inequality. We thus conclude that $\mathfrak{L}_c(K_5) = \mathfrak{U}_c(K_5)$.

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