A generalization of the alcove model and its applications
Cristian Lenart, Arthur Lubovsky

To cite this version:
A generalization of the alcove model and its applications

Cristian Lenart and Arthur Lubovsky

Abstract. The alcove model of the first author and Postnikov describes highest weight crystals of semisimple Lie algebras. We present a generalization, called the quantum alcove model, and conjecture that it uniformly describes tensor products of column shape Kirillov-Reshetikhin crystals, for all untwisted affine types. We prove the conjecture in types $A$ and $C$. We also present evidence for the fact that a related statistic computes the energy function.

1 Introduction

Kashiwara’s crystals \cite{Kashiwara(1991)} encode the structure of certain bases (called crystal bases) for highest weight representations of quantum groups $U_q(g)$ as $q$ goes to zero. The first author and Postnikov \cite{Lenart and Postnikov(2007), Lenart and Postnikov(2008)} defined the so-called alcove model for highest weight crystals associated to a semisimple Lie algebra $g$ (in fact, the model was defined more generally, for symmetrizable Kac-Moody algebras $g$). A related model is the one of Gaussent-Littelmann, based on LS-galleries \cite{Gaussent and Littelmann(2005)}. Both models are discrete counterparts of the celebrated Littelmann path model.

In this paper we define a generalization of the alcove model, which we call the quantum alcove model, as it is based on enumerating paths in the so-called quantum Bruhat graph of the corresponding finite Weyl group. This graph first appeared in connection with the quantum cohomology of flag varieties \cite{Fulton and Woodward(2004)}. The path enumeration is determined by the choice of a certain sequence of alcoves (an alcove path), like in the classical alcove model. If we restrict to paths in the usual Bruhat graph, we recover the classical alcove model. The mentioned paths in the quantum Bruhat graph first...
appeared in [Lenart(2012)], where they index the terms in the specialization \( t = 0 \) of the Ram-Yip formula [[Ram and Yip(2011)]] for Macdonald polynomials \( P_{\lambda}(X; q, t) \).

We define crystal operators in the quantum alcove model, both classical ones \( f_i, i > 0 \), and the affine one \( f_0 \). The main conjecture is that the new model uniformly describes tensor products of column shape Kirillov-Reshetikhin (KR) crystals [[Kirillov and Reshetikhin(1990)]], for all untwisted affine types. We prove the conjecture in types \( A \) and \( C \), by showing that the bijections constructed in [Lenart(2012)], from the objects of the quantum alcove model to tensor products of Kashiwara-Nakashima (KN) columns [[Kashiwara and Nakashima(1994)]], are affine crystal isomorphisms (indeed, a column shape KR crystal is realized by a KN column in these cases). The first author is working with S. Naito, A. Schilling, and M. Shimozono on a uniform proof of the conjecture for all untwisted affine types.

If the conjecture is true, then the quantum alcove model has the following two applications. The first one is to the energy function on a tensor product of KR crystals, which endows it with an affine grading. We present evidence for the fact that the so-called level statistic in the Ram-Yip formula mentioned above expresses the energy function. On another hand, the authors plan to realize the combinatorial above expresses the energy function. On another hand, the authors plan to realize the combinatorial R-matrix (i.e., the affine crystal isomorphism commuting two factors in a tensor product) by extending to the quantum alcove model the alcove model version of Schützenberger’s jeu de taquin on Young tableaux in [Lenart(2007)]; the latter is based on so-called Yang-Baxter moves.

2 Background

2.1 Root systems

Let \( \mathfrak{g} \) be a complex semisimple Lie algebra, and \( \mathfrak{h} \) a Cartan subalgebra, whose rank is \( r \). Let \( \Phi \subset \mathfrak{h}^* \) be the corresponding irreducible root system, \( \mathfrak{h}_\mathbb{R}^+ \subset \mathfrak{h} \) the real span of the roots, and \( \Phi^+ \subset \Phi \) the set of positive roots. The sign of the root \( \alpha \), denoted \( \text{sgn}(\alpha) \), is defined to be \( 1 \) if \( \alpha \in \Phi^+ \), and \(-1\) otherwise. Let \( \rho := \frac{1}{2} (\sum_{\alpha \in \Phi^+} \alpha) \). Let \( \alpha_1, \ldots, \alpha_r \in \Phi^+ \) be the corresponding simple roots. We denote \( \langle \cdot, \cdot \rangle \) the nondegenerate scalar product on \( \mathfrak{h}_\mathbb{R}^* \) induced by the Killing form. Given a root \( \alpha \), we consider the corresponding coroot \( \alpha^\vee := 2\alpha / \langle \alpha, \alpha \rangle \) and reflection \( s_\alpha \). If \( \alpha = \sum_i c_i \alpha_i \), then the height of \( \alpha \), denoted by \( \text{ht}(\alpha) \), is given by \( \text{ht}(\alpha) = \sum_i c_i \). We will denote by \( \bar{\alpha} \) the highest root in \( \Phi^+ \), and we let \( \theta = -\bar{\alpha} \).

Let \( W \) be the corresponding Weyl group, whose Coxeter generators are denoted, as usual, by \( s_i := s_{\alpha_i} \). The length function on \( W \) is denoted by \( \ell(\cdot) \). The Bruhat order on \( W \) is defined by its covers \( w < ws_\alpha \), for \( \alpha \in \Phi^+ \), if \( \ell(ws_\alpha) = \ell(w) + 1 \). The mentioned covers correspond to the labeled directed edges of the Bruhat graph on \( W \):

\[
w \xrightarrow{\alpha} ws_\alpha \quad \text{for} \ w < ws_\alpha. \tag{1}
\]

The weight lattice \( \Lambda \) is given by

\[
\Lambda = \{ \lambda \in \mathfrak{h}_\mathbb{R}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi \}. \tag{2}
\]

The weight lattice \( \Lambda \) is generated by the fundamental weights \( \omega_1, \ldots, \omega_r \), which form the dual basis to the basis of simple coroots, i.e., \( \langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij} \). The set \( \Lambda^+ \) of dominant weights is given by

\[
\Lambda^+ := \{ \lambda \in \Lambda : \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for any } \alpha \in \Phi^+ \}. \tag{3}
\]

Let \( \mathbb{Z}[\Lambda] \) be the group algebra of the weight lattice \( \Lambda \), which has the \( \mathbb{Z} \)-basis of formal exponents \( x^\lambda \), for \( \lambda \in \Lambda \), with multiplication \( x^\lambda \cdot x^\mu = x^{\lambda+\mu} \).
A generalization of the alcove model and its applications

Given $\alpha \in \Phi$ and $k \in \mathbb{Z}$, we denote by $s_{\alpha,k}$ the reflection in the affine hyperplane

$$H_{\alpha,k} := \{ \lambda \in h^*_R : \langle \lambda, \alpha^\vee \rangle = k \}. $$

These reflections generate the affine Weyl group $W_{\text{aff}}$ for the dual root system $\Phi^\vee := \{ \alpha^\vee | \alpha \in \Phi \}$. The hyperplanes $H_{\alpha,k}$ divide the real vector space $h^*_R$ into open regions, called alcoves. The fundamental alcove $A_\alpha$ is given by

$$A_\alpha := \{ \lambda \in h^*_R | 0 < \langle \lambda, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Phi^+ \}. $$

Define $w < ws_\alpha$, for $\alpha \in \Phi^+$, if $\ell(ws_\alpha) = \ell(w) - 2\langle \rho, \alpha^\vee \rangle + 1$. The quantum Bruhat graph [[Fulton and Woodward(2004)]] is defined by adding to the Bruhat graph (1) the following edges labeled by positive roots $\alpha$:

$$w \xrightarrow{\alpha} ws_\alpha \quad \text{for } w < ws_\alpha. $$

We will need the following properties of the quantum Bruhat graph [[Lenart, Naito, Schilling and Shimozono (2012)]].

**Lemma 2.1.** Let $w \in W$. We have $w^{-1}(\theta) > 0$ if and only if $w < s_\theta w$. We also have $w^{-1}(\theta) < 0$ if and only if $s_\theta w < w$.

**Proposition 2.2.** Let $w \in W$, let $\alpha$ be a simple reflection, $\beta \in \Phi^+$, and assume $s_\alpha w \neq ws_\beta$. Then $w < s_\alpha w$ and $w \rightarrow ws_\beta$ if and only if $ws_\beta < s_\alpha ws_\beta$ and $s_\alpha w \rightarrow s_\alpha ws_\beta$. Furthermore, in this context we have $w < ws_\beta$ if and only if $s_\alpha w < s_\alpha ws_\beta$.

**Proposition 2.3.** Let $w \in W$, $\beta \in \Phi^+$, and assume $s_\theta w \neq ws_\beta$. Then $w < s_\theta w$ and $w \rightarrow ws_\beta$ if and only if $ws_\beta < s_\theta ws_\beta$ and $s_\theta w \rightarrow s_\theta ws_\beta$.

### 2.2 Kirillov-Reshetikhin (KR) crystals

A $g$-crystal is a nonempty set $B$ together with maps $e_i, f_i : B \rightarrow B \cup \{0\}$ for $i \in I$ ($I$ indexes the simple roots, as usual, and $0 \notin B$), and $\omega : B \rightarrow \Lambda$. We require $b' = f_i(b)$ if and only if $b = e_i(b')$. The maps $e_i$ and $f_i$ are called crystal operators and are represented as arrows $b \rightarrow b'$; thus they endow $B$ with the structure of a colored directed graph. For $b \in B$, we set $e_i(b) = \max\{ k | e^k_i(b) \neq 0 \}$, $\varphi_i(b) = \max\{ k | f^k_i(b) \neq 0 \}$. Given two $g$-crystals $B_1$ and $B_2$, we define their tensor product $B_1 \otimes B_2$ as follows. As a set, $B_1 \otimes B_2$ is the Cartesian product of the two sets. For $b = b_1 \otimes b_2 \in B_1 \otimes B_2$, the weight function is simply $\omega(b) = \omega(b_1) + \omega(b_2)$. The crystal operator $f_i$ is given by

$$f_i(b_1 \otimes b_2) = \begin{cases} f_i(b_1) \otimes b_2 & \text{if } e_i(b_1) \geq \varphi_i(b_2), \\ b_1 \otimes f_i(b_2) & \text{otherwise}, \end{cases}$$

while $e_i(b)$ is defined similarly. The highest weight crystal $B(\lambda)$ of highest weight $\lambda \in \Lambda^+$ is a certain crystal with a unique element $w_\lambda$ such that $e_i(u_\lambda) = 0$ for all $i \in I$ and $\omega(u_\lambda) = \lambda$. It encodes the structure of the crystal basis of the $U_q(g)$-irreducible representation with highest weight $\lambda$ as $q$ goes to 0.

A Kirillov-Reshetikhin (KR) crystal [[Kirillov and Reshetikhin(1990)]] is a finite crystal $B^{r,s}$ for an affine algebra, associated to a rectangle of height $r$ and length $s$. We now describe the KR crystals $B^{r,1}$ for type $A_n^{(1)}$ and $C_n^{(1)}$, where $r \in \{1, 2, \ldots, n - 1\}$, and $s \in \{1, 2, \ldots, n\}$, respectively. As a classical type $A_n^{(1)}$ (resp. $C_n^{(1)}$) crystal, the KR crystal $B^{r,1}$ is isomorphic to the corresponding $B(\omega_r)$. 


In type $A_{n-1}^{(1)}$, $b \in B(\omega_r)$ is represented by a strictly increasing filling with entries in $[n] := \{1, \ldots, n\}$ of a height $r$ column. To apply $f_i$ on a tensor product of such elements, we need the following signature rule. For $1 \leq i \leq n-1$, the $i$-signature of a column filling is a word in $\{i,i+1\}$. The word is empty if the column contains both or neither letters, otherwise it is made up of one of the two letters from $\{i,i+1\}$ contained in the column. The 0-signature is computed the same way on the letters $\{n,1\}$. To apply a crystal operator $f_i$, first compute the $i$-signature of the filling by concatenating the $i$-signatures of each column, and then reduce the $i$-signature by deleting $(i+1)i$ terms successively (1$n$ terms if $i = 0$).

Eventually we obtain a reduced $i$-signature of the form $i \ldots i(i+1) \ldots (i+1)$ (the reduced 0-signature is of the form $n \ldots n1 \ldots 1$). Then the action of $f_i$ for $i > 0$ (resp. $e_i$) is defined by changing the rightmost $i$ to an $i+1$ (resp. leftmost $i+1$ to an $i$) in the corresponding column. Similarly $f_0$ (resp. $e_0$) changes the rightmost $n$ to a 1 (resp. leftmost 1 to an $n$) in the corresponding column, and sorts the column ascendingly.

**Example 2.4.** Let $n = 3$. The 0-signature of the filling \[
\begin{array}{ccc}
2 & 1 & 1 \\
3 & 2 & 0
\end{array}
\] is 311, which is already reduced. So we have $f_0 \left( \begin{array}{ccc} 2 & 1 & 1 \\ 3 & 2 & 0 \end{array} \right) = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 0 \end{array}$.

In type $C_n^{(1)}$, $B(\omega_r)$ is represented by Kashiwara-Nakashima (KN) columns \([\text{Kashiwara and Nakashima(1994)}]\) of height $r$, with entries in the set $\{1 < \cdots < n < \pi < \cdots < T\}$. The crystal operators are defined by a related signature rule.

Certain Demazure crystals for affine Lie algebras are isomorphic as classical crystals to tensor products of KR crystals \([\text{Fourier, Schilling and Shimozono(2007)}]\). Under this isomorphism only the KR arrows in Definition 2.5 below correspond to arrows in the related affine Demazure crystal.

**Definition 2.5.** An arrow $b \rightarrow f_i(b)$ is called a Demazure arrow if $i \neq 0$, or $i = 0$ and $\varphi_i(b) \geq 2$.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ be a partition, which is interpreted as a dominant weight in classical types; $\lambda'$ is the conjugate partition. Let $B^{\otimes \lambda} = \bigotimes_{i=1}^{\lambda_1} B^{\lambda_i,1}$. The energy function $D$ is a statistic on $B^{\otimes \lambda}$. It is defined by summing the local energies of column pairs. We will only need the following property of the energy function, which defines it as an affine grading on the crystal $B^{\otimes \lambda}$.

**Theorem 2.6 ([Schilling and Tingley(2011)])**. The energy is preserved by the classical crystal operators $f_i$. If $b \rightarrow f_0(b)$ is a Demazure arrow, then $D(f_0(b)) = D(b) - 1$.

It follows that the energy is determined up to a constant on the connected components of the subgraph of the affine crystal $B^{\otimes \lambda}$ containing only the Demazure arrows. Note that there is only one connected component if all the tensor factors are perfect crystals. For instance, $B^{k,1}$ is perfect in type $A_{n-1}^{(1)}$, but not in type $C_n^{(1)}$.

In types $A$ and $C$, and conjecturally in types $B$ and $D$, there is another statistic on $B^{\otimes \lambda}$, called the charge, which is obtained by translaing a certain statistic in the Ram-Yip formula for Macdonald polynomials (i.e., the level statistic in \([8]\)) to the model based on KN columns \([\text{Lenart(2012)}]\); this is done by using certain bijections, see Section 4. The charge statistic is related to the energy function by the following theorem.

**Theorem 2.7 ([Lenart and Schilling(2011)])**. Let $B^{\otimes \lambda}$ be a tensor product of KR crystals in type $A_{n-1}^{(1)}$ or type $C_n^{(1)}$. For all $b \in B^{\otimes \lambda}$, we have $D(b) = -\text{charge}(b)$.
The previous theorem is conjectured to also hold in types $B_n^{(1)}$ and $D_n^{(1)}$ [[Lenart and Schilling(2011)]]. The charge gives a much easier method to compute the energy than the recursive one based on Theorem 2.6.

3 The quantum alcove model

3.1 $\lambda$-chains and admissible subsets

We say that two alcoves are adjacent if they are distinct and have a common wall. Given a pair of adjacent alcoves $A$ and $B$, we write $A \xrightarrow{\beta} B$ if the common wall is of the form $H_{\beta,k}$ and the root $\beta \in \Phi$ points in the direction from $A$ to $B$.

**Definition 3.1** ([Lenart and Postnikov(2007)]). An alcove path is a sequence of alcoves $(A_0, A_1, \ldots, A_m)$ such that $A_{j-1}$ and $A_j$ are adjacent, for $j = 1, \ldots, m$. We say that an alcove path is reduced if it has minimal length among all alcove paths from $A_0$ to $A_m$.

Let $A_{\lambda} = A_0 + \lambda$ be the translation of the fundamental alcove $A_0$ by the weight $\lambda$.

**Definition 3.2** ([Lenart and Postnikov(2007)]). The sequence of roots $(\beta_1, \beta_2, \ldots, \beta_m)$ is called a $\lambda$-chain if

$$A_0 = A_0 \xrightarrow{\beta_1} A_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} A_m = A_{-\lambda}$$

is a reduced alcove path.

We now fix a dominant weight $\lambda$ and an alcove path $\Pi = (A_0, \ldots, A_m)$ from $A_0 = A_0$ to $A_m = A_{-\lambda}$. Note that $\Pi$ is determined by the corresponding $\lambda$-chain of positive roots $\Gamma := (\beta_1, \ldots, \beta_m)$. We let $r_i := s_{\beta_i}$, and let $\tilde{r}_i$ be the affine reflection in the hyperplane containing the common face of $A_{i-1}$ and $A_i$. For $i = 1, \ldots, m$, in other words, $\tilde{r}_i := s_{\beta_i, -l_i}$, where $l_i := |\{ j < i; \beta_j = \beta_i \}|$. We define $\bar{l}_i := \langle \lambda, \beta_i^\vee \rangle - l_i = |\{ j \geq i; \beta_j = \beta_i \}|$.

**Example 3.3.** Consider the dominant weight $\lambda = 3\varepsilon_1 + 2\varepsilon_2$ in the root system $A_2$ (cf. Section 4.1 and the notation therein). The corresponding $\lambda$-chain is $(\alpha_{23}, \alpha_{13}, \alpha_{23}, \alpha_{12}, \alpha_{13})$. The corresponding levels $l_i$ are $(0, 0, 1, 1, 0, 2)$ and $\bar{l}_i$ are $(2, 3, 1, 2, 1, 1)$. The alcove path is shown in Figure 1(a); here $A_0$ is shaded, and $A_0 - \lambda$ is the alcove at the end of the path.

Let $J = \{ j_1 < j_2 < \cdots < j_s \} \subseteq [m]$ be a subset of $[m]$. The elements of $J$ are called folding positions. We fold $\Pi$ in the hyperplanes corresponding to these positions and obtain a folded path, see Example 3.5 and Figure 1(b). Like $\Pi$, the folded path can be recorded by a sequence of roots, namely $\Delta = \Gamma(J) = (\gamma_1, \gamma_2, \ldots, \gamma_m)$; here $\gamma_k = r_{j_1}r_{j_2}\cdots r_{j_s}(\beta_k)$, with $j_s$ the largest folding position less than $k$. We define $\gamma_{\infty} := r_{j_1}r_{j_2}\cdots r_{j_s}(\rho)$. Upon folding, the hyperplane separating the alcoves $A_{k-1}$ and $A_k$ in $\Pi$ is mapped to $H_{|\gamma_k|, -\bar{l}_k^J} = \tilde{r}_{j_1}\tilde{r}_{j_2}\cdots \tilde{r}_{j_s}(H_{|\beta_k|, -l_k^J})$, for some $l_k^J$, which is defined by this relation.

Given $i \in J$, we say that $i$ is a positive folding position if $\gamma_i > 0$, and a negative folding position if $\gamma_i < 0$. We denote the positive folding positions by $J^+$, and the negative ones by $J^-$. We call $\mu = \mu(J) := -\tilde{r}_{j_1}\tilde{r}_{j_2}\cdots \tilde{r}_{j_s}(\lambda)$ the weight of $J$. We define

$$\text{level}(J) := \sum_{j \in J^-} \bar{l}_j.$$
**Definition 3.4.** A subset $J = \{ j_1 < j_2 < \cdots < j_s \} \subseteq [m]$ (possibly empty) is an admissible subset if we have the following path in the quantum Bruhat graph on $W$:

$$
\gamma_{j_1} \rightarrow r_{j_1} \gamma_{j_2} \rightarrow r_{j_1}r_{j_2} \gamma_{j_3} \cdots \gamma_{j_s} \rightarrow r_{j_1}r_{j_2} \cdots r_{j_s}.
$$

We call $\Delta = \Gamma(J)$ an admissible folding. We let $\mathcal{A} = \mathcal{A}(\lambda)$ be the collection of admissible subsets.

**Example 3.5.** We continue Example 3.3. Let $J = \{1, 2, 3, 5\}$, then $\Delta = \Gamma(J) = \{\alpha_{23}, \alpha_{12}, \alpha_{31}, \alpha_{23}, \alpha_{21}, \alpha_{13}\}$. The folded path is shown in Figure 1(b). We have $J^+ = \{1, 2\}$, $J^- = \{3, 5\}$, $\mu(J) = -\varepsilon_3$, and level$(J) = 2$. We check that $J$ is admissible in (23).

Given $J \subseteq [m]$ and $\alpha \in \Phi$, we will use the following notation:

$$
I_{\alpha} = I_{\alpha}(\Delta) := \{ i \in [m] \mid \gamma_i = \pm \alpha \}, \quad L_{\alpha} = L_{\alpha}(\Delta) := \{ l^\Delta \mid i \in I_{\alpha} \},
$$

$$
\hat{I}_{\alpha} = \hat{I}_{\alpha}(\Delta) := I_{\alpha} \cup \{ \infty \}, \quad \hat{L}_{\alpha} = \hat{L}_{\alpha}(\Delta) := L_{\alpha} \cup \{ l^\infty \},
$$

where $l^\infty := \langle \mu(J), \text{sgn}(\alpha)\alpha^\vee \rangle$. We will use $\hat{L}_{\alpha}$ to define the crystal operators on admissible subsets in the following sections. The following graphical representation of $\hat{L}_{\alpha}$ is useful for such purposes. Let

$$
\hat{L}_{\alpha} = \{ i_1 < i_2 < \cdots < i_n \leq m < i_{n+1} = \infty \} \quad \text{and} \quad \varepsilon_i := \begin{cases} 
1 & \text{if } i \not\in J \\
-1 & \text{if } i \in J \end{cases}.
$$

If $\alpha > 0$, we define the continuous piecewise-linear function $g_\alpha : \left[0, n + \frac{1}{2}\right] \rightarrow \mathbb{R}$ by

$$
g_\alpha(0) = -\frac{1}{2}, \quad g_\alpha(x) = \begin{cases} 
\text{sgn}(\gamma_{i_k}) & \text{if } x \in (k - 1, k - \frac{1}{2}), k = 1, \ldots, n \\
\varepsilon_i \text{sgn}(\gamma_{i_k}) & \text{if } x \in (k - \frac{1}{2}, k), k = 1, \ldots, n \\
\text{sgn}(\langle \gamma_\infty, \alpha^\vee \rangle) & \text{if } x \in (n, n + \frac{1}{2}).
\end{cases}
$$

---

**Fig. 1:** Unfolded and folded $\lambda$-chain
If $\alpha < 0$, we define $g_\alpha$ to be the graph obtained by reflecting $g_{-\alpha}$ in the $x$-axis. By [Lenart and Postnikov(2008)], we have
\[
\text{sgn}(\alpha) l_{\Delta}^k = g_\alpha \left( k - \frac{1}{2} \right), k = 1, \ldots, n, \quad \text{and} \quad \text{sgn}(\alpha) l_\infty^\Delta := \langle \mu(J), \alpha^\vee \rangle = g_\alpha \left( n + \frac{1}{2} \right).
\]

**Example 3.6.** We continue Example 3.5. The graphs of $g_{\alpha_2}$ and $g_\theta$ are given in Figure 2.

**3.2 Crystal operators**

We will now define crystal operators on the collection $A = A(\lambda)$ of admissible subsets corresponding to our fixed $\lambda$-chain. Let $J$ be such an admissible subset, let $\Delta$ be the associated admissible folding, and $L(\Delta) = (l_{\Delta}^i)_{i \in [m]}$ its level sequence. Also recall from Section 3.1 the definitions of the finite sequences $I_\alpha(\Delta), \widehat{I}_\alpha(\Delta), L_\alpha(\Delta), \text{ and } \widehat{L}_\alpha(\Delta)$, where $\alpha$ is a root, as well as the related notation. Let $\delta_{i,j}$ be the Kronecker delta function.

Throughout we fix $\alpha_p$, which is a simple root if $p > 0$, or $\theta$ if $p = 0$. We define $f_p$ on admissible subsets. We will use the convention $J \setminus \{\infty\} = J \cup \{\infty\} = J$. Let $M$ be the maximum of $g_{\alpha_p}$, and suppose that $M \geq \delta_{p,0}$. Let $m$ be the minimum index $i$ in $\widehat{I}_\alpha(\Delta)$ for which we have $\text{sgn}(\alpha_p) l_{\Delta}^i = M$.

**Proposition 3.7.** Given the above setup, the following hold.

1. If $m \neq \infty$, then $\gamma_m = \alpha_p$ and $m \in J$.
2. If $M > \delta_{p,0}$ then $m$ has a predecessor $k$ in $\widehat{I}_{\alpha_p}(\Delta)$ such that $\gamma_k = \alpha_p, k \notin J$, and $\text{sgn}(\alpha_p) l_{\Delta}^k = M - 1$.

Based on the previous proposition, we define
\[
f_p(J) := \begin{cases} 
(J \setminus \{m\}) \cup \{k\} & \text{if } M > \delta_{p,0} \\
0 & \text{otherwise}.
\end{cases}
\]

To define the crystal operator $e_p$, we assume that $M > \langle \mu(J), \alpha^\vee \rangle$. Let $k$ be the maximum index $i$ in $I_{\alpha_p}(\Delta)$ for which we have $\text{sgn}(\alpha_p) l_{\Delta}^i = M$, and let $m$ be the successor of $k$ in $\widehat{I}_{\alpha_p}(\Delta)$. 

![Fig. 2:](image-url)
Proposition 3.8. Given the above setup, the following hold.
(1) We have \( \gamma_k = \alpha_p \) and \( k \in J \).
(2) If \( m \neq \infty \) then
\[
\gamma_m = -\alpha_p, \ m \notin J, \text{ and } \text{sgn}(\alpha_p)\rho_{\alpha_p}^m = M - 1.
\]

We define
\[
e_p(J) := \begin{cases} 
(J \setminus \{k\}) \cup \{m\} & \text{if } M > \langle \mu(J), \alpha^\vee \rangle \\
0 & \text{otherwise}.
\end{cases}
\]

Note that \( f_p(J) = J' \) if and only if \( e_p(\mu(J')) = J \).

Example 3.9. We continue Example 3.6. We find \( f_2(J) \) by noting that \( \hat{I}_{\alpha_2} = \{1, 4, \infty\} \). From \( g_{\alpha_2} \) in Figure 2, we can see that \( \hat{I}_{\alpha_2} = \{0, 1\} \), so \( k = 4, m = \infty \), and \( f_2(J) = J \cup \{4\} = \{1, 2, 3, 4, 5\} \). We can see from Figure 2 that the maximum of \( g_0 = 1 \), hence \( f_0(J) = 0 \).

Theorem 3.10. If \( f_p(J) \) is defined, then it is also an admissible subset. Similarly for \( e_p(J) \).

Proof: Suppose \( p \neq 0 \). We consider \( f_p \) first. The cases corresponding to \( m \neq \infty \) and \( m = \infty \) can be proved in similar ways, so we only consider the first case. Let \( J = \{j_1 < j_2 < \ldots < j_s\} \), and let \( w_i = r_{j_1}r_{j_2}\ldots r_{j_s} \). Based on Proposition 3.7, let \( a < b \) be such that
\[
j_a < k < j_a + 1 < \cdots < j_b = m < j_b + 1;
\]
if \( a = 0 \) or \( b + 1 > s \), then the corresponding indices \( j_a \), respectively \( j_{b + 1} \), are missing. To show that \( (J \setminus \{m\}) \cup \{k\} \) is an admissible subset, it is enough to prove
\[
w_a \rightarrow w_ar_k \rightarrow w_ar_kr_{j_a+1} \rightarrow \ldots \rightarrow w_ar_kr_{j_b+1}r_{j_b-1} = w_b.
\]

By our choice of \( k \), we have
\[
w_a(\beta_k) = \alpha_p \iff w_a^{-1}(\alpha_p) = \beta_k > 0 \iff w_a < s_pw_a = w_ar_k.
\]

So we can rewrite (14) as
\[
w_a \rightarrow s_pw_a \rightarrow s_pw_{a+1} \rightarrow \ldots \rightarrow s_pw_{b-1} = w_b.
\]

We will now prove that (16) is a path in the quantum Bruhat graph. Observe
\[
s_pw_{i-1} = w_i \iff w_{i-1}(\beta_{j_i}) = \pm \alpha_p \iff j_i \in I_\alpha.
\]

Our choice of \( k \) and \( b \) implies that we have
\[
s_pw_{i-1} \neq w_i \text{ for } a < i < b
\]
(otherwise \( j_i \in I_\alpha \) for \( k < j_i < j_b \), and \( s_pw_{b-1} = w_b \) since \( j_b \in I_\alpha \). Since \( J \) is admissible, we have
\[
w_{i-1} \rightarrow w_i.
\]

With (15) as the base case, assume by induction that \( w_{i-1} \leq s_pw_{i-1} \). We can apply Proposition 2.2 to conclude that \( w_i < s_pw_i \) and
\[
s_pw_{i-1} \rightarrow s_pw_i \text{ for } a < i < b.
\]

The proof for \( e_p(J) \) is similar. The above proof follows through for \( p = 0 \) with \( \leq \) replaced by \( < \) with the help of Lemma 2.1 and Proposition 2.3.
3.3 Main conjecture

The following is our main conjecture about the quantum alcove model. The setup is that of untwisted affine root systems.

Conjecture 3.11. (1) $A(\lambda)$ is isomorphic to the subgraph of $B^{\otimes \lambda}$ containing only the Demazure arrows.

(2) If $b$ corresponds to $J$ under the isomorphism in (1), then the energy is given by $D(b) = -\text{level}(J)$.

The reason for which we restrict to the Demazure arrows is that the other 0-arrows in $B^{\otimes \lambda}$ are realized in the quantum alcove model by removing/adding more than one element of an admissible subset; this rule is not yet understood, but we hope to elucidate it in the future. Part (1) of the conjecture is addressed in Section 4 for types $A$ and $C$, whereas part (2) in these cases is essentially Theorem 2.7. There is even more evidence for the conjecture, namely the sets $A(\lambda)$ and $B^{\otimes \lambda}$ have the same cardinality in type $D$ and $E$ (assuming that only certain fundamental weights appear in $\lambda$ in type $E$). This follows by combining the specialization $t = 0$ of the Ram-Yip formula for Macdonald polynomials [Ram and Yip(2011)], with the relation between Macdonald polynomials and affine Demazure characters [Ion(2003)], as well as the relation between Demazure and KR crystals [Fourier and Littelmann(2006)]. In type $D$, it is also known that the generating functions of $-D(b)$ and level($J$) agree [Schilling and Tingley(2011), Corollary 9.5].

Assuming that part (1) of Conjecture 3.11 holds, part (2) can be approached by showing that the level statistic satisfies the recursive definition of energy in Theorem 2.6. The part of the recursion involving the crystal operators $f_i, i \neq 0$, is proved below.

Proposition 3.12. Let $\lambda$ be a dominant weight and $J \in A(\lambda)$. If $i \neq 0$ then $\text{level}(f_i(J)) = \text{level}(J)$.

Proof: By definition, $f_i(J)$ is obtained from $J$ by adding a folding position and possibly removing one, and both of these are positive. We will show that the other folding positions in $J$ do not change sign. This follows from the proof of Theorem 3.10. We continue using notation from that proof. For $w \in W$ and $\beta$ a positive root, $w(\beta) > 0$ if and only if $\ell(w) < \ell(ws_\beta)$. Hence $\gamma_{ij} = w_{i-1}(\beta_j) > 0$ if and only if $\ell(w_{i-1}) < \ell(w_i)$. By Proposition 2.2, the edge in (18) corresponding to $J$ is a cover if and only if the edge in (19) corresponding to $f_i(J)$ is a cover, so we don’t introduce any new negative folding positions.

4 The quantum alcove model in types $A$ and $C$

4.1 Type $A$

We start with the basic facts about the root system of type $A_{n-1}$. We can identify the space $\mathfrak{h}^*_\mathbb{R}$ with the quotient $V := \mathbb{R}^n / \mathbb{R}(1, \ldots, 1)$, where $\mathbb{R}(1, \ldots, 1)$ denotes the subspace in $\mathbb{R}^n$ spanned by the vector $(1, \ldots, 1)$. Let $\varepsilon_1, \ldots, \varepsilon_n \in V$ be the images of the coordinate vectors in $\mathbb{R}^n$. The root system is $\Phi = \{\alpha_{ij} := \varepsilon_i - \varepsilon_j : i \neq j, 1 \leq i, j \leq n\}$. The simple roots are $\alpha_i = \alpha_{i,i+1}$, for $i = 1, \ldots, n-1$. The highest root $\check{\alpha} = \alpha_n$, so $\theta = \alpha_n$. The weight lattice is $\Lambda = \mathbb{Z}^n / \mathbb{Z}(1, \ldots, 1)$. The fundamental weights are $\omega_i = \varepsilon_1 + \ldots + \varepsilon_i$, for $i = 1, \ldots, n-1$. A dominant weight $\lambda = \lambda_1 \varepsilon_1 + \ldots + \lambda_{n-1} \varepsilon_{n-1}$ is identified with the partition $(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n = 0)$ having at most $n-1$ parts. Note that $\rho = (n-1, n-2, \ldots, 0)$. Considering the Young diagram of the dominant weight $\lambda$ as a concatenation of columns, whose heights are $\lambda_1', \lambda_2', \ldots$, corresponds to expressing $\lambda$ as $\omega_{\lambda_1'} + \omega_{\lambda_2'} + \ldots$ (as usual, $\lambda'$ is the conjugate partition to $\lambda$).

The Weyl group $W$ is the symmetric group $S_n$, which acts on $V$ by permuting the coordinates $\varepsilon_1, \ldots, \varepsilon_n$. Permutations $w \in S_n$ are written in one-line notation $w = w(1) \ldots w(n)$. For simplicity, we use the same
notation \((i,j)\) with \(1 \leq i < j \leq n\) for the root \(\alpha_{ij}\) and the reflection \(s_{\alpha_{ij}}\), which is the transposition \(t_{ij}\) of \(i\) and \(j\).

We now consider the specialization of the alcove model to type \(A\). For any \(k = 1, \ldots, n-1\), we have the following \(\omega_k\)-chain, from \(A_n\) to \(A_{n-1}\), denoted by \(\Gamma(k)\) ([Lenart and Postnikov(2007)]):

\[
\begin{array}{cccccc}
(k, k+1), & (k, k+2), & \ldots, & (k, n), \\
(k-1, k+1), & (k-1, k+2), & \ldots, & (k-1, n), \\
 & \vdots & & \vdots & \\
(1, k+1), & (1, k+2), & \ldots, & (1, n) .
\end{array}
\]

**Example 4.1.** For \(n = 4\), \(\Gamma(2)\) can be visualized as obtained from the following broken column, by pairing row numbers in the top and bottom parts in the prescribed order.

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array} , \quad \Gamma(2) = \{(2, 3), (2, 4), (1, 3), (1, 4)\} .
\]

Note the top part of the above broken column corresponds to \(\omega_2\).

We construct a \(\lambda\)-chain \(\Gamma = (\beta_1, \beta_2, \ldots, \beta_m)\) as the concatenation \(\Gamma := \Gamma^1 \ldots \Gamma^\lambda\), where \(\Gamma^j = \Gamma(\lambda'_j)\). Let \(J = \{j_1 < \cdots < j_a\}\) be a set of folding positions in \(\Gamma\), not necessarily admissible, and let \(T\) be the corresponding list of roots of \(\Gamma\). The factorization of \(\Gamma\) induces a factorization on \(T\) as \(T = T^1 T^2 \ldots T^\lambda\), and on \(\Delta = \Gamma(J)\) as \(\Delta = \Delta^1 \ldots \Delta^\lambda\). We denote by \(T^1 \ldots T^j\) the permutation obtained via multiplication by the transpositions in \(T^1, \ldots, T^j\) considered from left to right. For \(w \in W\), written \(w = w_1 w_2 \ldots w_n\), let \(w[i, j] = w_i \ldots w_j\). To each \(J\) we can associate a filling of a Young diagram \(\lambda\).

**Definition 4.2.** Let \(\pi_j = \pi_j(T) = T^1 \ldots T^j\). We define the filling map, which produces a filling of the Young diagram \(\lambda\), by

\[
\text{fill}(J) = \text{fill}(T) = C_1 \ldots C_{\lambda_1} ; \quad \text{here } C_i = \pi_i[1, \lambda'_i].
\]

We need the circular order \(\prec_i\) on \([n]\) starting at \(i\), namely \(i \prec_i i + 1 \prec_i \ldots \prec_i n \prec_i 1 \prec_i \ldots \prec_i i - 1\). It is convenient to think of this order in terms of the numbers \(1, \ldots, n\) arranged on a circle clockwise. We make the convention that, whenever we write \(a \prec b \prec c \prec \ldots\), we refer to the circular order \(\prec = \prec_a\). We have the following description of the edges of the quantum Bruhat graph in type \(A\).

**Proposition 4.3** ([Lenart(2012)]). For \(1 \leq i < j \leq n\) we have an edge \(w^{(i,j)}\) if and only if there is no \(k\) such that \(i < k < j\) and \(w(i) < w(k) < w(j)\).

**Example 4.4.** We restate Examples 3.3 and 3.5 in more detail. Let \(n = 3\) and \(\lambda = (3, 2, 0)\), which is identified with \(3 \varepsilon_1 + 2 \varepsilon_2 = 2 \omega_2 + \omega_1\), and corresponds to the Young diagram \[
\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\]
We have

\[
\Gamma = \Gamma^1 \Gamma^2 \Gamma^3 = \Gamma(2) \Gamma(2) \Gamma(1) = \{(2, 3), (1, 3) | (2, 3), (1, 3) | (1, 2), (1, 3)\}.
\]
A generalization of the alcove model and its applications

where we underlined the roots in positions $J = \{1, 2, 3, 5\}$. Then

$$T = \{(2, 3), (1, 3), (2, 3), (1, 2)\},$$

and

$$\Gamma(J) = \Delta = \Delta^1 \Delta^2 \Delta^3 = \{(2, 3), (1, 2), (3, 1), (2, 3), (2, 1), (1, 3)\},$$

where we again underlined the folding positions. We write permutations in (9) as broken columns.

Based on Proposition 4.3, note that $J$ is admissible since

$$123 \lessdot 132 \lessdot 231 \lessdot 213 \lessdot 213.$$

By considering the top part of the last column in each segment and by concatenating these columns left to right, we obtain $\text{fill}(J)$, i.e., $\text{fill}(J) = 22113121$.

Definition 4.5. We define the sorted filling map $\text{sf}(J)$ by sorting ascendingly the columns of $\text{fill}(J)$.

Theorem 4.6 (Lenart(2012)). The map $\text{sf}$ is a bijection between $A(\lambda)$ and $B^{\otimes \lambda}$.

Theorem 4.7. The map $\text{sf}$ preserves the affine crystal structures, with respect to Demazure arrows. In other words, given $\text{sf}(J) = b$, there is a Demazure arrow $b \rightarrow f_i(b)$ if and only if $f_i(J) \neq 0$, and we have $f_i(b) = \text{sf}(f_i(J))$.

The main idea of the proof of Theorem 4.7 is the following. The signature of a filling, used to define the crystal operator $f_i$, can be interpreted as a graph similar to the graph of $g_i$, which is used to define the crystal operator on the corresponding admissible subsequence. The link between the two graphs is given by Lemma 4.8 below, called the level counting lemma, which we now explain.

Let $N_c(\sigma)$ denote the number of entries $c$ in a filling $\sigma$. Let $ct(\sigma) = (N_1(\sigma), \ldots, N_n(\sigma))$ be the content of $\sigma$. Let $\sigma[q]$ be the filling consisting of the columns 1, 2, $\ldots$, $q$ of $\sigma$. Recall the factorization of $\Delta$ illustrated in (22) and the levels $l^\Delta_k$ defined in (7).

Lemma 4.8 (Lenart(2011), Proposition 4.1). Let $J \subseteq [m]$, and $\sigma = \text{fill}(J)$. For a fixed $k$, let $\gamma_k = (c, d)$ be a root in $\Delta^{1+k}$. We have

$$\text{sgn}(\gamma_k) t^\Delta_k = \langle ct(\sigma[q]), \gamma_k^\vee \rangle = N_c(\sigma[q]) - N_d(\sigma[q]).$$

4.2 Type $C$

The results in type $A$ are paralleled in type $C$. The affine crystal $B^{\otimes \lambda}$ is realized as a tensor product of KN columns [Kashiwara and Nakashima(1994)], with a related signature rule. In [Lenart(2012), Theorem 6.1], the first author provides a bijection between $A(\lambda)$ and $B^{\otimes \lambda}$. This bijection can be shown to be an affine crystal isomorphism in the sense of Theorem 4.7.
References


