The ABC's of affine Grassmannians and Hall-Littlewood polynomials

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Abstract. We give a new description of the Pieri rule for k-Schur functions using the Bruhat order on the affine type-A Weyl group. In doing so, we prove a new combinatorial formula for representatives of the Schubert classes for the cohomology of affine Grassmannians. We show how new combinatorics involved in our formulas gives the Kostka-Foulkes polynomials and discuss how this can be applied to study the transition matrices between Hall-Littlewood and k-Schur functions.

Résumé. Nous présentons une nouvelle description, issue de l'ordre de Bruhat du groupe de Weyl affine de type A, de la règle de Pieri pour les fonctions k-Schur. Ce faisant, nous obtenons une nouvelle formule combinatoire pour les représentants des classes de Schubert de la cohomologie des Grassmannienne affines. Nous décrivons aussi comment notre approche permet d'obtenir les polynômes de Kostka-Foulkes et comment elle peut être appliquée à l'étude des matrices de transition entre les polynômes de Hall-Littlewood et les fonctions k-Schur.

Keywords: k-Schur functions, Pieri rule, Bruhat order, Macdonald polynomials, Hall-Littlewood polynomials, k-tableaux

1 Introduction

The dual k-Schur functions arose in [LM08] where it was shown that their coproduct encodes structure constants of the Verlinde fusion algebra for the $\widehat{\mathfrak{sl}}_n$ Wess–Zumino–Witten models [TUY89, Ver88] or equivalently, the 3-point Gromov-Witten invariants of genus zero (e.g. [Wit95]). It was further proven in [Lam08] that these functions represent cohomology classes of the affine Grassmannian. Dual k-Schur functions are defined as the weight generating functions of k-tableaux; a combinatorial object encoding successions of saturated chains in weak order on the affine Weyl group $\tilde{A}^{k+1}$ (see § 2). Here, we find that there is a natural realization of weak saturated chains of length ℓ as length $k-\ell$ saturated chains in the strong (Bruhat) order on $\tilde{A}^{k+1}$. This enables us to prove that the dual k-Schur functions $\mathfrak{S}_\lambda^{(k)}$ can be described in terms of a new combinatorial object called affine Bruhat counter-tableau, or ABC, that encodes successions of strong chains (see Definition 16). We prove, for a fixed positive integer k and $k+1$-core λ that

$$\mathfrak{S}_\lambda^{(k)} = \sum_A x^A \,, \tag{1}$$

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over the ABC's of inner shape λ.

Our new interpretation for $\{\mathfrak{S}_\lambda^{(k)}\}_{\lambda_1 \le k}$ puts these cohomology classes in the same combinatorial vein as the homology classes. That is, representatives for the homology classes of the affine Grassmannian are given by the k-Schur functions $s_\lambda^{(k)}$ of [LM07] which were defined in terms of chains in the strong order on $\tilde{A}^{k+1}$ by [LLMS10]. In fact, our work relies on a reformulation of the Pieri rule for $s_\lambda^{(k)}$; we show that this rule for computing the k-Schur expansion of the product $h_r s_\lambda^{(k)}$ can be described by certain saturated strong chains.

A strong motivation for introducing these new combinatorial ideas is to shed light on open problems in the theory of Macdonald polynomials. The origin of k-Schur functions was in the idea that there exists a more refined basis than Schur functions upon which Macdonald polynomials $H_\mu[X;q,t]$ expand positively. In particular, a family of polynomials $A_\lambda^{(k)}[X;t]$ was introduced by [LLM03] and it was conjectured that, for $\mu_1 \le k$,

$$H_\mu[X;q,t] = \sum_\lambda K_{\lambda\mu}^{(k)}(q,t)\, A_\lambda^{(k)}[X;t] \tag{2}$$

and $K_{\lambda\mu}^{(k)}(q,t) \in \mathbb{N}[q,t]$. When $k \ge |\lambda|$, $A_\lambda^{(k)}[X;t]$ reduces to the Schur function s_λ. In this case, the conjecture reduces to the conjecture of [Mac88] that $K_{\lambda\mu}^{(\infty)}(q,t)$ is a positive sum of monomials in q and t. The polynomiality of $K_{\lambda\mu}^{(\infty)}(q,t)$ was proven independently by [KN98, Sah96, Kno97, GT96, LV97] and the positivity was finally settled by [Hai01]. However, for generic k, this remains an open problem even in the Hall-Littlewood case $H_\mu[X;0,t]$.

It is conjectured that $A_\mu^{(k)}[X;1]$ are the k-Schur functions $s_\mu^{(k)}$ prompting our study of ABC combinatorics in the context of Hall-Littlewood polynomials. To this end, we introduce a simple spin statistic in § 5 on ABC's and conjecture that the $K_{\lambda\mu}^{(k)}(0,t)$ can be described as the spin generating function of ABC's. To provide evidence for our conjecture, we prove that the [LS78] formula for Kostka-Foulkes polynomials in terms of charge is equivalent to

$$K_{\lambda\mu}(t) = \sum_A t^{spin(A)}\,,$$

over all ABC's of weight μ and inner shape λ.

2 k-Schur Functions

Dual k-Schur functions are defined as the weight generating functions of k-tableaux. To understand the definition of these tableaux, let us set some notation. We identify each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ with its Ferrers shape (having λ_i lattice squares in the i^{th} row, from the bottom to top). A column-strict tableau is a filling of a shape with positive integers that weakly decrease along rows and strictly increase up columns. A p-core is a partition that does not contain any cell with hook-length p. We use $\mathcal{C}^p$ to denote the set of all p-cores. The content of cell (i,j) is $j-i$, and its p-residue is $j-i \bmod p$. The 5 residues of the 5-core (6,4,3,1,1,1) are

$$\begin{array}{|c|c|c|c|c|c|}
\hline 0 \\
\cline{1-1} 1 \\
\cline{1-1} 2 \\
\cline{1-3} 3 & 4 & 0 \\
\cline{1-4} 4 & 0 & 1 & 2 \\
\hline 0 & 1 & 2 & 3 & 4 & 0 \\
\hline
\end{array}\,.$$

The p-degree of a p-core λ, $deg^p(\lambda)$, is the number of cells in λ whose hook-length is smaller than p. In the example above, the core has a 5-degree of 11.

Hereafter we work with a fixed integer $k > 0$ and all cores (resp. residues) are $k+1$-cores (resp. $k+1$-residues) and deg^{k+1} will simply be written as deg.

Definition 1 *Let $\nu \in \mathcal{C}^{k+1}$ and let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a composition of $deg(\nu)$. A k-tableau of shape ν and k-weight α is a column-strict filling of ν with integers $1, 2, \ldots, r$ such that the collection of cells filled with letter i are labeled by exactly α_i distinct $k+1$-residues for $1 \le i \le r$.*

The 6-tableaux of weight $(3,3,3,1)$ and shape $(8,2,2) \in \mathcal{C}^7$ are

$$
\begin{array}{|c|c|cccccc}
\hline
3_5 & 3_6 \\
\hline
2_6 & 2_0 \\
\hline
1_0 & 1_1 & \multicolumn{1}{|c|}{1_2} & \multicolumn{1}{c|}{2_3} & \multicolumn{1}{c|}{3_4} & \multicolumn{1}{c|}{3_5} & \multicolumn{1}{c|}{3_6} & \multicolumn{1}{c|}{4_0} \\
\hline
\end{array}
\qquad\qquad
\begin{array}{|c|c|cccccc}
\hline
3_5 & 4_6 \\
\hline
2_6 & 3_0 \\
\hline
1_0 & 1_1 & \multicolumn{1}{|c|}{1_2} & \multicolumn{1}{c|}{2_3} & \multicolumn{1}{c|}{2_4} & \multicolumn{1}{c|}{3_5} & \multicolumn{1}{c|}{3_6} & \multicolumn{1}{c|}{3_0} \\
\hline
\end{array}\ .
$$

Note that when k is larger than the biggest hook-length in ν, a k-tableau T of shape ν and weight α is a column-strict tableau of weight α since no two diagonals of T can have the same residue.

The dual k-Schur functions are the k-tableaux generating functions: for any $\lambda \in \mathcal{C}^{k+1}$,

$$\mathfrak{S}_\lambda^{(k)} = \sum_{\substack{T:k\text{-tableau} \\ \text{shape}(T)=\lambda}} x^{k\text{-}weight(T)} . \tag{3}$$

These arose in the context of the quantum cohomology of Grassmannians in [LM08], their definition was then generalized by [Lam06] to give a family of affine Stanley symmetric functions, and it was shown by [Lam08] that they represent Schubert cohomology classes of affine Grassmannians. The term affine Schur function is also used for dual k-Schur function.

2.1 Weak k-Pieri Rule

Our new formulation for $\mathfrak{S}_\lambda^{(k)}$ in (1) is derived along the same lines that led to the introduction of k-tableaux in [LM05]. This route starts with a family of symmetric functions related to $\mathfrak{S}_\lambda^{(k)}$ by duality. To be precise, the set of $\{\mathfrak{S}_\lambda^{(k)}\}_{\lambda \in \mathcal{C}^{k+1}}$ forms a basis for

$$\Lambda/\mathcal{I} \qquad \text{where} \quad \mathcal{I} = \langle m_\lambda \rangle_{\lambda_1 > k} . \tag{4}$$

This quotient is naturally paired with the subring $\Lambda^k = \mathbb{Z}[h_1, \ldots, h_k]$ of symmetric functions under the Hall-inner product,

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu} .$$

The basis dual to the set of $\{\mathfrak{S}_\lambda^{(k)}\}_{\lambda \in \mathcal{C}^{k+1}}$ turns out to be the k-Schur function basis $\{s_\lambda^{(k)}\}_{\lambda \in \mathcal{C}^{k+1}}$, functions conjectured to be the $t = 1$ case of the atoms that arose in the Macdonald polynomial study of [LLM03]. By the duality of $\{h_\lambda\}$ and $\{m_\lambda\}$, (3) implies that the close examination of the expansion

$$h_\mu = \sum_\lambda K_{\lambda\mu}^{(k)} \, s_\lambda^{(k)} \tag{5}$$

should reveal the k-tableaux. In fact, starting with $s_\emptyset^{(k)} = 1$, k-tableaux are precisely the objects that encode the combinatorial Pieri rule for k-Schur functions given in [LM07]. That is, for $\lambda \in \mathcal{C}^{k+1}$ and $0 < \ell \leq k$,

$$h_\ell s_\lambda^{(k)} = \sum_{\nu \in H_{\lambda,\ell}^{(k)}} s_\nu^{(k)}\,,$$

over a specified set of cores $H_{\lambda,\ell}^{(k)}$ described by certain saturated chains in the weak order on the affine Weyl group $\tilde{A}^{k+1}$. Recall that weak order on $\tilde{A}^{k+1}$ can be realized on cores by the covering relation

$$\beta \lessdot \nu \iff \nu = \beta + \text{all addable corners of one fixed residue.}$$

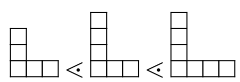
Definition 2 *For $0 < \ell \leq k$ and $k+1$-cores λ and ν, the skew shape ν/λ is a* weak ℓ-strip *if there is a weak saturated chain of cores*

$$\lambda = \gamma^0 \lessdot \gamma^1 \lessdot \cdots \lessdot \gamma^\ell = \nu\,,$$

where ν/λ is a horizontal strip and the rightmost cell of γ^i/γ^{i-1} is to the left of that in γ^{i+1}/γ^i for all $i = 1, \ldots, \ell-1$.

It should be observed that if there exists in the previous definition such a weak saturated chain then it is unique.

Example 3 *The skew shape $(4,1,1,1)/(3,1,1)$ of 4-cores is a weak 2-strip as there is the saturated chain*



The terms appearing in the summand of the k-Pieri rule are simply the set of weak ℓ-strips. That is,

$$H_{\lambda,\ell}^{(k)} = \{\nu : \nu/\lambda \text{ is a weak } \ell\text{-strip}\}.$$

2.2 Strong k-Pieri Rule

Recall that the strong (Bruhat) order on affine type A is realized on cores by the covering relation:

$$\rho \lessdot_B \gamma \iff \rho \subseteq \gamma \text{ and } deg(\gamma) = deg(\rho) + 1.$$

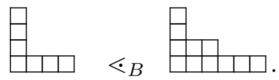
An important fact about strong covers is useful in our study. An *n-ribbon R* is a skew diagram λ/μ consisting of n rookwise connected cells such that there is no 2×2 shape contained in R. We refer to the southeasternmost cell of a ribbon as its *head*.

Lemma 4 *[LLMS10] Let $\rho \lessdot_B \gamma$ be cores. Then*

1. *Each connected component of ρ/γ is a ribbon.*
2. *The components are translates of each other and their heads have the same residue.*

Definition 5 *(γ, c) is a marked strong cover of a $k+1$-core ρ if $\rho \lessdot_B \gamma$ and c is the content of the head of a ribbon in γ/ρ.*

Example 6 *Consider the 4-cores $\rho = (4, 1, 1, 1), \gamma = (6, 3, 1, 1)$.*



$(\gamma, 5)$ is a marked strong cover of ρ, as is $(\gamma, 1)$.

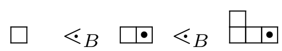
Definition 7 *For $0 < \ell \leq k$ and $k + 1$-cores λ and γ, a strong ℓ-strip from λ to γ is a saturated chain in the strong order*

$$\lambda = \gamma^0 \lessdot_B \gamma^1 \lessdot_B \ldots \lessdot_B \gamma^\ell = \gamma$$

together with a content vector $c = (c_1, c_2, \cdots, c_\ell)$ such that

1. *(γ^i, c_i) is a marked strong cover of γ^{i-1},*
2. *$c_{i-1} < c_i$ for $2 \leq i \leq \ell$.*

Example 8 *The following sequence of marked strong covers is a strong 2-strip from (1) to $(3, 1)$*



since the content vector $c = (1, 2)$ is increasing.

The strong ℓ-strips help to encode the combinatorial strong Pieri rule for the dual k-Schur functions. That is, for $\lambda \in \mathcal{C}^{k+1}$ and $0 < \ell \leq k$,

$$h_\ell \mathfrak{S}_\lambda^{(k)} = \sum_\gamma d_\gamma \mathfrak{S}_\gamma^{(k)},$$

where d_γ is the number of strong ℓ-strips from λ to γ.

3 Bottom Strong Strips

We have discovered that there is a natural association between weak strips and certain strong strips. In this association, a saturated chain in weak order of length ℓ is identified with a saturated chain in strong order of length $k - \ell$.

Our notion comes from a close examination of the core $(k + \lambda_1, \lambda)$ constructed from any $\lambda \in \mathcal{C}^{k+1}$.

Definition 9 *For $0 < \ell \leq k$ and $k + 1$-cores λ and ν, the skew shape $(k + \lambda_1, \lambda)/\nu$ is a* bottom strong ℓ-strip *if there is a saturated chain of cores*

$$\nu = \nu^0 \lessdot_B \nu^1 \lessdot_B \cdots \lessdot_B \nu^{k-\ell} = (k + \lambda_1, \lambda),$$

where

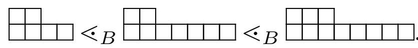
1. *$(k + \lambda_1, \lambda)/\nu$ is a horizontal strip.*
2. *The bottom rightmost cell of ν^i (which is always in the first row) is also a cell in ν^i/ν^{i-1}, for $1 \leq i \leq k - \ell$.*

It turns out that in the previous definition if such a chain exists then it is unique.

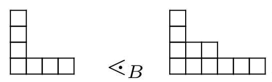
Remark 10 *Every bottom strong ℓ-strip $(k+\lambda_1,\lambda)/\nu$ is a strong ℓ-strip if we take the content of ν^i to be the content of its bottom rightmost cell in the corresponding saturated chain of cores*

$$\nu=\nu^0 \lessdot_B \nu^1 \lessdot_B \cdots \lessdot_B \nu^{k-\ell}=(k+\lambda_1,\lambda)\,.$$

Example 11 *The skew shape $(8,3)/(4,2)$ of 6-cores is a bottom strong 3-strip as there is the saturated chain*



Example 12 *The skew shape $(6,3,1,1)/(4,1,1,1)$ of 4-cores is a bottom strong 2-strip as there is the saturated chain*



This is an example of a bottom strong 2-strip which is also a weak 2-strip (compare to Example 3). In fact, there is a close connection between bottom ℓ-strips and weak ℓ-strips.

Theorem 13 *For $0<\ell\le k$ and $k+1$-cores λ and ν,*

$$(k+\lambda_1,\lambda)/\nu \text{ is a bottom strong } \ell\text{-strip} \iff \nu/\lambda \text{ is a weak } \ell\text{-strip}.$$

One immediate application of this result is that the k-Pieri rule can instead be given in terms of bottom strong ℓ-strips.

Corollary 14 *For any $0<\ell\le k$ and $\lambda\in\mathcal{C}^{k+1}$,*

$$h_\ell\, s_\lambda^{(k)} = \sum_{\substack{\nu\in\mathcal{C}^{k+1}\\ (k+\lambda_1,\lambda)/\nu:\ \text{bottom strong } \ell\text{-strip}}} s_\nu^{(k)}.$$

Example 15 *When $k=3$, the set of ν such that $(6,3,1,1)/\nu$ is a bottom strong 2-strip is*

$$\{(3,3,1,1),(5,2,1),(4,1,1,1)\}\,,$$

and thus

$$h_2 s^{(3)}_{(3,1,1)} = s^{(3)}_{(3,3,1,1)} + s^{(3)}_{(5,2,1)} + s^{(3)}_{(4,1,1,1)}.$$

4 Affine Bruhat Counter-Tableaux

The iteration of Corollary 14 gives rise to a new combinatorial structure whose enumeration describes the coefficients in

$$h_\lambda=\sum_\nu K^{(k)}_{\nu\lambda} s^{(k)}_\nu\,. \tag{6}$$

To describe this structure, first let us set more notations. Recall that a counter-tableau A is a filling of a skew shape with numbers that strictly decrease up columns and weakly decrease along the rows. Given a counter-tableau A, let $A^{(x)}$ denote the subtableau made up of the rows of A weakly higher than row x (where row 1 is this time the *highest* row). Let $A_{>i}$ denote the restriction of A to letters strictly larger than i where empty cells in a skew are considered to contain ∞.

Definition 16 *For a composition α whose entries are not larger than k, a skew counter-tableau A is an affine Bruhat counter-tableau (or ABC) of k-weight α if*

$$(k+\lambda_1^{(i-1)}, \lambda^{(i-1)})/\lambda^{(i)} \text{ is a bottom strong } \alpha_i\text{-strip for all } 1 \le i \le \ell(\alpha)\,,$$

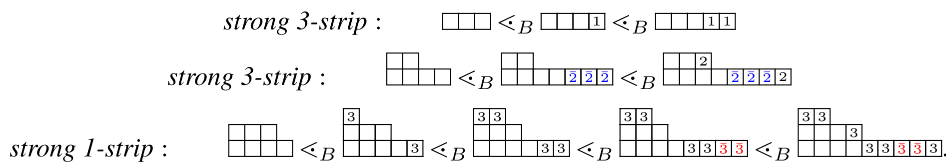
where $\lambda^{(x)} = \mathrm{shape}(A^{(x)}_{>x})$. The inner shape of A is $\lambda^{(\ell(\alpha))}$.

In the construction of a counter-tableau, we start with the empty shape $\lambda^{(0)}$, successively adding bottom strong strips and tiling $(k+\lambda_1^{(i-1)}, \lambda^{(i-1)})/\lambda^{(i)}$ with i-ribbons at each step.

Example 17 *With $k=5$, an ABC of 5-weight $(3,3,1)$ is*

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
3 & 3 & 2 & 1 & 1 \\
\hline
 & & & 3 & \bar{2} & \bar{2} & \bar{2} & 2 \\
\hline
 & & & & 3 & 3 & \bar{3} & \bar{3} & 3 \\
\hline
\end{array},$$

since



The black letters are ribbons of size one, red letters make a ribbon of size two and blue letters make a ribbon of size 3 (or for those with out color the ribbons are depicted with a bar).

For a composition α whose entries are not larger than k, there is a known bijection between the set of all standard k-tableaux of k-weight α and the set of all ABC's of k-weight α. Letting $K^{(k)}_{\nu\lambda}$ denote the number of affine Bruhat counter-tableaux of inner shape ν and k-weight λ, it is not hard to see that it is this set of ABC's being enumerated by the coefficients in (6) and by duality we can prove that the dual k-Schur functions are the ABC generating functions.

Theorem 18 *For any $\lambda \in \mathcal{C}^{k+1}$,*

$$\mathfrak{S}^{(k)}_\lambda = \sum_A x^A$$

where the sum is over all affine Bruhat counter-tableaux of inner shape λ, and $x^A = x^{k\text{-}weight(A)}$.

5 Hall-Littlewood Expansions

A motivation for Theorem 13 is in its application to the study of the integral form of Macdonald polynomials (e.g. [Mac95, Ber09]) and the k-Schur expansion coefficients $K^{(k)}_{\lambda\mu}(q,t)$ in (2). For $\mu \vdash n$, it is known that the Macdonald polynomial $H_\mu[X;q,t]$ reduces to h_{1^n} when $q=t=1$. Thus,

$$K^{(k)}_{\lambda\mu}(1,1) = K^{(k)}_{\lambda 1^n}$$

are the coefficients in (6) when $\mu = (1^n)$. We have seen that these count the number of ABC's of shape λ with k-weight (1^n) (called *standard* ABC's). Assuming the positivity of $K^{(k)}_{\lambda\mu}(q,t)$, this leads us to believe there exists a pair of statistics (non-negative integers) $a(A)$ and $b(A)$ associated to each ABC so that

$$K^{(k)}_{\lambda\mu}(q,t) = \sum_{\substack{A=\text{standard ABC}\\ \text{inner shape}(A)=\lambda}} t^{a_\mu(A)} q^{b_\mu(A)} .$$

As a special case, when $k = n$, these statistics would give a combinatorial formula for the q,t-Kostka polynomials arising in the Schur expansion of the Macdonald polynomials – a long-standing open problem in the field. Setting $q = 0$, the Macdonald polynomials reduce to Hall-Littlewood polynomials. In fact, when $k = n$, [LS78] give a beautiful combinatorial formula for the Kostka-Foulkes polynomials $K^{(n)}_{\lambda\mu}(0,t)$ by defining a statistic on column-strict tableaux. It is interesting to note that k-tableaux reduce to column-strict tableaux when $k = n$, whereas the ABC's are truly of a different spirit even when $k = n$. We thus start by introducing a statistic on ABC's when $k = n$.

Here, we will restrict our attention to *standard* ABC's – those of weight (1^n). It turns out that a standard ABC has only 1 or 2-ribbons. In fact, when $k = n$, there is a simple statistic that gives the Kostka-Foulkes polynomials.

Definition 19 *For each ABC A of n-weight 1^n, let*

$$spin(A) = \sum_i i\,\chi(\textit{row } i \textit{ has a 2-ribbon and it is not east of a 2-ribbon in row } i+1)\,,$$

where row 1 is the bottommost row of A, and where χ is the indicator function evaluating to zero when the statement is false and to 1 otherwise. Recall that a ribbon is east of another ribbon if the head of the former is to the east of the latter's.

Example 20 *Consider the ABC of 5-weight 1^5*

$$A = \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & 1 & 1 & 1 & 1 \\ \cline{1-6} 4 & 3 & 2 & 2 & 2 & 2 \\ \hline & 5 & 3 & 3 & \bar{3} & \bar{3} & 3 \\ \hline & & 4 & 4 & 4 & \bar{4} & \bar{4} \\ \hline & & 5 & \bar{5} & \bar{5} & 5 & 5 \\ \hline \end{array} .$$

2-ribbons are colored (or for those without color, 2-ribbons are depicted with a bar). Of these 3 2-ribbons, only the one in row 1 and row 3 contribute to the spin since the 2-ribbon in row 2 is east of that in row 3. The spin is thus 1+3=4.

Theorem 21 *For $\lambda \vdash n$,*

$$K_{\lambda,1^n}(0,t) = K^{(n)}_{\lambda 1^n}(0,t) = \sum_{\substack{A:\, ABC \text{ of } n\text{-weight } 1^n\\ \text{inner shape}(A)=\lambda}} t^{spin(A)} . \qquad (7)$$

Example 22 *The set of all ABC's of 3-weight 1^3 are*

$$\left\{ \begin{array}{|c|c|c|c|} \hline & 1 & 1 \\ \hline & 2 & \bar{2} & \bar{2} \\ \hline & \bar{3} & \bar{3} & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 3 & 1 & 1 \\ \hline & 2 & \bar{2} & \bar{2} \\ \hline & & 3 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|} \hline 2 & 1 & 1 \\ \cline{1-4} & 3 & 2 & 2 \\ \hline & & \bar{3} & \bar{3} & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|} \hline 2 & 1 & 1 \\ \cline{1-4} 3 & 3 & 2 & 2 \\ \hline & & & 3 & 3 \\ \hline \end{array} \right\}$$

The spins of these ABC's, respectively, are {3, 2, 1, 0}. We thus have that

$$H_{1^3}[X,t] = t^3\, s_{(1,1,1)} + (t^2+t)\, s_{(2,1)} + s_{(3)}\,.$$

The beauty of this formulation is that it gives a hint into finding a statistic to describe the more general Hall-Littlewood coefficients $K^{(k)}_{\lambda,1^n}(t)$. To do this, we need only the extra concept of an *offset* in an ABC. In an ABC, for $i > 1$, an i-ribbon R is an *offset* if there is a lower i-ribbon filled with the same letter as R whose head has the same residue as the head of R. The number of offsets in the ABC is denoted *off*(ABC).

Example 23 *Consider the ABC of* 3*-weight* 1^5.

$$A = \begin{array}{|c|c|c|c|c|c|}
\hline
3 & 1 & 1 & & & \\
\hline
 & 2 & \bar{2} & \bar{2} & & \\
\hline
 & 4 & 3 & 3 & & \\
\hline
 & \bar{5} & \bar{5} & 4 & 4 & \\
\hline
 & & & 5 & \bar{5} & \bar{5} \\
\hline
\end{array}$$

Here we see that A has only one offset $\boxed{\bar{5}\,|\,\bar{5}}$ *in the second row from the bottom. Observe that it is an offset since there is another 2-ribbon of the same residue in the first row with the same letter. This tells us that off*$(A) = 1$.

Definition 24 *Let A be an ABC of k-weight 1^n and define*

$$spin^k(A) = \mathit{off}(A) + \sum_i i\,\chi(\textit{row } i \textit{ has a 2-ribbon and it is not east of a 2-ribbon in row } i+1)\,.$$

When $k = n$, an ABC never has any offsets and thus the above definition reduces to Definition 19.

Conjecture 25 *For any $k+1$-core λ where $deg(\lambda) = n$,*

$$K^{(k)}_{\lambda 1^n}(0,t) = \sum_{\substack{A:\ ABC \text{ of weight } 1^n \\ \text{inner shape}(A)=\lambda}} t^{spin^k(A)}\,. \tag{8}$$

Example 26 *The set of all ABC's of 2-weight 1^4 are*

$$\left\{
\begin{array}{|c|c|c|c|}
\hline
 & 1 & & \\
\hline
 & \bar{2} & \bar{2} & \\
\hline
 & & 3 & \\
\hline
 & & \bar{4} & \bar{4} \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|c|}
\hline
4 & 1 & & \\
\hline
 & \bar{2} & \bar{2} & \\
\hline
 & 4 & 3 & \\
\hline
 & & & 4 \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|c|c|}
\hline
2 & 1 & & & \\
\hline
 & 3 & 2 & & \\
\hline
 & \bar{4} & \bar{4} & 3 & \\
\hline
 & & & \bar{4} & \bar{4} \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|c|c|}
\hline
2 & 1 & & & \\
\hline
4 & 3 & 2 & & \\
\hline
 & & 4 & 3 & \\
\hline
 & & & & 4 \\
\hline
\end{array}
\right\}$$

Hence, our conjecture checks out against the expansion of $H_{1^4}[X,t]$ in terms of 2-Schur functions:

$$H_{1^4}[X,t] = s^{(2)}_{(2,2,1,1)} + t^2 s^{(2)}_{(3,1,1)} + t^3 s^{(2)}_{(3,1,1)} + t^4 s^{(2)}_{(4,2)}\,.$$

6 Related and Future work

Central to the proof of the [LS78] formula for Hall-Littlewood polynomials is the *plactic monoid*, [LS81]. The plactic monoid is the quotient of the free monoid A^* on the totally ordered set A by *Knuth relations*, defined on words so $w \sim w'$ if and only if they are sent to the same tableau under RSK-insertion (see [Knu70]). This establishes a bijection $T \mapsto [w] \in (A^*/\sim)$ and the monoid can be viewed as the set of tableaux on letters in A with multiplication defined by insertion. The ring of symmetric functions Λ can be identified with a subring of the monoid algebra $\mathbb{Z}[q,t][A^*/\sim]$ by sending the Schur function s_λ to the sum of all tableaux with shape λ. Although the plactic monoid is noncommutative, the combinatorial nature of computation is more evident.

A refinement of the plactic monoid to a structure on k-tableaux that can be applied to combinatorial problems involving k-Schur functions is partially given in [LLMS] by a bijection compatible with the RSK-bijection. A deeper understanding of this highly intricate bijection is underway. Towards this effort, Lapointe and Pinto have discovered a statistic on k-tableaux that is compatible with the bijection [LP]. We conjecture that their statistic matches the spin of our ABC's and are working to put ABC's in a context that simplifies the bijection.

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