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# *Asymptotical behaviour of roots of infinite Coxeter groups I*
## *(extended abstract)*

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**Abstract.** Let $W$ be an infinite Coxeter group, and $\Phi$ be the root system constructed from its geometric representation. We study the set $E$ of limit points of "normalized" roots (representing the directions of the roots). We show that $E$ is contained in the isotropic cone $Q$ of the bilinear form associated to $W$, and illustrate this property with numerous examples and pictures in rank 3 and 4. We also define a natural geometric action of $W$ on $E$, for which $E$ is stable. Then we exhibit a countable subset $E_2$ of $E$, formed by limit points for the dihedral reflection subgroups of $W$; we explain how $E_2$ can be built from the intersection with $Q$ of the lines passing through two roots, and we establish that $E_2$ is dense in $E$.

**Résumé.** Soit $W$ un groupe de Coxeter infini, et $\Phi$ le système de racines construit à partir de sa représentation géométrique. Nous étudions l'ensemble $E$ des points d'accumulation des racines "normalisées" (représentant les directions des racines). Nous montrons que $E$ est inclus dans le cône isotrope $Q$ de la forme bilinéaire associée à $W$, et nous illustrons cette propriété à l'aide de nombreux exemples et images en rang 3 et 4. Nous définissons une action géométrique naturelle de $W$ sur $E$, pour laquelle $E$ est stable. Puis nous présentons un sous-ensemble dénombrable $E_2$ de $E$, constitué des points d'accumulation associés aux sous-groupes de réflexion diédraux de $W$; nous expliquons comment $E$ peut être construit à partir des points d'intersection de $Q$ avec les droites passant par deux racines, et nous montrons que $E_2$ est dense dans $E$.

**Keywords:** Coxeter group, root system, limit point, accumulation set.

## Introduction

When dealing with Coxeter groups, one of the most powerful tools we have at our disposal is the notion of root systems. In the case of a finite Coxeter group $W$—i.e., a finite reflection group—, roots are representatives of normal vectors for the euclidean reflections in $W$. Thinking about finite Coxeter groups and their associated *finite* root systems allows the use of arguments from Euclidean geometry and finite group theory, which makes finite root systems well studied, see for instance [Hum90, Ch.1], and the references therein.

To deal with infinite Coxeter groups, we usually distinguish two classes: affine reflection groups, and the other infinite but not affine Coxeter groups. Information about root systems associated to affine Coxeter groups are also well studied: an affine root system can be realized in an affine Euclidean space as a finite root system up to translations, see for instance [Hum90, Ch.4]. For the other *infinite* (non affine) Coxeter groups, in comparison, very little is known. A first observation is that even the term *infinite root system* seems to designate different objects, depending on whether associated to Lie algebras (see [Kac90, LN04]), Kac-Moody Lie algebras (see [MP89]) or a Coxeter group via its geometric representation (see [Hum90, Ch.5]). While all definitions of root systems are related to a given bilinear form, the bilinear forms considered in the case of Kac-Moody algebras or Lie algebras are different from the one in the classical definition of a root system for infinite Coxeter groups. In particular, this difference lies in the possibility to change the

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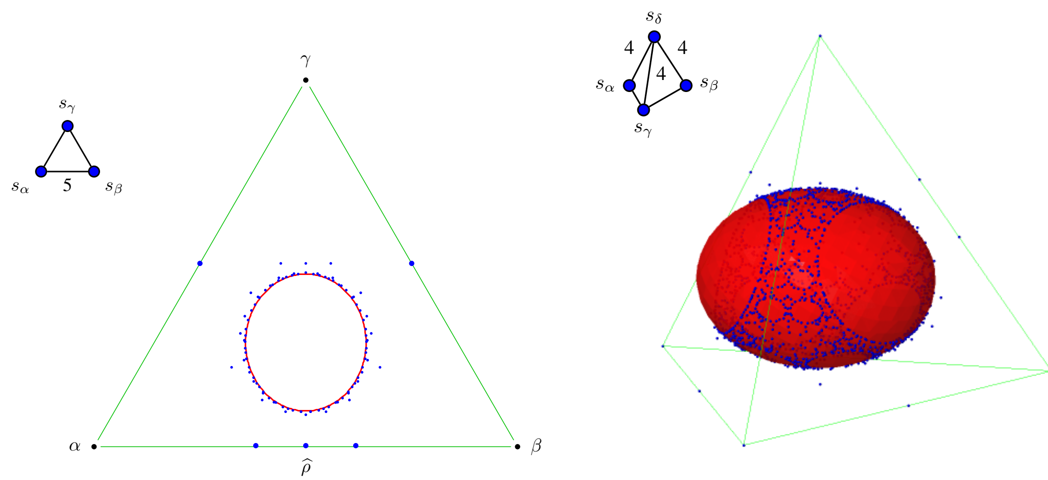
value of the bilinear form on a pair of reflections whose product has infinite order. In this vein, D. Krammer ([Kra09], see also [BD10]) described more general geometric representations of a Coxeter group and of root systems (that we take up in Section 1), which has been followed by several studies about infinite root systems of Coxeter groups (see for instance [BD10, Dye10, Dye11, Fu11]).

While investigating a conjecture on biclosed sets of positive roots (conjecture 2.5 in [Dye11]), we felt that the main difficulty to explore this question was that we needed to know more about how the roots of an infinite root system are geometrically distributed over the space. Using the mathematics software system Sage, we obtain the following pictures (Figures 1(a) and 1(b)), which suggests that roots have a very interesting asymptotical behaviour. It is the study of this behaviour we initiate in this article.




(a) The first 100 normalized positive roots, around the isotropic cone $Q$, for the rank 3 Coxeter group with the depicted graph.

(b) The first 1665 normalized positive roots, around the isotropic cone $Q$, for the rank 4 Coxeter group with the depicted graph.

**Figure 1:** Root systems for two infinite Coxeter group computed via the computer algebra system Sage.

Let us explain what we see in these pictures. First, we fix a geometric action of $W$ on a finite dimensional real vector space $V$, which implies the data of a symmetric bilinear form $B$, and a simple system $\Delta$, which is a basis for $V$ (see Section 1). In Section 2, we first show that the norm of an (injective) sequence of roots diverges to infinity. So in order to visualize limits of roots, we define $V_1$ to be the affine hyperplane spanned by the points corresponding to the simple roots: Figures 1(a) and (b) live in $V_1$ and the triangle (resp. tetrahedron) is the convex hull of the simple roots. The blue dots are the intersection of $V_1$ with the rays spanned by the roots, and we call them *normalized roots*.

Our first result (Theorem 2.7) is that the set $E$ of accumulation points of these normalized roots is contained in the isotropic cone $Q = \{v \in V \mid B(v,v) = 0\}$ of the quadratic form associated to $B$ (in red in Figure 1). We have been made aware that M. Dyer discovered independently this property in his research on the imaginary cone of Coxeter groups, see [Dye] and Remark 2.8. However, we state this result and its proof in an affine context, which is slightly different from M. Dyer's framework, and allows us to visualize pictures until rank $4$. Through them, we see new geometric properties emerging; in Sections 2.3 and 3 we describe two of these properties of $E$ which we feel should motivate further works on the subject:

1. The geometric action of $W$ on $V$ induces an action on $E$, for which $E$ is stable (Proposition 2.11). This is an action of $W$ simply given by the following process: for $\alpha \in \Delta$ and $x \in E$, the image $s_\alpha \cdot x$ of $x$ is the intersection point (other than $x$, if possible) of $Q$ with the line passing through the points $\alpha$ and $x$.

2. The set $E$ is the closure of the set of accumulation points obtained from the dihedral reflection

subgroups of $W$ only (Theorem 3.2). In other words, $E$ is the closure of the set of all the points you obtain by intersecting $Q$ with the lines in $V_1$ passing through two normalized roots.

In a forthcoming article ([DHR]), the first and third authors, together with M. Dyer, will show that $E$ is the closure of the orbit of a finite set of accumulation points, and will make some connections with the notions of root posets and of dominance order via M. Dyer's imaginary cone (cf. [Dye]). In Section 4 we present possible future works and open problems.

Note that the pictures we obtain could be reminiscent of the framework of quasicrystals constructed from extensions of noncrystallographic Coxeter groups (see [PT02] for example), but as far as we know there is no direct link.

**Figures.** The pictures were realized using the TEX-package TikZ, and computed by dint of the computer algebra system *Sage* [S$^+$11].
**Note.** This article is an extended abstract of the preprint [HLR11].

# 1 Geometric representations of a Coxeter group

We consider a Coxeter system $(W, S)$. Recall that $S \subseteq W$ is a set of generators for $W$, subject only to relations of the form $(st)^{m_{s,t}} = 1$, where $m_{s,t} \in \mathbb{N}^* \cup \{\infty\}$ is attached to each pair of generators $s, t \in S$, with $m_{s,s} = 1$ and $m_{s,t} \geq 2$ for $s \neq t$. We write $m_{s,t} = \infty$ if the product $st$ has infinite order. In the following we suppose $S$ finite, and denote by $n = |S|$ the rank of $W$. The theory of Coxeter groups is a rich one, and we recall here only what is needed for the purpose of this article. For more details, see [Hum90, BB05, Kan01, Bou68], and the references therein.

## *1.1 The classical geometric representation of a Coxeter group*

Coxeter groups are modelled to be the abstract combinatorial counterpart of reflection groups, i.e., groups generated by reflections. It is well known that any finite Coxeter group can be represented geometrically as a (finite) reflection group. This property still holds for infinite Coxeter groups, for some adapted definition of reflection that we first recall below. For $B$ a symmetric bilinear form on a real vector space $V$ (of finite dimension), and $\alpha \in V$ such that $B(\alpha, \alpha) \neq 0$, we denote by $s_\alpha$ the following map:

$$s_\alpha(v) = v - 2\frac{B(\alpha, v)}{B(\alpha, \alpha)}\, \alpha, \quad \text{for any } v \in V. \tag{1}$$

We denote by $H_\alpha := (\mathbb{R}\alpha)^\perp$ the orthogonal of the line $\mathbb{R}\alpha$ for the form $B$. Since $B(\alpha, \alpha) \neq 0$, note that we have $H_\alpha \oplus \mathbb{R}\alpha = V$. It is straightforward to check that $s_\alpha$ fixes $H_\alpha$, that $s_\alpha(\alpha) = -\alpha$, and $s_\alpha$ also preserves the form $B$, so it lies in the associated orthogonal group $\mathrm{O}_B(V)$. We call $s_\alpha$ the *$B$-reflection associated to* $\alpha$ (or simply reflection whenever $B$ is clear). When $B$ is a scalar product, this is of course the usual definition of a reflection.

Let us now recall this *classical geometric representation* (following [Hum90, §5.3-5.4]). Consider a real vector space $V$ of dimension $n$, with basis $\Delta = \{\alpha_s \mid s \in S\}$. We define a symmetric bilinear form $B$ by:

$$B(\alpha_s, \alpha_t) = \begin{cases} -\cos\left(\frac{\pi}{m_{s,t}}\right) & \text{if } m_{s,t} < \infty \\ -1 & \text{if } m_{s,t} = \infty \end{cases}.$$

Then any element $s$ of $S$ acts on $V$ as the $B$-reflection associated to $\alpha_s$ (as defined in Equation 1), i.e., $s(v) = v - 2B(\alpha_s, v)\,\alpha_s$ for $v \in V$. It is known that this induces a *faithful* action of $W$ on $V$, which preserves the form $B$; thus we denote by the same letter an element of $W$ and its associated element of $\mathrm{O}_B(V)$.

## *1.2 Root system and reflection subgroups of a Coxeter group*

The root system of $W$ is a way to encode the reflections of the Coxeter group, i.e., the conjugates of elements of $S$ (called simple reflections). The elements of $\Delta = \{\alpha_s \mid s \in S\}$ are called *simple roots* of $W$, and the *root system* $\Phi$ of $W$ is defined as the orbit of $\Delta$ under the action of $W$. By construction, any root $\rho \in \Phi$ gives rise to the reflection $s_\rho$ of $W$, which is conjugate to some $s_\alpha \in S$.

A reflection subgroup of $W$ is a subgroup of $W$ generated by reflections; so it can be built from a subset of $\Phi$. It turns out that any such reflection subgroup is again a Coxeter group, with some canonical generators ([Dye90, Deo89]). A major drawback of the classical geometric representation we described above is that it is not "functorial" with respect to the reflection subgroups: it can happen that the representation on some reflection subgroups $W'$ of $W$, induced by the geometric representation, is not the same as the geometric representation of $W'$ as a Coxeter group. A simple example is given below.

**Example 1.1** (Reflection subgroups of rank 2)**.** Consider the Coxeter group of rank 3 with $S = \{s_\alpha, s_\beta, s_\gamma\}$ and $m_{s_\alpha,s_\beta} = 5$, $m_{s_\beta,s_\gamma} = m_{s_\alpha,s_\gamma} = 3$ (whose Coxeter diagram is on Figure 1(a)). Take the root $\rho = s_\alpha s_\beta(\alpha) = s_\beta s_\alpha(\beta)$. We compute $\rho = \frac{1+\sqrt{5}}{2}(\alpha+\beta)$. Consider the reflection subgroup $W'$ generated by $s_\gamma$ and $s_\rho$. The product $s_\gamma s_\rho$ has infinite order, so $W'$ is an infinite dihedral group, with Coxeter generators $s_\gamma$ and $s_\rho$. But, if $B$ denotes the bilinear form associated to the Coxeter group $W$, we get: $B(\gamma,\rho) = -\frac{1+\sqrt{5}}{2} \neq -1$. So, the restriction to $W'$ of the representation of $W$ does not correspond to the usual geometric representation of $W'$ as an infinite dihedral group. In Example 2.4 we will describe a geometric interpretation of this fact, visible in Figure 1(a).

## 1.3 Other geometric representations

In order to solve this issue, we can relax the requirements on the bilinear form B used to represent the group $W$. Actually, an even more general setting is adapted here: the notion of a based root system (used for instance in [How96], [Kra09], [BD10, §3]).

**Definition 1.2.** Let $V$ be a real vector space, equipped with a bilinear form $B$. Consider a subset $\Delta$ of $V$ such that:

(i) $\Delta$ is positively independent: if $\sum_{\alpha\in\Delta}\lambda_\alpha\alpha = 0$ with all $\lambda_\alpha \geq 0$, then all $\lambda_\alpha = 0$;

(ii) for all $\alpha,\beta\in\Delta$, with $\alpha\neq\beta$, $B(\alpha,\beta) \in \,]-\infty,-1] \cup \{-\cos\left(\frac{\pi}{k}\right), k\in\mathbb{Z}_{\geq 2}\}$;

(iii) for all $\alpha\in\Delta$, $B(\alpha,\alpha)=1$.

Denote by $S := \{s_\alpha \mid \alpha\in\Delta\}$ the set of $B$-reflections associated to elements in $\Delta$ (see Equation 1). Let $W$ be the subgroup of $\mathrm{O}_B(V)$ generated by $S$, and $\Phi$ be the orbit of $\Delta$ under the action of $W$. Then $(\Phi,\Delta)$ is called *a based root system in* $(V,B)$ *with associated Coxeter system* $(W,S)$.

Indeed, with the notations above, $(W,S)$ is a Coxeter system, where the order of $s_\alpha s_\beta$ is $k$ whenever $m_{\alpha,\beta} = -\cos(\frac{\pi}{k})$, and $\infty$ if $m_{\alpha,\beta}\leq -1$.

Other classical properties of root systems hold here. Denote by $\mathrm{cone}(\Delta)$ the convex cone consisting of all positive linear combinations of elements of $\Delta$. If we define $\Phi^+ := \Phi\cap\mathrm{cone}(\Delta)$ (called the set of *positive roots*), then we have: $\Phi = \Phi^+ \sqcup (-\Phi^+)$.

Note that both loosenings (i) and (ii) of the usual notion of a root system are necessary to get a nice functorial behaviour with respect to inclusion of reflection subgroups (see [BD10]). However, for simplification purposes, we will here only use the generalization (ii), and still suppose that $\Delta$ is a basis, although the results remain valid in full generality. Throughout this paper, we will thus call *Coxeter root system* $(\Phi,\Delta)$ a based root system in the sense of Definition 1.2, with the additional requirement that $\Delta$ must be a basis for $V$. So the data of a Coxeter root system corresponds to the data of a Coxeter group together with one of its geometric representation.

Note that if all $m_{s,t}$ (called the *labels* of the group) are finite, then the only possible representation is the classical one. In particular, when the form $B$ is positive definite, then $\Phi$ is a *finite* Coxeter root system and contains no more information than the associated finite Coxeter group. We say that $(\Phi,\Delta)$ is an *affine Coxeter root system* when the form $B$ is positive semidefinite, but not definite. Note that traditionally, the Coxeter group itself is said to be affine if its classical geometric representation is affine.

**Example 1.3** (Irreducible affine root systems)**.** A dihedral infinite group $W$ has not only an affine representation. If $\Phi$ is an infinite root system of rank 2, with simple roots $\alpha$, $\beta$, then $B(\alpha,\beta)\leq -1$, and $\Phi$ is affine if and only if $B(\alpha,\beta) = -1$ (i.e., when $\Phi$ corresponds to the classical geometric representation of $W$). We will see a geometric description of these two cases in Figure 2. However, note that if $\Phi$ is irreducible of rank $\geq 3$, then $\Phi$ is affine if and only if $W$ is an affine Coxeter group (because there is no label $\infty$ in an irreducible affine Coxeter graph of rank $\geq 3$).

Using results of Bonnafé-Dyer, it is easy to prove that the following properties hold for any rank 2 reflection subgroup of a Coxeter group $W$. These properties will be essential in the next sections.

**Proposition 1.4.** *Let $(\Phi, \Delta)$ be a Coxeter root system, with bilinear form $B$, and Coxeter group $W$. We denote by $Q$ the isotropic cone: $Q := \{v \in V \mid q(v) = 0\}$, where $q(v) = B(v, v)$. Let $\rho_1 \neq \rho_2$ be two roots in $\Phi^+$. Denote by $W'$ the subgroup of $W$ generated by the two reflections $s_{\rho_1}$ and $s_{\rho_2}$, and by $\Phi'$ the corresponding subset of $\Phi$: $\Phi' := \{\rho \in \Phi \mid s_\rho \in W'\}$. Then:*

*(i)* $\Phi'$ *is a Coxeter root system of rank* $2$*, with Coxeter group* $W'$*; denote by* $\alpha, \beta$ *its simple roots.*

*(ii)* $\Phi'$ *is infinite if and only if the plane* $\mathrm{span}(\rho_1, \rho_2)$ *intersects* $Q \setminus \{0\}$*, if and only if* $B(\alpha, \beta) \leq -1$.

*(iii)* $\Phi'$ *is affine if and only if* $\mathrm{span}(\rho_1, \rho_2) \cap Q$ *is a line, if and only if* $B(\alpha, \beta) = -1$.

# 2 Limit points of roots

Let $\Phi$ be a Coxeter root system, with associated Coxeter group $W$ (as defined in Section 1.3). When $W$ is finite, $\Phi$ is also finite and the distribution of the roots in the space $V$ is well studied. However, when $W$ is infinite, the root system is infinite and we have, as far as we know, not many tools to study the distribution of the roots over $V$. One way to go is to look at the asymptotical behaviour of the roots. This section deals with a first step of this study. Note that, since $\Phi = \Phi^+ \sqcup (-\Phi^+)$, it is enough to deal with the positive roots, which are inside the simplicial cone $\mathrm{cone}(\Delta)$. In order to get a first grip of what could happen, we begin with some enlightening examples.

## 2.1 Examples: roots and normalized roots in rank $2$, $3$, $4$

**Example 2.1** (Rank 2: affine and non-affine representation of infinite dihedral groups)**.** Let $\Phi$ be a Coxeter root system of rank 2, as defined in Section 1.3. We get a rank 2-Coxeter group $W$, geometrically represented in a 2-dimensional vector space $V$ (together with a bilinear form $B$); $V$ is generated by two simple roots $\alpha, \beta$. Consider the case where $W$ is an infinite dihedral group, so $B(\alpha, \beta) \leq -1$.

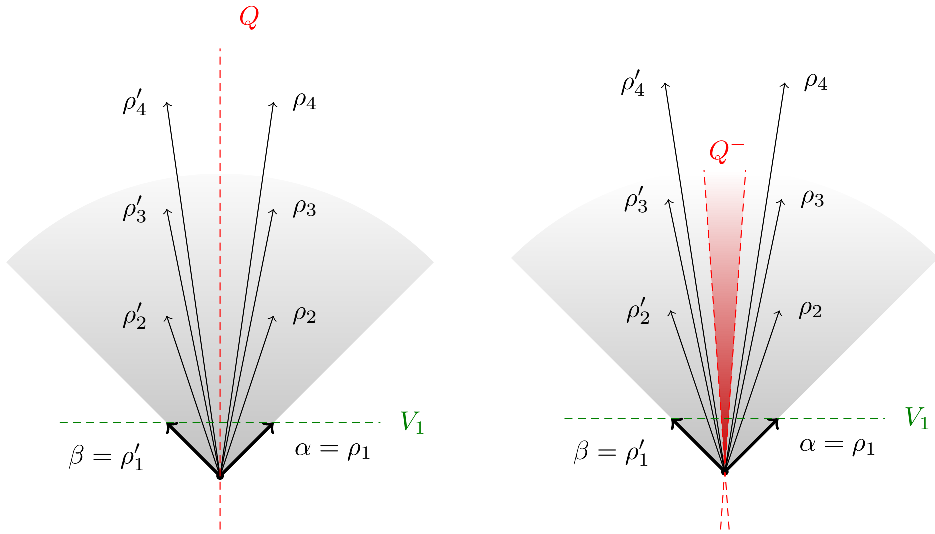
Suppose first that $B(\alpha, \beta) = -1$, i.e., that $\Phi$ is affine and that we use the classical geometric representation. Then any positive root has the following form: $\rho_n = (n+1)\alpha + n\beta$, or $\rho'_n = n\alpha + (n+1)\beta$, for $n \in \mathbb{N}$. If we fix a (Euclidean) norm on $V$ (e.g., such that $\{\alpha, \beta\}$ is an orthonormal basis), then the norm of the roots tend to infinity, but their directions tend to the line generated by $(\alpha + \beta)$ (see Fig. 2a). Note that this line is precisely the isotropic cone of the bilinear form $B$, i.e., the set

$$Q := \{v \in V \mid q(v) = 0\}\text{ , where } q(v) = B(v, v)\ .$$

In the more general geometric representation of $W$, $\Phi$ can be non-affine, i.e., $B(\alpha, \beta) = k < -1$ . The isotropic cone $Q$ is then constituted of two lines (generated by $(-k \pm \sqrt{k^2 - 1})\alpha + \beta$). If we draw the roots, we note that, again, their norms diverge to infinity and their directions tend to the two directions of the lines of $Q$ (see Fig. 2 b).

Let us go back to the general case of an infinite Coxeter root system of rank $n$. As we noted in the simple example of dihedral groups, the roots themselves have no limit points, we are rather interested in the asymptotical behaviour of their directions. In order to talk properly about limits of directions, we want to "normalize" the roots and construct "unit vectors" representing each root. One simple way to do so is to intersect the line generated by a root with the hyperplane $V_1 := \{v \in V \mid \sum_{\alpha \in \Delta} v_\alpha = 1\}$, where the $v_\alpha$'s are the coordinates of $v$ in the basis $\Delta$ of simple roots; so $V_1$ is the affine hyperplane containing the $n$ simple roots (seen as points). Then, set $V_0 := \{v \in V \mid |v|_1 = 0\}$ and $V_0^+ := \{v \in V \mid |v|_1 > 0\}$, where $|v|_1 := \sum_{\alpha \in \Delta} v_\alpha$. Note that $|..|_1$ is not a norm on $V$ (but it is when restricted to $\mathrm{conv}(\Delta)$). Since all the positive roots are in the half-space $V_0^+$, the following normalization map can be applied to $\Phi^+$:

$$\begin{array}{ccc} V_0^+ & \to & V_1 \\ v & \mapsto & \widehat{v} := \frac{v}{|v|_1}. \end{array}$$




(a) $B(\alpha, \beta) = -1$ (b) $B(\alpha, \beta) = -1.01 < -1$

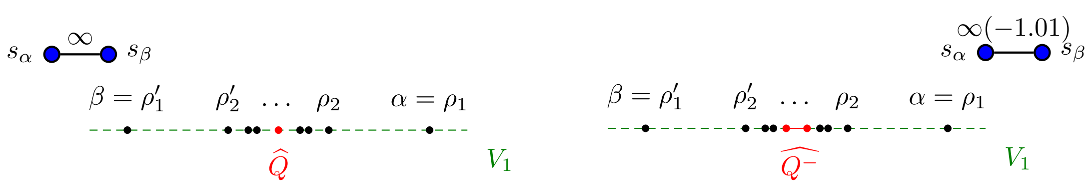
**Figure 2:** Picture of the isotropic cone $Q$ and the first positive roots of an infinite Coxeter root system of rank 2. (a): in the (classical) affine representation. (b): in a non-affine representation (the red part $Q^-$ denotes the set $\{v \in V \mid q(v) < 0\}$).

For any subset $A$ of $V_0^+$, we write $\widehat{A}$ for the set $\{\widehat{a} \mid a \in A\}$. Here we are particularly interested in the set $\widehat{\Phi^+}$ of *normalized roots*.

**Remark 2.2.** Obviously we could have considered the roots in the projective space $\mathbb{P}^1(V)$ instead. The principal advantage to consider explicitly $V_1$ is to visualize positive roots in an affine subspace of dimension $n-1$, inside an $n$-simplex. Indeed, the simple roots are in $V_1$, so all the normalized roots lie in their convex hull $\mathrm{conv}(\Delta)$, which is an $n$-simplex in $V_1$. Note that as a convex polytope, $\mathrm{conv}(\Delta)$ is closed and compact, which is practical when studying sequences of roots. From now on, in examples, we will only draw the normalized roots, inside the $n$-simplex.

The aim of this work is to study the accumulation set of $\widehat{\Phi^+}$, i.e., the set of limit points of normalized roots. We will first examine its relation with the isotropic cone $Q$.

**Example 2.3** (Normalized roots in the dihedral case)**.** In the infinite dihedral case, the "normalized" version of Figure 2 is Figure 3: there is one or two limit points of normalized roots (depending on whether $B(\alpha, \beta) = -1$ or not), and the set of limit points is always equal to the intersection $Q \cap V_1 = \widehat{Q}$.




(a) $B(\alpha, \beta) = -1$ (b) $B(\alpha, \beta) = -1.01 < -1$

**Figure 3:** Picture of the isotropic cone $\widehat{Q}$ and the first normalized roots of an infinite Coxeter root system of rank 2. (a): in the (classical) affine representation. (b): in a non-affine representation.

**Notation.** The graph we draw to characterize a Coxeter root system is the same as the classical Coxeter graph, except that, when the label of the edge $s_\alpha$—$s_\beta$ is $\infty$, we specify in parenthesis the value of $B(\alpha, \beta)$ if it is not $-1$ (i.e., when we do not consider the classical representation).

We give now some examples and pictures in rank 3 and 4.

**Example 2.4** (Rank 3)**.** In Figures 1(a) (in Introduction) and 4 through 7, we draw the (normalized) isotropic cone $\widehat{Q}$ (in red), the 3-simplex $\mathrm{cone}(\Delta)$ (in green), and the first normalized roots (in blue), for six different Coxeter root systems of rank 3. Note that the notion of depth used in the captions is a measure of the "complexity" of a root, which will be defined in Section 2.2.

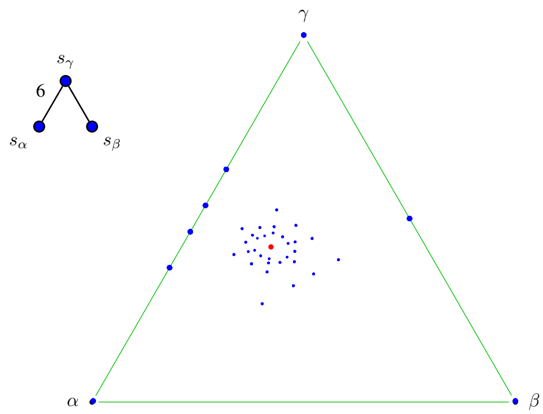



**Figure 4:** Picture of $\widehat{Q}$ and the first normalized roots (with depth $\leq 12$) for the Coxeter root system of type $\widetilde{G_2}$ (affine).

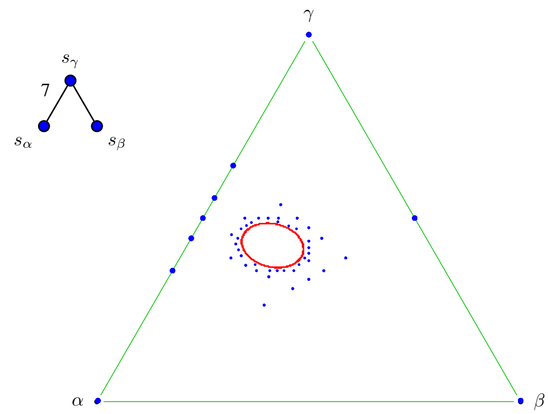



**Figure 5:** Picture of $\widehat{Q}$ and the first normalized roots (with depth $\leq 10$) for the Coxeter root system with labels $2, 3, 7$.

We see that the normalized roots seem again to tend quickly towards $\widehat{Q}$. In the affine cases (corresponding to groups of type $\widetilde{A_2}$, $\widetilde{B_2}$, and $\widetilde{G_2}$ drawn in Figure 4), $\widehat{Q}$ contains only one point, which is the intersection of the line $V^\perp$ (the radical of $B$) with $V_1$. Otherwise, $\widehat{Q}$ is always a conic, because the signature of $B$ is $(2,1)$.

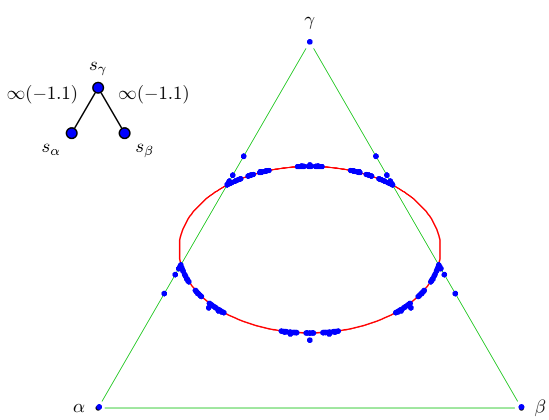



**Figure 6:** Picture of $\widehat{Q}$ and the first normalized roots (with depth $\leq 10$) for the Coxeter root system with labels $2, \infty(-1.1), \infty(-1.1)$.

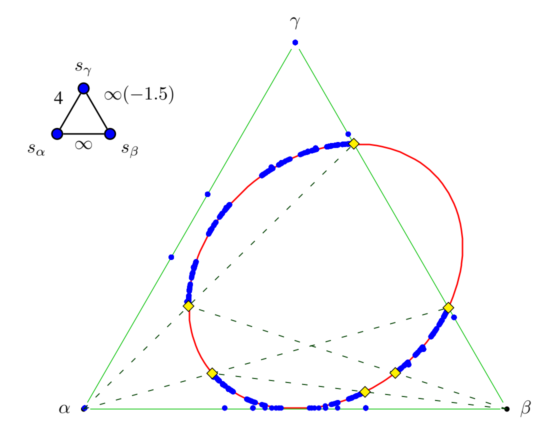



**Figure 7:** Picture of $\widehat{Q}$ and the first normalized roots (with depth $\leq 8$) for the Coxeter root system with labels $\infty, \infty(-1.5), 4$. The meaning of the yellow points and the dashed lines will be explained in Sections 2.3 and 4.

Note that we can see inside the pictures some rank 2 root subsystems, corresponding to dihedral reflection subgroups. The roots corresponding to such a reflection subgroup, generated by two reflections $s_{\rho_1}$ and $s_{\rho_2}$, lie in the line containing the roots $\rho_1$ and $\rho_2$. And because of Proposition 1.4, the subgroup is infinite if and only if $\widehat{Q}$ is intersected by this line. In Figure 1(a), we see that for the group with labels $5, 3, 3$, the line joining $\gamma$ and $\widehat{\rho} = \frac{\alpha+\beta}{2}$ intersects the ellipse in two points, as predicted by Example 1.1.

In particular, the behaviour for standard parabolic dihedral subgroups is seen on the facets of the simplex, where three situations can occur. The ellipse $\widehat{Q}$ can either cut a facet $[\alpha, \beta]$ in two points, or be tangent, or not intersect it, whether $B(\alpha, \beta) < -1, = -1$, or $> -1$ respectively; see in particular Figures 6 and 7.

**Remark 2.5.** Note also that when $\widehat{Q}$ is included in the simplex, it seems that the limit points of normalized roots cover the whole ellipse, whereas in the other cases the behaviour looks more intricate. We will talk more about this phenomenon in Section 4.1.

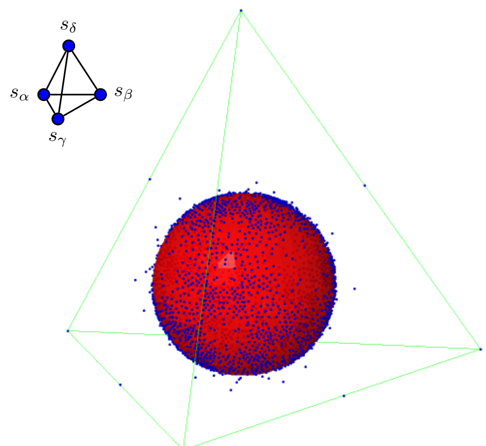



**Figure 8:** Picture of $\widehat{Q}$ and the first normalized roots (with depth $\leq 8$) for the Coxeter root system with diagram the complete graph with labels 3.

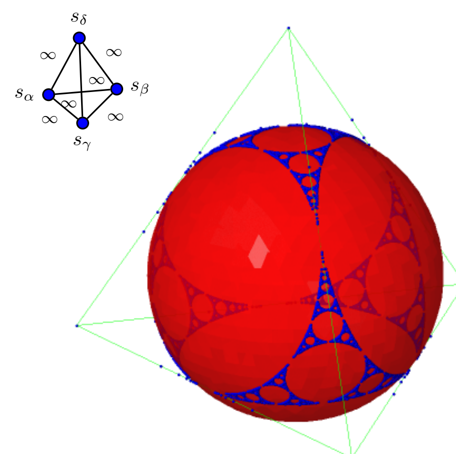



**Figure 9:** Picture of $\widehat{Q}$ and the first normalized roots (with depth $\leq 8$) for the Coxeter root system with diagram the complete graph with labels $\infty$.

**Example 2.6** (Rank 4)**.** In Figures 1(b) (in Introduction), and 8-9, we draw similar pictures for some Coxeter root systems of rank 4, together with the tetrahedron $\mathrm{conv}(\Delta)$. Analogous properties seem to be true: the limit points are in $\widehat{Q}$, and the way how $\widehat{Q}$ cuts a facet depend on whether the associated standard parabolic subgroup of rank 3 is infinite non affine, affine, or finite. Moreover, Remark 2.5 still holds, and Figures 1(b) and 9 makes appear a nice fractal behaviour that we will try to describe in Section 4.1.

## *2.2 The limit points of normalized roots lie in the isotropic cone*

The following theorem summarizes our first observations.

**Theorem 2.7.** *Consider an injective sequence of positive roots $(\rho_n)_{n\in\mathbb{N}}$, and suppose that $(\widehat{\rho_n})$ converges to a limit $\ell$. Then:*

*(i) the norm $||\rho_n||$ tends to infinity (for any norm on $V$);*

*(ii) the limit $\ell$ lies in $\widehat{Q}$.*

*In other words, the accumulation set of the set $\widehat{\Phi^+}$ of normalized roots is contained in the isotropic cone.*

**Remark 2.8.** M. Dyer proved independently this property in the context of his work on imaginary cone [Dye][(i)], extending a study of V. Kac (in the framework of Weyl groups of Lie algebras), who states that the convex hull of our limit points corresponds to the closure of the imaginary cone (see [Kac90], Lemma 5.8 and Exercise 5.12).

Note that this theorem has the following consequence (which can of course be proved more directly using the fact that $W$ is discrete in $\mathrm{GL}(V)$ [Hum90, Prop. 6.2]):

**Corollary 2.9.** *The set of roots of a Coxeter group is discrete.*

[(i)] M. Dyer, personal communication, September 2011.

Let us give a brief outline of the proof of the theorem. We first need to recall the notion of depth of a root. The depth of a positive root is a natural way to measure the "complexity" of this root, as constructed from the simple roots: for $\rho \in \Phi^+$,

$$\mathrm{dp}(\rho) = 1 + \min\{k \mid \rho = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}(\alpha_{k+1}),\ \text{for } \alpha_1, \dots, \alpha_k, \alpha_{k+1} \in \Delta\}$$

(see [BB05, §4.6] for details). By construction, the number of positive roots of bounded depth is finite. Consider an injective sequence $(\rho_n)_{n\in\mathbb{N}}$ of positive roots, as in Theorem 2.7. Then $\mathrm{dp}(\rho_n)$ diverges to infinity as $n \to \infty$. So, to prove the first item of the theorem, it is sufficient to show that when the depth of a sequence of roots tends to infinity, so does the norm of the roots. This is done using the following proposition, which clarifies the relation between norm and depth.

**Proposition 2.10.** *Let $(\Phi, \Delta)$ be a Coxeter root system, as defined in Section 1.3. We take for the norm $||\cdot||$ the Euclidean norm for which $\Delta$ is an orthonormal basis for $V$. Then, with the above notations, we have the following properties:*

*(i)* $\exists \kappa > 0,\ \forall \alpha \in \Delta,\ \forall \rho \in \Phi^+,\ B(\alpha, \rho) \neq 0 \ \Rightarrow\ |B(\alpha, \rho)| \geq \kappa$.

*(ii)* $\exists \lambda > 0,\ \forall \rho \in \Phi^+,\ ||\rho||^2 \geq 1 + \lambda(\mathrm{dp}(\rho) - 1)$.

The first point is an adaptation of a classical result, see e.g.[BB05, Prop. 4.5.5]. Item (ii) can be proved by induction on $\mathrm{dp}(\rho)$, using (i) and some well known properties of the depth.

Then, Theorem 2.7(ii) is a direct consequence of the fact that $||\rho_n||$ tends to infinity: we have $q(\ell) = \lim_{n\to+\infty} q(\widehat{\rho_n}) = \lim_{n\to+\infty} \frac{1}{|\rho_n|_1^2} = 0$, since $|\cdot|_1$ coincides on $\mathrm{cone}(\Delta)$ with a norm ($L_1$-norm).

We denote by $E(\Phi)$ (or simply $E$ when there is no possible confusion) the set of *accumulation points* (or *limit points*) of $\widehat{\Phi^+}$, i.e., the set constituted by the possible limits of injective sequences of normalized roots. As $\widehat{\Phi^+}$ is included in the simplex $\mathrm{conv}(\Delta)$ (which is closed), the theorem implies that:

$$E(\Phi) \subseteq Q \cap \mathrm{conv}(\Delta) = \widehat{Q} \cap \mathrm{cone}(\Delta)\ .$$

The reverse inclusion is not always true : we saw some examples of this fact in Section 2.1, for $\mathrm{rk}(W) = 4$, or even for $\mathrm{rk}(W) = 3$ whenever some $B(\alpha, \beta) < -1$. We will address the question of a more precise description of $E(\Phi)$ in Section 4.1.

## 2.3 Geometric action of $W$ on $E$

The geometric action of the group $W$ on $V$ induces a natural action on a part of $V_1$, using the normalization map. For the action to be well-defined, the elements on which $W$ acts have to stay in $V_0^+$ after action of any $w \in W$. So we define :

$$D = \bigcap_{w\in W} w(V_0^+) \cap V_1\ ,\ \text{and, for } x \in D \text{ and } w \in W,\ w \cdot x := \widehat{w(x)}\ .$$

It is straightforward to check that this is a well-defined action of $W$ on $D$. The following statement is the motivation for defining this action:

**Proposition 2.11.** *Let $E$ and $D$ as defined above. Then $E$ is contained in $D$, and $E$ is stable by the action of $W$.*

The aim behind studying the set $E \subseteq V_1$ is to be able to represent "limit points of roots" in an affine space. Now we can also study an action of $W$ on the affine space $V_1$. It turns out that this action of $W$ on $E$ is geometric in essence:

**Proposition 2.12.** *Let $\alpha \in \Delta$, and $x \in \widehat{Q}$. Denote by $L(\alpha, x)$ the line containing $\alpha$ and $x$. Then $s_\alpha \cdot x$ is the unique intersect point of the line $L(\alpha, x)$ with $Q$, other than $x$ (if it exists; otherwise $s_\alpha \cdot x = x$).*

In Figure 7, consider the two yellow points on the side $[\beta, \gamma]$: we drew their images by the action of $s_\alpha$ and $s_\beta s_\alpha$.

**Remark 2.13.** Obviously, this action is not faithful when $\Phi$ is an affine root system (since $E$ is a singleton). We do not know whether this action is faithful for infinite non affine root systems.

# 3 Construction of a dense subset of $E$ from dihedral reflection subgroups

As in the previous sections, we fix an infinite Coxeter root system $(\Phi, \Delta)$, and denote $E = E(\Phi)$ the accumulation set of its normalized roots. In this section we construct a nice countable subset of $E$, easy to describe, and we give a skeleton of the proof that it is dense in $E$.

This set is constructed from the limit points of roots of rank 2-subgroups of $W$. Consider two positive roots $\rho_1 \neq \rho_2$, and define the rank 2 dihedral subgroup $W' := \langle s_{\rho_1}, s_{\rho_2} \rangle$ and $\Phi' := \{\rho \in \Phi \mid s_\rho \in W'\}$. From Proposition 1.4, we know that $\Phi'$ is a Coxeter root system of rank 2, associated to the dihedral group $W'$. Denote by $\alpha, \beta$ its simple roots, and suppose that $B(\alpha, \beta) \leq -1$, i.e., that $W'$ is infinite and the line $L(\rho_1, \rho_2)$ intersects $Q$ (see Prop. 1.4(ii)). Inside $V_1$, similarly, $L(\widehat{\rho_1}, \widehat{\rho_2})$ intersects $\widehat{Q}$. We write $L(\widehat{\rho_1}, \widehat{\rho_2}) \cap \widehat{Q} = \{u, v\}$ (where $u = v$ if and only if $B(\alpha, \beta) = -1$).

Then, because of the rank 2 picture (see Section 2.1), we know that the set $E(\Phi')$ of limit points of normalized roots of $W'$ is equal to $\{u, v\}$. This leads to the following natural definition.

**Definition 3.1.** We denote by $E_2$ the subset of $E$ formed by the union of the sets $E(\Phi')$, for $\Phi'$ any root subsystem of rank 2. Equivalently:

$$E_2 := \bigcup_{\rho_1, \rho_2 \in \Phi^+} L(\widehat{\rho_1}, \widehat{\rho_2}) \cap \widehat{Q}\,,$$

where $L(\widehat{\rho_1}, \widehat{\rho_2})$ denotes the line containing $\widehat{\rho_1}$ and $\widehat{\rho_2}$.

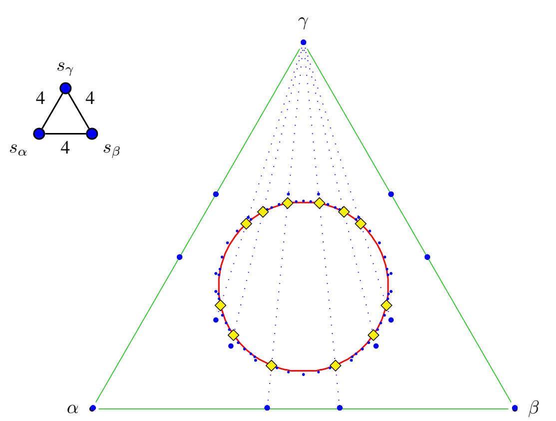



**Figure 10:** Geometric construction of $E_2$, for the Coxeter root system with labels 4, 4, 4 (the first roots in blue, some elements of $E_2$ in yellow diamonds).

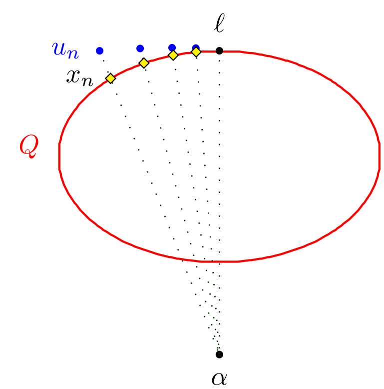



**Figure 11:** Construction of a sequence of elements of $E_2$ ($x_n$, in yellow diamonds) from a sequence of elements of $\widehat{\Phi^+}$ ($u_n$, in blue), both converging to $\ell \in E$.

Note that $E_2$ is countable, and geometrically much more easy to describe than the whole set $E$. Figure 10 gives an example of construction of some points of $E_2$. Surprisingly, $E_2$ still carries all the information of $E$, as implied by the following theorem.

**Theorem 3.2.** *Let $\Phi$ be an (infinite) Coxeter root system, and $E$ its set of limit points of normalized roots. Then the reunion $E_2$ of all limit points arising from dihedral reflection subgroups is dense in $E$.*

The basic idea of the proof is sketched in Figure 11. For $\ell \in E$, there exists a sequence $(u_n)_{n \in \mathbb{N}}$ in $\widehat{\Phi^+}$ converging to $\ell$. We construct a sequence of elements of $E_2$ converging to $\ell$ as well, using intersection points of the lines $L(\alpha, u_n)$ with $\widehat{Q}$, for an $\alpha \in \Delta$ well chosen. The core of the proof is to show that it is always possible to find such a simple root that makes the construction work (see details in [HLR11]).

# 4 Further works and open questions

We already pointed out (Remark 2.13) the question of the faithfulness of the action defined in 2.3. Let us describe roughly two other interesting problems.

### 4.1 Fractal description of $E$

We know that $E$ is contained in $\widehat{Q}$, but we could obtain a more precise inclusion. First of all, $E$ is also included in $\operatorname{conv}(\Delta)$. Now consider for example Figure 7, and suppose that we can act by $W$ on the whole of $\widehat{Q}$, with the action of Section 2.3, i.e., that $\widehat{Q} \subseteq D$ (we do not know if this is true in general). Thus, as there are no limit points in the red arc which is outside of the triangle $\operatorname{conv}(\Delta)$, there is also no limit points on its image by $s_\alpha$, which is the smaller red arc on the bottom left (cut out by the two first dashed lines); and not either in the subsequent image by $s_\beta$, as evidenced in the picture. So $E$ seems to be contained in a self-similar fractal subset $F$ of $\widehat{Q}$, obtained by removing from $\widehat{Q}$ all these iterated arcs. The rank $4$ pictures are even more convincing of this property; see in particular Figures 1(b) and 9, where the fractal $F$ obtained (an ellipsoid cut out by an infinite number of planes) is an Appolonian gasket drawn on $\widehat{Q}$.

We conjecture that $E$ is actually equal to $F$. In the particular case where $\widehat{Q}$ is contained in $\operatorname{conv}(\Delta)$ —e.g., Figure 8, and all the cases of a rank 3 Coxeter group with classical representation, as Figures 1(a) and 5—, this means that $E$ fills the whole of $\widehat{Q}$.

### 4.2 Limit points for parabolic subgroups

Let $I$ be a subset of $\Delta$, $\Phi_I$ its orbit under $W$ (i.e., a parabolic root subsystem), and $V_I$ the span of $I$. The set $E(\Phi_I)$ of accumulation points of $\widehat{\Phi_I^+}$ is of course included in $E(\Phi) \cap V_I$, and it is natural to ask whether the reverse inclusion is true. The answer is no in general, as shown by the following counterexample.

**Example 4.1.** Take the rank $5$ root system $\Phi$ with $\Delta = \{\alpha, \beta, \gamma, \delta, \varepsilon\}$ and the labels $m_{\alpha,\beta} = m_{\delta,\varepsilon} = \infty$, $m_{\beta,\gamma} = m_{\gamma,\delta} = 3$, and the others equal to 2. Take $I = \Delta \setminus \{\gamma\}$, so that $W_I$ is the product of two infinite dihedral groups. Then we have $E(\Phi_I) = \{\frac{\alpha+\beta}{2}, \frac{\delta+\varepsilon}{2}\}$. But if we consider $\rho_n = (s_\alpha s_\beta s_\varepsilon s_\delta)^n(\gamma)$, it is easy to check that $\widehat{\rho_n}$ tends to $\frac{\alpha+\beta+\delta+\varepsilon}{4}$, so lies in $E(\Phi) \cap V_I$.

In a subsequent paper [DHR], we will describe a subset of $E$ for which the restriction of this parabolic property holds.

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