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FPSAC 2012, Nagoya, Japan DMTCS proc. **AR**, 2012, 827–838

Bijections for lattice paths between two boundaries

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Abstract. We prove that on the set of lattice paths with steps $N = (0, 1)$ and $E = (1, 0)$ that lie between two boundaries B and T, the two statistics 'number of E steps shared with B' and 'number of E steps shared with T' have a symmetric joint distribution. We give an involution that switches these statistics, preserves additional parameters, and generalizes to paths that contain steps $S = (0, -1)$ at prescribed x-coordinates. We also show that a similar equidistribution result for other path statistics follows from the fact that the Tutte polynomial of a matroid is independent of the order of its ground set. Finally, we extend the two theorems to k-tuples of paths between two boundaries, and we give some applications to Dyck paths, generalizing a result of Deutsch, and to pattern-avoiding permutations.

Résumé. On montre que, sur l'ensemble des chemins avec des pas $N = (0, 1)$ et $E = (1, 0)$ qui se trouvent entre deux chemins donnés B et T, les deux statistiques "nombre des pas E en commun avec B" et "nombre des pas E en commun avec T" ont une distribution conjointe symétrique. On donne une involution qui échange ces deux statistiques, préserve quelques autres paramètres additionelles, et admet une généralisation à des chemins avec des pas $S = (0, -1)$ dans des positions données. On montre aussi un autre résultat d'équidistribution similaire, lié au polynôme de Tutte d'un matroïde. Finalement, on étend les deux théorèmes à k-tuples de chemins entre deux frontières, et on donne quelques applications aux chemins de Dyck, en généralisant un résultat de Deutsch, et aux permutations avec des motifs exclus.

Keywords: lattice path, combinatorial statistic, involution, Tutte polynomial, matroid

1 Introduction

Directed lattice paths are fundamental combinatorial objects. One reason is that they have have applications to statistical physics, algebra and computer science. Another reason is that many enumeration questions can be restated in terms of lattice paths inside a certain region.

Perhaps the most frequently occurring lattice paths are Dyck paths. It is well known that they are counted by the Catalan numbers, and hundreds of Dyck path statistics have been studied in the literature. For example, a frequently quoted result of Deutsch [5] states that on Dyck paths of a given length, the number of returns has the same distribution as the height of the first peak, and in fact the joint distribution of these two statistics is symmetric (see Section 5.1 for definitions and details). One of the motivations of this paper came from the realization that this symmetry property holds for a much larger family of lattice paths, namely paths with unit north and east steps that lie between two arbitrary fixed boundaries,

†Partially supported by NSF grant DMS-1001046.

which are themselves paths of the same type with common endpoints. While Dyck paths have a lot of beautiful properties that have been thoroughly studied, little is known about paths between two arbitrary boundaries, and so it is surprising that such a symmetry result holds in general.

Our second motivation is that, when generalized to k-tuples of non-crossing paths between two boundaries, our results have implications to existing work in the literature involving watermelon configurations, flagged semistandard Young tableaux, k-triangulations, and pattern-avoiding permutations, as explained in Section 5.

Let B and T be two lattice paths in $\mathbb{N}^2$ with north steps ($N = (0, 1)$) and east steps ($E = (1, 0)$) from the origin to some prescribed point $(x, y) \in \mathbb{N}^2$ such that B is weakly below T, i.e., no point of B is strictly above or strictly left of T. Let $\mathcal{P}(B, T)$ be the set of lattice paths with north and east steps from the origin to (x, y) that lie between B and T, i.e., weakly above B and weakly below T. Thus, the paths B and T are the *lower* and *upper* boundaries of the paths in $\mathcal{P}(B, T)$.

Our goal is to show that several natural statistics on lattice paths in $\mathcal{P}(B, T)$ have a symmetric distribution. Formally, a *statistic* on such lattice paths is simply a function from $\mathcal{P}(B, T)$ to $\mathbb{N}$. Two k-tuples of statistics $(f_1, f_2, \dots, f_k)$ and $(g_1, g_2, \dots, g_k)$ have the same *joint distribution* over $\mathcal{P}(B, T)$, denoted $(f_1, f_2, \dots, f_k) \sim (g_1, g_2, \dots, g_k)$, if

$$\sum_{P \in \mathcal{P}(B,T)} x_1^{f_1(P)} \dots x_k^{f_k(P)} = \sum_{P \in \mathcal{P}(B,T)} x_1^{g_1(P)} \dots x_k^{g_k(P)}.$$

The distribution of $(f_1, f_2, \dots, f_k)$ is *symmetric* over $\mathcal{P}(B, T)$ if $(f_1, f_2, \dots, f_k) \sim (f_{\pi(1)}, f_{\pi(2)}, \dots, f_{\pi(k)})$ for every permutation π of $[k] = \{1, 2, \dots, k\}$.

We consider statistics counting the following special steps of paths $P \in \mathcal{P}(B, T)$:

- a *bottom contact* is an east step that is also a step of B,
- a *top contact* is an east step that is also a step of T,
- a *left contact* is a north step that is also a step of T,
- a *right contact* is a north step that is also a step of B.

We denote the number of bottom (top, left, right) contacts of P by $b(P)$ (respectively $t(P)$, $\ell(P)$, $r(P)$).

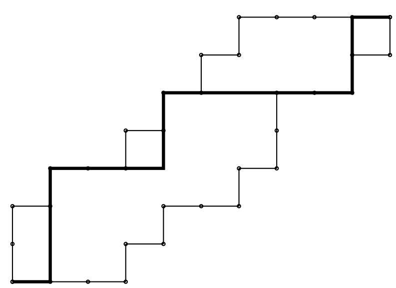

Fig. 1: A path $P \in \mathcal{P}(B, T)$ with $b(P) = 3$, $t(P) = 4$, $\ell(P) = 2$, and $r(P) = 1$.

We will give bijective proofs of the following two results.

Theorem 1.1 *The distribution of the pair (b, t) over $\mathcal{P}(B, T)$ is symmetric.*

Theorem 1.2 *The pairs (b, ℓ) and (t, r) have the same joint distribution over $\mathcal{P}(B, T)$.*

We point out that is not true that $(b, t, \ell) \sim (t, b, r)$ in general. In Section 2 we provide a direct, relatively simple involution proving a refined and generalized version of Theorem 1.1. The refinement consists of keeping track of the sequence of y-coordinates of the east steps that are not contacts, while the generalization allows the paths to have south steps at prescribed x-coordinates. The analogous refinement and generalization of Theorem 1.2 do not hold, and in fact our proof of Theorem 1.1, given in Section 3, is quite indirect and very different from that of Theorem 1.2. Namely, we show that both

$$\sum_{P \in \mathcal{P}(B,T)} x^{b(P)} y^{\ell(P)} \quad \text{and} \quad \sum_{P \in \mathcal{P}(B,T)} x^{t(P)} y^{r(P)}$$

can be interpreted as the Tutte polynomial of the lattice path matroid associated with $\mathcal{P}(B, T)$, as defined in [2]. To do so, we use the definition of the Tutte polynomial in terms of *activities*, which relies on a linear ordering on the ground set of the matroid. The independence of the Tutte polynomial of this ordering then implies Theorem 1.2. In the full paper [7] we provide a bijective proof for this independence property.

In Section 4, the two above theorems are generalized to k-tuples of non-crossing paths. Let $\mathcal{P}^k(B, T)$ be the set of k-tuples $\mathbf{P} = (P_1, P_2, \ldots, P_k)$ such that $P_i \in \mathcal{P}(B, T)$ for all i, and P_i is weakly below P_{i+1} for $1 \le i \le k - 1$. Let $P_0 = B$ and $P_{k+1} = T$. For $0 \le i \le k$, denote by $t_i = t_i(\mathbf{P})$ the number of east steps where P_i and P_{i+1} coincide. We provide a bijective proof of the following generalization of Theorem 1.1, for any fixed $k \ge 1$.

Theorem 1.3 *The distribution of $(t_0, t_2, \ldots, t_k)$ over $\mathcal{P}^k(B, T)$ is symmetric.*

To generalize Theorem 1.2, define the *bottom contacts of* $\mathbf{P}$ to be the bottom contacts of P_1, and denote their number by $b(\mathbf{P}) = t_0(\mathbf{P})$. Similarly, let $r(\mathbf{P})$ be the number of right contacts of P_1, and denote by $t(\mathbf{P}) = t_k(\mathbf{P})$ (resp. $\ell(\mathbf{P})$) be the number of top (resp. left) contacts of P_k.

Theorem 1.4 *The pairs (b, ℓ) and (t, r) have the same joint distribution over $\mathcal{P}^k(B, T)$.*

In Section 5 we show some consequences of our work to Dyck paths, watermelon configurations, semistandard Young tableaux, and pattern-avoiding permutations.

2 The symmetry $(b, t) \sim (t, b)$ for a single path

In this section we construct an involution that proves a generalized version of Theorem 1.1. It applies to a more general set of paths, and also gives a refined result by preserving the sequence of y-coordinates of the east steps that are not contacts.

Let $\widetilde{\mathcal{P}}(B, T)$ be the set of lattice paths from the origin to (x, y) with north, east and south ($S = (0, -1)$) steps, lying weakly above B and weakly below T. Given such a lattice path P, the *descent set* of P is the set of x-coordinates where south steps occur. For a fixed subset $\mathcal{D} \subset \mathbb{N}$, denote by $\widetilde{\mathcal{P}}(B, T, \mathcal{D})$ the set of paths $P \in \widetilde{\mathcal{P}}(B, T)$ having descent set $\mathcal{D}$. Note that $\widetilde{\mathcal{P}}(B, T, \emptyset) = \mathcal{P}(B, T)$ by defintion.

Furthermore, for a given sequence $\mathbf{H}$ of integers, let $\widetilde{\mathcal{P}}(B, T, \mathcal{D}, \mathbf{H})$ (respectively $\mathcal{P}(B, T, \mathbf{H})$) be the subset of $\widetilde{\mathcal{P}}(B, T, \mathcal{D})$ (respectively $\mathcal{P}(B, T)$) containing those paths whose sequence of y-coordinates of the east steps that are not bottom or top contacts equals $\mathbf{H}$. Fig. 4 shows some paths in $\widetilde{\mathcal{P}}(B, T, \mathcal{D}, \mathbf{H})$ with $B = EENEEEENNN$, $T = NNNEEEENEE$, $\mathcal{D} = \{2\}$, and $\mathbf{H} = 2\,2\,3$.

Theorem 2.1 *For any set $\mathcal{D}$ and any sequence $\mathbf{H}$ of integers, the distribution of (b,t) over $\widetilde{\mathcal{P}}(B,T,\mathcal{D},\mathbf{H})$ is symmetric.*

Remark. Without the refinement involving $\mathbf{H}$, a non-bijective proof of this result has been found independently by Guo Niu Han [8]. See also the remark after Theorem 4.1.

In what follows, we encode a path in $\widetilde{\mathcal{P}}(B,T)$ by the sequence of y-coordinates of its east steps, except that we record bottom contacts using $\mathbf{b}$'s and top contacts using $\mathbf{t}$'s. For example, the first path in Fig. 4 is encoded by $2\mathbf{t}23\mathbf{t}$.

The main ingredient in the proof of Theorem 2.1 is a transformation ϕ between certain subsets of $\widetilde{\mathcal{P}}(B,T,\mathcal{D},\mathbf{H})$, mapping paths with e bottom and f top contacts to paths with $e+1$ bottom and $f-1$ top contacts. The map ϕ is a bijection between subsets of paths satisfying certain restrictions. This bijection relies on two constructions. The first one, which we denote by μ, transforms a sequence of e $\mathbf{b}$'s and f $\mathbf{t}$'s into a sequence of $e+1$ $\mathbf{b}$'s and $f-1$ $\mathbf{t}$'s. The second construction modifies a given path by changing a single top contact into a bottom contact.

2.1 A transformation on words

Let us first describe the map μ, which is defined on sequences of $\mathbf{b}$'s and $\mathbf{t}$'s. We say that a word $c_1c_2\ldots c_{2n}$ over the alphabet $\{\mathbf{b},\mathbf{t}\}$ is a Dyck word if it contains the same number of $\mathbf{b}$'s and $\mathbf{t}$'s, and in every prefix $c_1c_2\ldots c_i$ with $1\le i\le 2n$, the number of $\mathbf{b}$'s never exceeds the number of $\mathbf{t}$'s.

Definition 2.2 Let $\mathbf{c}=c_1c_2\ldots c_{e+f}$ be a word over the alphabet $\{\mathbf{b},\mathbf{t}\}$. Any such $\mathbf{c}$ can be factorized uniquely as

$$\mathbf{c}=D_1\mathbf{b}D_2\mathbf{b}\ldots\mathbf{b}D_j\mathbf{t}D_{j+1}\mathbf{t}D_{j+2}\mathbf{t}\ldots\mathbf{t}D_m, \tag{1}$$

where each D_i for $1\le i\le m$ is a (possibly empty) Dyck word. In such a factorization, the letters $\mathbf{b}$ and $\mathbf{t}$ which are not part of a Dyck word are called *unmatched* letters. Suppose that there is at least one unmatched $\mathbf{t}$. Then, define $\mu(\mathbf{c})$ to be the word obtained from $\mathbf{c}$ by replacing the leftmost unmatched $\mathbf{t}$ with a $\mathbf{b}$, that is,

$$\mu(\mathbf{c})=D_1\mathbf{b}D_2\mathbf{b}\ldots\mathbf{b}D_j\mathbf{b}D_{j+1}\mathbf{t}D_{j+2}\mathbf{t}\ldots\mathbf{t}D_m. \tag{2}$$

The above factorization can be visualized by representing a word with a path, drawing an north-east step $(1,1)$ for each letter $\mathbf{t}$ and a south-east $(1,-1)$ for each letter $\mathbf{b}$. In Fig. 2, the effect of the map μ applied to the word $\mathbf{bttbtbbbttbttbtbtt}$ is shown. In this example, the Dyck words D_2, D_4 and D_5 are non-empty, and they are indicated by the dotted areas.

We omit the proofs of the following lemmas due to space constraints, but they appear in the full paper [7]. In different context, the map μ belongs to mathematical folklore (see for example [13, p. 26]).

Lemma 2.3 *Let e,f,u be nonnegative integers with $u\ge\max\{f-e,e-f+2\}$. The map μ is a bijection between*

(i) the set of words with e $\mathbf{b}$'s and f $\mathbf{t}$'s having at least u unmatched letters, and
(ii) the set of words with $e+1$ $\mathbf{b}$'s and $f-1$ $\mathbf{t}$'s having at least u unmatched letters.

Lemma 2.4 *Let $e<f$. For $0\le i\le f-e$, let $\mathcal{W}_i$ be the set of words with $e+i$ $\mathbf{b}$'s and $f-i$ $\mathbf{t}$'s having at least $f-e$ unmatched letters. Specifically, $\mathcal{W}_0$ is the set of all words with e $\mathbf{b}$'s and f $\mathbf{t}$'s, and $\mathcal{W}_{f-e}$ is the set of all words with f $\mathbf{b}$'s and e $\mathbf{t}$'s. Then the map μ produces a sequence of bijections*

$$\mathcal{W}_0\xrightarrow{\mu}\mathcal{W}_1\xrightarrow{\mu}\ldots\xrightarrow{\mu}\mathcal{W}_{f-e}.$$

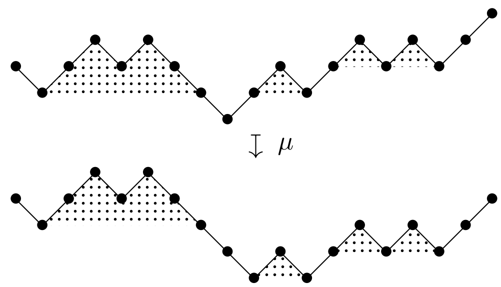

Fig. 2: A visual description of the map μ.

2.2 A transformation on paths with one contact

Our next goal is to translate μ, which is a map on words, into a transformation on paths. For this purpose, we first give a construction for path fragments containing a single contact. In the following, $I = [x_1, x_2]$ is the interval bounded by two integers $x_1 < x_2$. We denote by $\widetilde{\mathcal{P}}_I(B,T)$ the set of path fragments obtained by restricting paths $P \in \widetilde{\mathcal{P}}(B,T)$ to the segment that lies strictly between the vertical lines $x = x_1$ and $x = x_2$. In other words, $\widetilde{\mathcal{P}}_I(B,T)$ is the set of paths which start on the line $x = x_1$, end on the line $x = x_2$, start and end with an east step, and lie weakly between B and T. Define $\widetilde{\mathcal{P}}_I(B,T,\mathcal{D},\mathbf{H})$ similarly by restricting paths in $P \in \widetilde{\mathcal{P}}(B,T,\mathcal{D},\mathbf{H})$ to I.

Definition 2.5 Let Q be a path in $\widetilde{\mathcal{P}}_I(B,T)$ with exactly one top contact and no bottom contacts. Its east steps can be decomposed uniquely as $Q = WX\mathbf{t}YZ$, where

- X is maximal such that there is no descent after any of its steps and no (right) endpoint of any of its steps lies on B, and
- Y is maximal such that there is a descent before each of its steps.

Let h_X (respectively h_Y) be $-\infty$ if X (respectively Y) is empty, and otherwise the y-coordinate of its last (respectively first) east step. Define

$$\phi_I(Q) = \begin{cases} WXY\mathbf{b}Z & \text{if } h_X \le h_Y, \\ W\mathbf{b}XYZ & \text{if } h_X > h_Y. \end{cases}$$

Remark. In the case of paths with no descents, the definition of ϕ_I is simpler: writing Q as $Q = WX\mathbf{t}Z$, where X is maximal not touching B, we have $\phi_I(Q) = W\mathbf{b}XZ$.

Lemma 2.6 *The map ϕ_I is a bijection from the set of paths in $\widetilde{\mathcal{P}}_I(B,T,\mathcal{D},\mathbf{H})$ with exactly one top contact and no bottom contacts to the set of paths in $\widetilde{\mathcal{P}}_I(B,T,\mathcal{D},\mathbf{H})$ with exactly one bottom contact and no top contacts.*

Proof: Let $Q \in \widetilde{\mathcal{P}}_I(B,T,\mathcal{D},\mathbf{H})$ be a path with one top contact and no bottom contacts. It is clear that $\phi_I(Q) \in \widetilde{\mathcal{P}}_I(B,T,\mathbf{H})$, and that this path has one bottom contact and no top contacts. Let us now

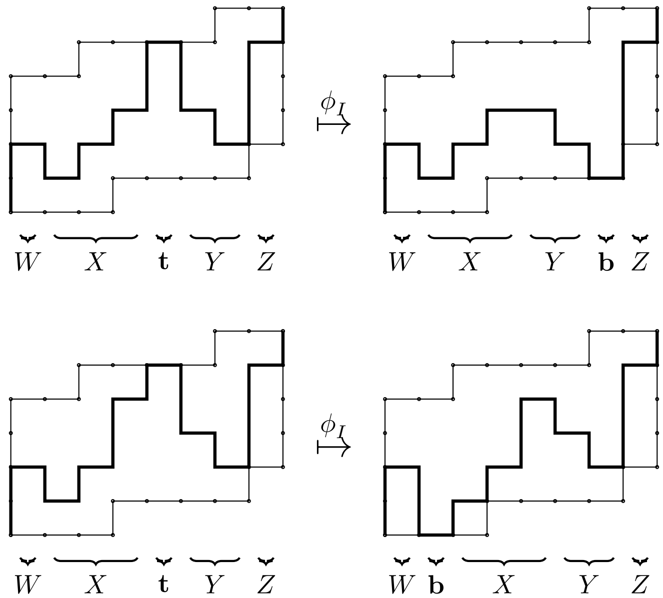

Fig. 3: Two examples of ϕ_I: one where $h_X \leq h_Y$ (top) and one where $h_X > h_Y$ (bottom). In both cases, $I = [0, 8]$.

check that Q and $\phi_I(Q)$ have the same descent set. Suppose that $Q = WX\mathbf{t}YZ$, and consider the case $h_X \leq h_Y$, so $\phi_I(Q) = WXY\mathbf{b}Z$, that is, the block $\mathbf{t}Y$ in Q becomes $Y\mathbf{b}$ in $\phi_I(Q)$. By the choice of Y and because there cannot be a descent just before a top contact, it is clear that Q has no descent just before and just after the block $\mathbf{t}Y$, and there are descents at all positions inside the block. Let us check that this is also the case for the block $Y\mathbf{b}$ in $\phi_I(Q)$.

- Just before $Y\mathbf{b}$: if X is non-empty, then $h_Y \geq h_X > -\infty$, which implies that Y is non-empty and there is no descent between X and Y; if X is empty, then either W is empty or its last east step has its right endpoint on B, so there is no descent just before $Y\mathbf{b}$ either.
- Just after $Y\mathbf{b}$: there cannot be a descent just after a bottom contact.
- Inside $Y\mathbf{b}$: by the definition of Y, there are descents at all positions inside Y; at the position between Y and $\mathbf{b}$ (if Y is non-empty), there is a descent because the last east step of Y was not a bottom contact in Q, so its y-coordinate is strictly larger than that of $\mathbf{b}$ in $\phi_I(Q)$.

The arguments for the case that $h_X > h_Y$ and $\phi_I(Q) = W\mathbf{b}XYZ$ are very similar and thus omitted.

To show that ϕ_I is invertible we will exhibit its inverse. The description of ϕ_I^{-1} is analogous to that of ϕ_I if we rotate the paths by 180 degrees. Explicitly, let $Q' \in \widetilde{\mathcal{P}}_I(B, T, \mathcal{D}, \mathbf{H})$ be a path with exactly one bottom contact and no top contacts. Its east steps can be decomposed uniquely as $Q' = RS\mathbf{b}UV$, where

- S is maximal such that there is a descent after each of its steps, and
- U is maximal such that there is no descent before any of its steps and no (left) endpoint of any of its east steps lies on T.

Let h_S (h_U) be $+\infty$ if S (respectively U) is empty, and otherwise the y-coordinate of its last (respectively first) east step. Define

$$\phi_I^{-1}(Q) = \begin{cases} R\mathbf{t}SUV & \text{if } h_S \le h_U, \\ RSU\mathbf{t}V & \text{if } h_S > h_U. \end{cases}$$

Let us check ϕ_I^{-1} is indeed the inverse of ϕ_I. Suppose that $Q = WX\mathbf{t}YZ$ and $h_X \le h_Y$, and let $Q' = \phi_I(Q) = WXY\mathbf{b}Z$. When applying ϕ_I^{-1} to Q', the decomposition $Q' = RS\mathbf{b}UV$ has $S = Y$, by definition of S and the fact that there is no descent just before Y but there are descents in all the positions inside $Y\mathbf{b}$. Additionally, $h_S \le h_U$ because there was no descent just after Y in Q. Thus, $\phi_I^{-1}(Q') = R\mathbf{t}SUV = Q$. The case that $h_X > h_Y$ is similar. □

2.3 The maps ϕ and Φ

Now we are ready to define the map ϕ on arbitrary paths in $\widetilde{\mathcal{P}}(B,T)$.

Definition 2.7 Let $P \in \widetilde{\mathcal{P}}(B,T)$ be a path with e bottom and f top contacts, and let $\mathbf{c} = c_1c_2\dots c_{e+f}$ be the word over $\{\mathbf{b},\mathbf{t}\}$ obtained by recording the top and bottom contacts of P from left to right, except for the steps that are simultaneously top and a bottom contacts, which are not recorded. We call $\mathbf{c}$ the *sequence of contacts* of P. Suppose that $\mathbf{c}$ contains some unmatched $\mathbf{t}$, and let i be such that c_i is the last unmatched $\mathbf{t}$, which becomes a $\mathbf{b}$ in $\mu(\mathbf{c})$.

Let $I = [x_1, x_2]$ be the maximal interval of x-coordinates that contains the contact $c_i = \mathbf{t}$ and no other contact of P. Let $Q \in \widetilde{\mathcal{P}}_I(B,T)$ be the fragment of P between with x-coordinates in the interval I. Define $\phi(P)$ to be the path obtained from P by replacing the fragment Q with $\phi_I(Q)$. Note that the sequence of contacts of $\phi(P)$ is $\mu(\mathbf{c})$.

The following lemmas are proved in the full paper [7].

Lemma 2.8 *Let e, f, u be nonnegative integers with $u \ge \max\{f-e, e-f+2\}$. The map ϕ is a bijection between*

(i) the set of paths in $\widetilde{\mathcal{P}}(B,T,\mathcal{D},\mathbf{H})$ whose sequence of contacts has e $\mathbf{b}$'s, f $\mathbf{t}$'s, and at least u unmatched letters, and

(ii) the set of paths in $\widetilde{\mathcal{P}}(B,T,\mathcal{D},\mathbf{H})$ whose sequence of contacts has $e+1$ $\mathbf{b}$'s, $f-1$ $\mathbf{t}$'s, and at least u unmatched letters.

Lemma 2.9 *Let $e < f$. For $0 \le i \le f-e$, let $\mathcal{R}_i$ be the set of paths in $\widetilde{\mathcal{P}}(B,T,\mathcal{D},\mathbf{H})$ whose sequence of contacts has $e+i$ $\mathbf{b}$'s, $f-i$ $\mathbf{t}$'s, and at least $f-e$ unmatched letters. Specifically, $\mathcal{R}_0$ is the set of all paths in $\widetilde{\mathcal{P}}(B,T,\mathcal{D},\mathbf{H})$ having e bottom contacts and f top contacts, and $\mathcal{R}_{f-e}$ is the set of all paths in $\widetilde{\mathcal{P}}(B,T,\mathcal{D},\mathbf{H})$ having f bottom contacts and e top contacts. Then the map ϕ produces a sequence of bijections*

$$\mathcal{R}_0 \xrightarrow{\phi} \mathcal{R}_1 \xrightarrow{\phi} \dots \xrightarrow{\phi} \mathcal{R}_{f-e}.$$

We can now describe the bijection Φ that proves Theorem 2.1, which in turn generalizes Theorem 1.1.

Definition 2.10 For $P \in \widetilde{\mathcal{P}}(B,T)$, define $\Phi(P) = \phi^{f-e}(P)$, where $e = b(P)$ and $f = t(P)$.

Lemma 2.11 *The map Φ is an involution on $\widetilde{\mathcal{P}}(B,T)$ that preserves the descent set, as well as the sequence of y-coordinates of the east steps that are not contacts, and satisfies $b(\Phi(P)) = t(P)$ and $t(\Phi(P)) = b(P)$.*

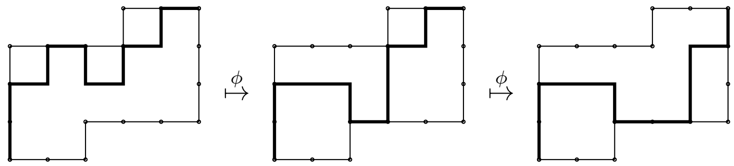

Fig. 4: The map Φ applied to a path with two top contacts and no bottom contact. In this example, ϕ first performs the transformation $\phi_{[0,3]}$, followed by $\phi_{[3,5]}$.

3 The symmetry $(b, \ell) \sim (t, r)$ for a single path

In this section we prove Theorem 1.2. Although this theorem looks superficially similar to Theorem 1.1, we have not found a comparable 'natural' bijective proof. Instead, our theorem below is an easy consequence of work of Anna de Mier, Joseph Bonin and Marc Noy [2], and also Federico Ardila [1].

Again, let B and T be lattice paths in $\mathbb{N}^2$ with north and east steps from the origin to (x, y) such that B is weakly below T. The paths in this section have no south steps. We encode a path $P \in \mathcal{P}(B, T)$ as the subset $\hat{P}$ of $\mathcal{N} = \{1, 2, \dots, x + y\}$ given by the indices of the north steps in P. For example, the path P in Fig. 1 is specified by the subset $\hat{P} = \{2, 3, 4, 8, 9, 15, 16\} \subseteq [17]$.

Definition 3.1 The set $\mathcal{B} = \{\hat{P} : P \in \mathcal{P}(B, T)\}$ is the set of bases of a matroid with *ground set* $\mathcal{N}$, called a *lattice path matroid*.

Let $\prec$ by an arbitrary linear order on $\mathcal{N}$, and let $\hat{P} \in \mathcal{B}$. Then an element $e \notin \hat{P}$ is *externally active* with respect to $(\hat{P}, \prec)$ if $\nexists n \in \hat{P}$ such that $n \prec e$ and $\hat{P} \setminus n \cup e \in \mathcal{B}$. Similarly, an element $n \in \hat{P}$ is *internally active* with respect to $(\hat{P}, \prec)$ if $\nexists e \in \mathcal{N} \setminus \hat{P}$ such that $e \prec n$ and $\hat{P} \setminus n \cup e \in \mathcal{B}$.

The *internal activity* of $\hat{P}$ is the number of its internally active elements, and the *external activity* is the number of its externally active elements. The *Tutte polynomial* of the matroid with set of bases $\mathcal{B}$ is the generating polynomial for the internal and external activities of its bases:

$$\sum_{\hat{P} \in \mathcal{B}} x^{\text{internal activity of } \hat{P}} y^{\text{external activity of } \hat{P}}. \tag{3}$$

The Tutte polynomial for matroids was introduced by Henry Crapo [4]. He also showed that it is independent of the order on the ground set, and thus well defined. This property is the main ingredient in the proof of Theorem 1.2, which we restate below for convenience. In [7] we give an activity-preserving bijection on the bases of a matroid relative to different orderings. This bijection works for any matroid, and in particular it gives a bijective proof of Theorem 3.2, but it is iterative and rather tedious to apply. It would be interesting to find a direct bijective proof of this theorem.

Theorem 3.2 *The pairs* (b, ℓ) *and* (t, r) *have the same joint distribution over* $\mathcal{P}(B, T)$.

Proof: Let $\prec$ be the usual order $1 \prec 2 \prec 3 \prec \cdots$ of the ground set $\mathcal{N}$ of the matroid described above. As shown in [2, Theorem 5.4], an internally active edge of $\hat{P} \in \mathcal{B}$ with respect to this order is a left contact of $P \in \mathcal{P}(B, T)$, and an externally active edge of $\hat{P}$ is a bottom contact of P. It follows from equation (3) that the Tutte polynomial of the matroid equals $\sum_{P \in \mathcal{P}(B,T)} x^{b(P)} y^{\ell(P)}$.

Since the Tutte polynomial is independent of the ordering on the ground set, it can also be obtained as follows. Let now $\prec$ be the order $\cdots \prec 3 \prec 2 \prec 1$. With respect to this order, an internally active edge of $\hat{P} \in \mathcal{B}$ is a right contact of $P \in \mathcal{P}(B,T)$, and an externally active edge of $\hat{P}$ is a top contact of P. It follows that the Tutte polynomial of the matroid also equals $\sum_{P\in\mathcal{P}(B,T)} x^{t(P)}y^{r(P)}$. $\square$

4 A k-tuple of paths between two boundaries

In this section we show how to extend Theorems 1.1 and 1.2 to families of k non-crossing paths. In both cases, the idea is to repeatedly apply the theorems for single paths.

4.1 The symmetry $(b,t) \sim (t,b)$

Here we extend Theorem 1.1 to k-tuples of paths. Although we do not allow paths with south steps as in the more general Theorem 2.1, we are able to partially incorporate the refinement keeping track of the y-coordinates of the non-contact east steps.

Recall from Section 1 that, given $\mathbf{P} = (P_1, P_2, \ldots, P_k) \in \mathcal{P}^k(B,T)$, with the convention that $P_0 = B$ and $P_{k+1} = T$, we denote by $t_i = t_i(\mathbf{P})$ the number of east steps where P_i and P_{i+1} coincide, for $0 \le i \le k$. Such east steps will be called *top contacts of* P_i. For $s \ge 1$, let $m_s = m_s(\mathbf{P})$ be the number of positive integers x such that some path among $P_1, \ldots, P_k$ has an east step $(x-1,s) \to (x,s)$ which does not coincide with an east step of either B or T. Let $\mathbf{m}(\mathbf{P}) = (m_1, m_2, \ldots)$. For a fixed sequence $\mathbf{m}$ of nonnegative integers, let $\mathcal{P}^k(B,T,\mathbf{m})$ be the set of k-tuples $\mathbf{P} \in \mathcal{P}^k(B,T)$ with $\mathbf{m}(\mathbf{P}) = \mathbf{m}$.

Theorem 4.1 *For any sequence $\mathbf{m}$ of nonnegative integers, the distribution of $(t_0, t_1, \ldots, t_k)$ over $\mathcal{P}^k(B,T,\mathbf{m})$ is symmetric.*

Remark. Without the refinement involving $\mathbf{m}$, non-bijective proofs of this theorem have been given by Carlos Nicolas [11, Theorem 3] and Guo Niu Han [8].

Proof sketch: It suffices to show that $(t_0, \ldots, t_{i-1}, t_i, \ldots, t_k) \sim (t_0, \ldots, t_i, t_{i-1}, \ldots, t_k)$ for any i with $1 \le i \le k$. Fix such an i, and let $\mathbf{P} = (P_1, P_2, \ldots, P_k) \in \mathcal{P}^k(B,T,\mathbf{m})$. Regarding P_i as a path in $\mathcal{P}(P_{i-1}, P_{i+1})$, we can apply the bijection Φ from Definition 2.10 to it, obtaining a path $\Phi(P_i) \in \mathcal{P}(P_{i-1}, P_{i+1})$. Let $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_k)$, where $Q_i = \Phi(P_i)$ and $Q_j = P_j$ for $j \ne i$.

By definition, $t_j(\mathbf{P}) = t_j(\mathbf{Q})$ and $m_j(\mathbf{P}) = m_j(\mathbf{Q})$ for $j \notin \{i-1, i\}$. By Lemma 2.11, the number of east steps where P_i and P_{i+1} (respectively P_{i-1}) coincide equals the number of east steps where $\Phi(P_i)$ and P_{i-1} (respectively P_{i+1}) coincide, so $t_i(\mathbf{P}) = t_{i-1}(\mathbf{Q})$ and $t_{i-1}(\mathbf{P}) = t_i(\mathbf{Q})$. It remains to show that $m_{i-1}(\mathbf{P}) + m_i(\mathbf{P}) = m_{i-1}(\mathbf{Q}) + m_i(\mathbf{Q})$, which we do in [7]. $\square$

4.2 The symmetry $(b,\ell) \sim (t,r)$

Given $\mathbf{P} = (P_1, P_2, \ldots, P_k) \in \mathcal{P}^k(B,T)$, letting $P_0 = B$ and $P_{k+1} = T$, a *left contact of* P_i is a north step that coincides with a north step of P_{i+1}. We denote by $\ell_i = \ell_i(\mathbf{P})$ the number of left contacts of P_i. Note that $b(\mathbf{P}) = t_0(\mathbf{P})$, $t(\mathbf{P}) = t_k(\mathbf{P})$, $\ell(\mathbf{P}) = \ell_k(\mathbf{P})$ and $r(\mathbf{P}) = \ell_0(\mathbf{P})$.

Theorem 4.2 *The pairs (b,ℓ) and (t,r) have the same joint distribution over $\mathcal{P}^k(B,T)$.*

Proof: As in the proof of Theorem 4.1, each P_i, for $1 \le i \le k$, can be regarded as a path in $\mathcal{P}(P_{i-1}, P_{i+1})$. In this setting, the statistics involved in the statement of Theorem 3.2 are $b(P_i) = t_{i-1}(\mathbf{P})$, $\ell(P_i) = \ell_i(\mathbf{P})$, $t(P_i) = t_i(\mathbf{P})$ and $r(P_i) = \ell_{i-1}(\mathbf{P})$. Applying Theorem 3.2 to $P_i \in \mathcal{P}(P_{i-1}, P_{i+1})$, it follows that there is a bijection between families $\mathbf{P} \in \mathcal{P}^k(B,T)$ with $t_{i-1}(\mathbf{P}) = e$ and $\ell_i(\mathbf{P}) = f$, and families $\mathbf{P} \in \mathcal{P}^k(B,T)$ with $t_{i-1}(\mathbf{P}) = e$ and $\ell_i(\mathbf{P}) = f$, which preserves the statistics t_j and ℓ_j for all $j \notin \{i-1, i\}$.

Given $\mathbf{P} \in \mathcal{P}^k(B,T)$ with $b(\mathbf{P}) = t_0(\mathbf{P}) = e$ and $\ell(\mathbf{P}) = \ell_k(\mathbf{P}) = f$, one can apply this bijection first to P_1, then to P_2 in the resulting tuple, and successively up to P_k. This composition gives a bijection between families $\mathbf{P} \in \mathcal{P}^k(B,T)$ with $b(\mathbf{P}) = e$ and $\ell(\mathbf{P}) = f$, and families $\mathbf{P} \in \mathcal{P}^k(B,T)$ with $\ell_{k-1}(\mathbf{P}) = f$ and $t(\mathbf{P}) = t_k(\mathbf{P}) = e$. On the latter set, one can now apply the bijection to P_{k-1}, then to P_{k-2}, and successively down to P_1, proving that families $\mathbf{P} \in \mathcal{P}^k(B,T)$ with $\ell_{k-1}(\mathbf{P}) = f$ and $t(\mathbf{P}) = e$ are in turn in bijection with families $\mathbf{P} \in \mathcal{P}^k(B,T)$ with $r(\mathbf{P}) = \ell_0(\mathbf{P}) = f$ and $t(\mathbf{P}) = e$. □

5 Corollaries and Applications

5.1 Dyck paths and generalizations

In the particular case that $B = E^n N^n$ and $T = (NE)^n$, the statistics b and t become two familiar Dyck path statistics: the height of the first peak and the number of returns to the x-axis, respectively. A bijective proof of the fact that these statistics are equidistributed on Dyck paths was given by Deutsch [5], who later also gave an involution [6] proving the symmetry of their joint distribution. Our involution Φ, when restricted to the case of Dyck paths, is quite different from Deutsch's involution, which is defined recursively. The symmetry of these Dyck paths statistics can also be proved using standard generating function techniques, based on the usual recursive decomposition of Dyck paths. However, neither these techniques nor Deutsch's involution seems to extend to the general setting of Theorem 1.1. In addition to providing an extension to paths between arbitrary boundaries B and T, our involution can also be used to prove the following statement (see [7]).

Corollary 5.1 *Let $B = E^x N^y$, and let T be a path weakly above B which begins with a north step. Then, for $i, j \geq 0$, the number of paths in $\mathcal{P}(B,T)$ with i bottom and j top contacts equals the number of paths with north and east steps from $(i+j, 2)$ to (x,y) staying weakly below T and thus depends only on $i+j$.*

For $B = E^n N^n$ and $T = (NE)^n$, Corollary 5.1 specializes to the fact that the number of Dyck paths of fixed semilength $n+1$ with $j+1$ returns whose first peak has height $i+1$ depends only on the sum $i+j$. In this case, the number of paths from $(i+j, 2)$ to (n,n) weakly below T is the ballot number

$$\frac{i+j}{n}\binom{2n-i-j-1}{n-i-j}.$$

5.2 Watermelon configurations

As a consequence of Theorem 1.3, we can recover a theorem of Richard Brak and John Essam [3, Corollary 1] concerning certain families of k non-intersecting paths called *watermelon configurations*.

Definition 5.2 A *watermelon configuration* of *length* x and *deviation* y is a family of k non-intersecting lattice paths with north-east $(1,1)$ and south-east $(1,-1)$ steps, starting at $(0, 2i)$ and terminating at $(x, y+2i)$ for $0 \leq i \leq k-1$, not going below the x-axis.

Brak and Essam derive the following statement using manipulations of a determinant.

Theorem 5.3 ([3]) *The number of watermelon configurations of length x and deviation y whose bottom path has e returns to the x-axis is the same as the number of families of k non-intersecting paths where the lower $k-1$ paths form a watermelon configuration of length x and deviation y, and the top path terminates at $(x-e-1, y+2k+e-3)$.*

Christian Krattenthaler [10, Proposition 6] gives a bijective proof by transforming the configurations into certain semi-standard Young tableaux and applying a variant of jeu de taquin. However, as we show in [7], one obtains a more straightforward proof by interpreting it as a special case of Theorem 1.3.

5.3 Flagged semistandard Young tableaux

The case of watermelon configurations with deviation 0 (also known as fans of Dyck paths) is particularly interesting. Carlos Nicolas [11, Conjecture 1] discovered experimentally that the distribution of degrees of k consecutive vertices in a k-triangulation of the n-gon equals the distribution of $(t_0, t_1, \dots, t_{k-1})$ over $\mathcal{P}^k(B,T)$ with $B = E^{n-2k-1}N^{n-2k-1}$ and $T = (NE)^{n-2k-1}$. An important special case has been shown by Luis Serrano and Christian Stump [12, Theorem 4.4], who prove that the degree of a fixed vertex in the set of k-triangulations of the n-gon is equidistributed with the number of bottom contacts of families in $\mathcal{P}^k(B,T)$.

As it turns out, much more seems to be true. Namely, let $B = E^xN^y$ and let T be arbitrary. The region enclosed by B and T can be interpreted as a Young diagram λ by rotating it 180 degrees. Recall that the *content* of a semistandard Young tableau of shape λ is $\mu = (\mu_1, \mu_2, \dots)$ where μ_i is the number of entries i in the tableau. Using this notation, the following theorem is proved in the full length version of this extended abstract [7].

Theorem 5.4 *For B, T and λ as above, there is an explicit bijection between k-tuples of paths $\mathbf{P} \in \mathcal{P}^k(B,T)$ with $t_i(\mathbf{P}) = t_i$ ($0 \le i \le k$) and $m_s(\mathbf{P}) = m_s$ ($1 \le s < y$), and semistandard Young tableaux of shape λ and content*

$$(\lambda_1 - t_0, \lambda_1 - t_1, \dots, \lambda_1 - t_k, \lambda_2 - m_1, \lambda_3 - m_2, \dots, \lambda_y - m_{y-1})$$

whose entries in row j are at bounded by $k+j$ for $1 \le j \le y$.

Using the Edelman-Greene correspondence as described in [12], we show in [7] that Theorem 5.4 implies Carlos Nicolas' conjecture [11, Conjecture 1].

5.4 Pattern-avoiding permutations

We now describe an application of Theorem 2.1 to pattern-avoiding permutations. Let $\mathcal{S}_n$ denote the set of permutations of $[n]$.

Definition 5.5 Let $\pi \in \mathcal{S}_n$. We say that $\pi(i)$ is a *left-to-right minimum* (*left-to-right maximum*) of π if $\pi(i) < \pi(j)$ (respectively, $\pi(i) > \pi(j)$) for all $j < i$. For $1 < i < n$, we say that π has an *occurrence of the (dashed) pattern* 2-31 at position i if there is a $j < i$ such that $\pi(i+1) < \pi(j) < \pi(i)$.

Proposition 5.6 *The set of permutations in $\mathcal{S}_n$ with e left-to-right minima, f left-to-right maxima, and having occurrences of the pattern* 2-31 *exactly at positions $\mathcal{D}$ is in bijection with the set of paths in $\widetilde{\mathcal{P}}(B,T,\mathcal{D})$ with e bottom contacts and f top contacts, where $B = E^nN^n$ and $T = (EN)^n$.*

Proof sketch: Given a path $P \in \widetilde{\mathcal{P}}(B,T,\mathcal{D})$, let $p_1, p_2, \ldots, p_n$ be the sequence of y-coordinates of its east steps from left to right. We associate to P a permutation $\pi \in \mathcal{S}_n$ as follows: $\pi(n) = p_n + 1$ and, for each i from $n-1$ to 1, let $\pi(i)$ be the (p_i+1)-st smallest number in $[n] \setminus \{\pi(i+1), \pi(i+2), \ldots, \pi(n)\}$.
□

Corollary 5.7 *Let $\mathcal{D} \subseteq [n-1]$. In the set of permutations $\pi \in \mathcal{S}_n$ having occurrences of* 2-31 *exactly at positions $\mathcal{D}$, the joint distribution of the statistics 'number of left-to-right minima' and 'number of left-to-right maxima' is symmetric.*

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