

# Witness structures and immediate snapshot complexes

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In this paper we introduce and study a new family of combinatorial simplicial complexes, which we call *immediate snapshot complexes*. Our construction and terminology are strongly motivated by theoretical distributed computing, as these complexes are combinatorial models of the standard protocol complexes associated to immediate snapshot read/write shared-memory communication model.

In order to define the immediate snapshot complexes we need a new combinatorial object, which we call a *witness structure*. These objects are indexing the simplices in the immediate snapshot complexes, while a special operation on them, called *ghosting*, describes the combinatorics of taking simplicial boundary. In general, we develop the theory of witness structures and use it to prove several combinatorial as well as topological properties of the immediate snapshot complexes.

**Keywords:** collapses, distributed computing, combinatorial algebraic topology, immediate snapshot, protocol complexes

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## 1 The motivation for the study of immediate snapshot complexes

This section contains the motivation for introducing immediate snapshot complexes, coming from theoretical distributed computing. The reader interested primarily in the actual family of combinatorial simplicial complexes, which we define in this paper, and the mathematics thereof, may skip this section altogether and return to it at any later point which is deemed to be convenient.

One of the core theoretical models, which is used to understand the shared-memory communication between a finite number of processes is the so-called *immediate snapshot read/write model*, abbreviated to IS model. In this model, a number of processes are set to communicate by means of a shared memory. Each process has an assigned register, and each process can perform two types of operations: *write* and *snapshot*. The write operation, by definition, writes the entire state of the process into its assigned register; the snapshot operation reads the entire memory in one atomic step. The order in which a process performs these operations is determined by the distributed protocol, whose execution is asynchronous, satisfying an additional condition. Namely, we assume that at each turn a group of processes becomes active. First this group simultaneously writes its values to the memory, then it simultaneously performs an atomic read of the entire memory. The word simultaneously here means that these processes first all write in some

arbitrary order, and this order does not matter, after which they do snapshots in some order, and this order does not matter either. In any case, each execution can be encoded by a sequence of groups of processes which become active at each turn.

A closely related computational model is called *iterated immediate snapshot model*, or IIS model for short. In that model each process uses fresh memory each time it becomes active again. A lot of research is dedicated to the IIS model, which has a major advantage in that the corresponding simplicial model is much simpler. Furthermore, the solvability in IS is equivalent to the solvability in IIS, so for these purposes the models are interchangeable. The questions of round-complexity for solvable tasks are however very different, as there is a substantial overhead factor when simulating IS using IIS. More details on these computational models, the associated protocol complexes and their equivalence with each other and with other models can be found in a recent book [HKR14], as well as in [AP12, AW04, Hav04, HS99].

In this paper we are motivated by the standard full-information distributed protocols for  $n + 1$  processes indexed  $0, \dots, n$ , where the  $k$ -th process runs for  $r_k$  rounds and then stops. When saying that the  $k$ -th process runs for  $r_k$  rounds, what we actually mean is that the  $k$ -th process becomes active exactly  $r_k$  times; each time it is included into some group of processes, which simultaneously become active and perform a write followed by a snapshot, as described above. Let the associated protocol complex be called  $P(r_0, \dots, r_n)$ . Our first contribution is to give a rigorous purely combinatorial definition of  $P(r_0, \dots, r_n)$ . To do this, we introduce new combinatorial objects, which we call *witness structures* and use them as a language to define and to analyze this family of simplicial complexes. The special case  $r_0 = \dots = r_n = 1$  corresponds to the so-called *standard chromatic subdivision* of a simplex, which was previously considered in the literature, see, e.g., [Ko12, HKR14, Ko15a]. The cases where for some  $i$ , we have  $r_i \geq 2$  are new. We note, that the iterated standard chromatic subdivision is related to the above mentioned IIS model, and has been a very useful simplicial model to analyze solvability and complexity of distributed tasks, see, e.g., recent work on the weak symmetry breaking, [Ko15b, Ko16a, Ko16b, Ko17].

Let us briefly sketch the plan of the article. In Section 2 we define witness structures, and a few operations on them, where *ghosting* is the most important one. That operation encodes the combinatorics of taking simplicial boundary in the immediate snapshot complexes. The reason we chose the term *ghosting* is because the operation removes the information of a set of processes. However, these processes do not disappear completely, and parts of their information appears in the total pattern anyway, reflected in the views of other active processes; so these processes remain like ghosts hidden in the background. In Section 3, we use the language of witness structures to define the immediate snapshot complexes, and prove the important Reconstruction Lemma. In Section 4 we look at the first properties of the immediate snapshot complexes. In particular, we prove that they are pure, we look at some enumerative combinatorics connected to these complexes, and we show how the standard chromatic subdivision can be seen as an instance of immediate snapshot complex construction. In Section 5 we describe a canonical decomposition of the immediate snapshot complexes, and prove that topologically, they are pseudomanifolds with boundary. Finally, in Section 6, we explain why our immediate snapshot complexes provide correct combinatorial model for the protocol complexes of standard protocols in the immediate snapshot read/write computation model.

This paper grew out of the first half of author's preprint [Ko14a]. The second half of [Ko14a] is focused on showing that the immediate snapshot complexes are homeomorphic to closed balls. Specifically, we show that there exists a homeomorphism from an appropriate standard simplex to the immediate snapshot complex, such that the image of every subsimplex of the standard simplex is a subcomplex of the immediate snapshot complex. This part has already appeared in print, see [Ko14b].

## 2 The language of witness structures

### 2.1 Some notations

In general, we will not define standard notions related to combinatorial simplicial complexes, and rather refer to standard monographs, see [Hat02, Ko07, M84]. However, we do want to fix some notations. We let  $\mathbb{Z}_+$  denote the set of nonnegative integers  $\{0, 1, 2, \dots\}$ . For a nonnegative integer  $n$ , we shall use  $[n]$  to denote the set  $\{0, \dots, n\}$ , with an additional convention that  $[-1] = \emptyset$ . For a finite subset  $S \subset \mathbb{Z}_+$ , such that  $|S| \geq 2$ , we let  $\text{smax } S$  denote the *second* largest element, i.e.,  $\text{smax } S := \max(S \setminus \{\max S\})$ . For a set  $S$  and an element  $a$ , we set

$$\chi(a, S) := \begin{cases} 1, & \text{if } a \in S; \\ 0, & \text{otherwise.} \end{cases}$$

Whenever  $(X_i)_{i=1}^t$  is a family of topological spaces, we set  $X_I := \bigcap_{i \in I} X_i$ . Also, when no confusion arises, we identify one-element sets with that element, and write  $p$  instead of  $\{p\}$ .

Furthermore, we need some poset terminology. Recall, that for a poset  $P$  and an element  $x \in P$ , one sets  $P_{<x} := \{y \in P \mid y < x\}$ . In general, a subset  $Q \subseteq P$  is called an *ideal* if for any  $x \in Q$  and  $y \in P$ , such that  $y \leq x$ , we have  $y \in Q$ . Furthermore, for any subset  $A \subseteq P$ , we let  $I(A, P)$  denote the set  $\{y \in P \mid \exists x \in A, \text{ such that } y \leq x\}$ ; clearly  $I(A, P) = \bigcup_{x \in A} P_{\leq x}$  and is always an ideal.

### 2.2 Witness prestructures and structures

#### 2.2.1 Definition and examples

We start by defining our main combinatorial gadget.

**Definition 2.1** A **witness prestructure** is a finite sequence of pairs of finite subsets of  $\mathbb{Z}_+$ , denoted  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ , with  $t \geq 0$ , satisfying the following conditions:

- (P1)  $W_i, G_i \subseteq W_0$ , for all  $i = 1, \dots, t$ ;
- (P2)  $G_i \cap G_j = \emptyset$ , for all  $0 \leq i < j \leq t$ ;
- (P3)  $G_i \cap W_j = \emptyset$ , for all  $0 \leq i \leq j \leq t$ ;
- (P4)  $(W_i, G_i) \neq (\emptyset, \emptyset)$ , for all  $1 \leq i \leq t$ .

A witness prestructure is called **stable** if in addition the following condition is satisfied:

- (S) if  $t \geq 1$ , then  $W_t \neq \emptyset$ .

A **witness structure** is a witness prestructure satisfying the following strengthening of conditions (S) and (P4):

- (W) the subsets  $W_1, \dots, W_t$  are all nonempty.

It is often useful to depict a witness prestructure in form of a table, see Figure 2.1. The intuition behind this notation is as follows. The first column is essentially for book-keeping purposes only, as the set  $W_0 \cup G_0$  tells us which processes are present at all. The set  $G_0$  contains the processes which are not

$W_0$	$W_1$	$W_2$	$\dots$	$W_t$
$G_0$	$G_1$	$G_2$	$\dots$	$G_t$

**Fig. 2.1:** Table presentation of a witness (pre)structure.

active and which have not been witnessed by anybody. For  $k \geq 1$  the  $k$ -th column tells us about the information we have at  $k$ -th turn. The set  $W_k$  describes the processes which are active at this turn, while the set  $G_k$  describes the processes which became the ghosts at this turn. The reader should note that the word *turn* does not mean the same here as in the execution of the protocol. It can happen that such a large number of processes becomes ghosts that there are several turns in the original protocol where the active processes do not witness anything. Rather, the columns of the table on Figure 2.1 should be thought as the *witnessing turns*.

The axioms in Definition 2.1 can now be given intuitive interpretation. The axiom (P1) says that  $W_0$  records processes which are active at some point. Note that we allow processes in

$$W_0 \setminus (W_1 \cup \dots \cup W_t \cup G_0 \cup \dots \cup G_t),$$

these correspond to processes which are only required to execute 0 times but still declare themselves active. Axiom (P2) says that each process can become a ghost at at most one point in time. Axiom (P3) says that once a process has become a ghost it cannot be listed as active at a later point in time. Axiom (W) (which subsumes (P4) and (S)) says that at each witnessing step we have some process which is active.

The 3 possibilities provided by Definition 2.1 are illustrated on Figure 2.2. On that figure, the witness structure  $\sigma_3$  encodes the following view of process number 1: the process has been active once, at which point it has witnessed processes 2 and 3, and it has also witnessed that process 2 has witnessed process 0 at some earlier stage of the execution. The process 1 did not witness process 4 and it also did not witness anyone who has witnessed 4; however the witness structure still records the fact that process 4 exists. The witness structure  $\sigma_3$  will index a vertex in the corresponding immediate snapshot simplicial complex.

[4]	1	$\emptyset$	2, 3	$\emptyset$	[3]	$\emptyset$	2	$\emptyset$	1	[3]	2	1
5	$\emptyset$	1	4	3	4	0	$\emptyset$	2	3	4	0	2, 3

**Fig. 2.2:** A witness prestructure  $\sigma_1$ , a stable witness prestructure  $\sigma_2$ , and a witness structure  $\sigma_3$ .

Note, that every witness prestructure with  $t = 0$  is a witness structure. On the other hand, if  $W_0 = \emptyset$ , then conditions (P1) and (P4) imply that  $t = 0$ . In this case, only the set  $G_0$  carries any information, and we call this witness structure *empty*.

Let us remark here that the main objects we would like to study are the witness structures. The reason being that they index simplices in the immediate snapshot complexes. In order to reflect the incidence structure of the immediate snapshot complexes we need to study the so-called *ghosting operation* on the witness structures, including the effect of the iteration of ghosting. It is technically much easier to do that in the more general context of prestructures, which is the main reason why we introduce these objects.

### 2.2.2 Data associated to witness prestructures

**Definition 2.2** Let  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  be an arbitrary witness prestructure. We define the following data associated to  $\sigma$ :

- the set  $W_0 \cup G_0$  is called the **support** of  $\sigma$  and is denoted by  $\text{supp } \sigma$ ;
- the **ghost set** of  $\sigma$  is the set  $G(\sigma) := G_0 \cup \dots \cup G_t$ ;
- the **active set** of  $\sigma$  is the complement of the ghost set

$$A(\sigma) := \text{supp } (\sigma) \setminus G(\sigma) = W_0 \setminus (G_1 \cup \dots \cup G_t);$$

- the **dimension** of  $\sigma$  is

$$\dim \sigma := |A(\sigma)| - 1 = |W_0| - |G_1| - \dots - |G_t| - 1.$$

For the examples on Figure 2.2 we get  $\text{supp } \sigma_1 = [5]$ ,  $\text{supp } \sigma_2 = \text{supp } \sigma_3 = [4]$ ,  $G(\sigma_1) = \{1, 3, 4, 5\}$ ,  $G(\sigma_2) = G(\sigma_3) = \{0, 2, 3, 4\}$ ,  $A(\sigma_1) = \{0, 2\}$ ,  $A(\sigma_2) = A(\sigma_3) = \{1\}$ ,  $\dim \sigma_1 = 1$ , and  $\dim \sigma_2 = \dim \sigma_3 = 0$ .

By definition, the dimension of a witness prestructure  $\sigma$  is between  $-1$  and  $|\text{supp } \sigma| - 1$ . Let us analyze the witness structures which have special dimension. To start with, if  $\dim(\sigma) = -1$ , then  $|A(\sigma)| = 0$ , i.e.,  $A(\sigma) = \emptyset$ , hence  $W_0 = G_1 \cup \dots \cup G_t$ . On the other hand, if  $t \geq 1$ , by (P3) we have

$$W_t \cap (G_1 \cup \dots \cup G_t) = \emptyset,$$

and by (P1) we have  $W_0 \supseteq W_t$ , implying that  $W_t = \emptyset$ . Hence, by (P1) and (P4), the only witness structures of dimension  $-1$  are empty, i.e., of the form  $\sigma = ((\emptyset, G_0))$ .

Note, that by (P3), the set  $W_k$  is in general disjoint from  $G_0 \cup \dots \cup G_k$ , for all  $0 \leq k \leq t$ , so we have

$$W_k \subseteq A(\sigma) \cup G_{k+1} \cup \dots \cup G_t.$$

Furthermore, it is easy to characterize all witness structures  $\sigma$  of dimension 0. In this case, we have  $|A(\sigma)| = 1$ . We let  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  and let  $p$  denote the unique element of  $A(\sigma)$ , then  $\sigma$  has dimension 0 if and only if

$$W_k \subseteq \{p\} \cup G_{k+1} \cup \dots \cup G_t, \text{ for all } k = 0, \dots, t.$$

In particular, we must of course have  $W_t = \{p\}$ , and we shall call  $p$  the *color* of the witness structure  $\sigma$ .

At the opposite extreme, a witness structure  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  has dimension  $|\text{supp } \sigma| - 1$  if and only if  $G_0 = \dots = G_t = \emptyset$ . In such a situation, we shall frequently use the short-hand notation  $\sigma = (W_0, W_1, \dots, W_t)$ .

### 2.2.3 Traces and alternative definition of witness structures

For brevity of some formulas, we set  $W_{-1} := W_0 \cup G_0 = \text{supp } \sigma$ .

**Definition 2.3** For a prestructure  $\sigma$  and an arbitrary  $p \in \text{supp } \sigma$ , we set

$$\text{Tr}(p, \sigma) := \{0 \leq i \leq t \mid p \in W_i \cup G_i\},$$

and call it the **trace** of  $p$ . Furthermore, for all  $p \in \text{supp } \sigma$ , we set

$$\text{last}(p, \sigma) := \max\{-1 \leq i \leq t \mid p \in W_i\}.$$

When the choice of  $\sigma$  is unambiguous, we shall simply write  $\text{Tr}(p)$  and  $\text{last}(p)$ . Note that  $0 \in \text{Tr}(p)$ , hence  $\text{Tr}(p) \neq \emptyset$ . Note furthermore, that if  $p \in A(\sigma)$ , then  $\text{Tr}(p) = \{0 \leq i \leq t \mid p \in W_i\}$ , while  $p \in G(\sigma)$  implies  $\text{Tr}(p) \setminus \max \text{Tr}(p) = \{0 \leq i \leq t \mid p \in W_i\}$  and  $p \in G_{\max \text{Tr}(p)}$ .

To get a better grasp on the witness structures, as well as operations in them, the following alternative definition, which uses traces, is often of value.

**Definition 2.4** *Let  $A$  and  $G$  be disjoint finite subsets of  $\mathbb{Z}_+$ , and let  $\{\text{Tr}(p)\}_{p \in A \cup G}$  be a family of finite subsets of  $\mathbb{Z}_+$ . Set  $t := \max \bigcup_{p \in A \cup G} \text{Tr}(p)$ . The triple  $(A, G, \{\text{Tr}(p)\}_{p \in A \cup G})$  is called a **witness prestructure** if the following conditions are satisfied:*

(T1)  $0 \in \text{Tr}(p)$ , for all  $p \in A \cup G$ ;

(T2)  $\bigcup_{p \in A \cup G} \text{Tr}(p) = [t]$ .

A witness prestructure is called **stable** if it satisfies an additional condition:

(TS) if  $A = \emptyset$ , then  $\text{Tr}(p) = \{0\}$ , for all  $p \in G$ , else  $\max \bigcup_{p \in A} \text{Tr}(p) = t$ .

A stable witness prestructure is called **witness structure** if the following strengthening of Condition (TS) is satisfied:

(TW) for all  $1 \leq k \leq t$  either there exists  $p \in A$  such that  $k \in \text{Tr}(p)$ , or there exists  $p \in G$  such that  $k \in \text{Tr}(p) \setminus \max \text{Tr}(p)$ .

We shall call the form of the presentation of the witness prestructure as a triple  $(A, G, \{\text{Tr}(p)\}_{p \in A \cup G})$  its *trace form*.

Intuitively, each trace set  $\text{Tr}(p)$  records the turns in which process  $p$  is involved, either actively or as a ghost. The axiom (T1) means that each process is recorded at step 0 which only serves the bookkeeping purpose of telling us which processes do we have. Axiom (T2) means that some process is recorded at each step, so there are no ‘‘skips’’. Axioms (TS) and (TW) are very close to our previous axioms (S) and (W). Axiom (TS) says that some non-ghost process is active at the last turn, whereas axiom (TW) says that this is the case at every turn.

**Proposition 2.5** *There is a natural bijection between the objects described by the Definitions 2.1 and 2.4.*

**Proof:** The translation between the two descriptions is as follows. First, assume  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  is a witness prestructure according to Definition 2.1. Set  $A := A(\sigma)$ ,  $G := G(\sigma)$ , and for each  $p \in A \cup G$ , set  $\text{Tr}(p)$  to be the trace of  $p$  as given by Definition 2.3. The condition (T1) is then satisfied since  $\text{supp } \sigma = A(\sigma) \cup G(\sigma) = W_0 \cup G_0$ . The condition (T2) is implied by (P4). If  $\sigma$  is a stable prestructure, then  $\max_{p \in A} \text{last}(p) = t$ , hence the condition (TS) is satisfied. Finally, if  $\sigma$  is a witness structure, then  $W_i \neq \emptyset$  for all  $i = 1, \dots, t$ . Assume  $p \in W_i$ . If  $p \in A(\sigma)$ , then  $i \in \text{Tr}(p)$ , else  $p \in G(\sigma)$ , and  $i \in \text{Tr}(p) \setminus \max \text{Tr}(p)$ . In any case, the condition (TW) is satisfied.

Reversely, assume we are given a triple  $(A, G, \{\text{Tr}(p)\}_{p \in A \cup G})$  as in Definition 2.4. We set

$$t := \max \bigcup_{p \in A \cup G} \text{Tr}(p),$$

and for all  $0 \leq k \leq t$ , we set

$$G_k := \{p \in G \mid k = \max \text{Tr}(p)\},$$

$$W_k := \{p \in A \cup G \mid k \in \text{Tr}(p)\} \setminus G_k.$$

In particular,  $G_0 = \{p \in G \mid \text{Tr}(p) = \{0\}\}$ , and by (T1) we have  $W_0 = (A \cup G) \setminus G_0$ . It follows that  $W_i, G_i \subseteq W_0$ , for all  $i = 1, \dots, t$ , and (P1) is satisfied. Furthermore, we have  $W_i \cap G_i = \emptyset$ , and  $G_i \cap G_j = \emptyset$ , for  $i \neq j$ , by construction, so (P2) is satisfied. Still by construction, we have  $G_i \cap W_j = \{p \in G \mid i = \max \text{Tr}(p), j \in \text{Tr}(p)\}$ , which is clearly empty when  $i < j$ .

The condition (TS) implies (S), since it implies that there exists an element  $p \in A$ , such that  $\max \text{Tr}(p) = t$ , so  $p \in W_t$ . Finally, (TW) implies (W), since both  $p \in A, k \in \text{Tr}(p)$  and  $p \in G, k \in \text{Tr}(p) \setminus \max \text{Tr}(p)$ , imply  $p \in W_k$ . We leave it to the reader to verify that the translations described above are inverses of each other.  $\square$

For example, for the prestructure  $\sigma_1$  on Figure 2.2, we get  $\text{Tr}(0) = \text{Tr}(5) = \{0\}$ ,  $\text{Tr}(1) = [2]$ ,  $\text{Tr}(2) = \text{Tr}(4) = \{0, 3\}$ , and  $\text{Tr}(3) = \{0, 3, 4\}$ .

## 2.3 Operations on witness prestructures

### 2.3.1 Stabilization of witness prestructures

In order to gain a more clear understanding of the composition of the ghosting operation with itself, we will break that operation into two parts: stabilization modulo a set and taking canonical form. We start by noting, that any witness prestructure can be made stable using the following operation.

**Definition 2.6** Let  $\sigma = (A, G, \{\text{Tr}(p)\}_{p \in A \cup G})$  be a witness prestructure. We set

$$q := \begin{cases} \max \bigcap_{p \in A} \text{Tr}(p), & \text{if } A \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

The **stabilization** of  $\sigma$  is the witness prestructure  $\text{st}(\sigma)$  whose trace form is  $(A, G, \{\text{Tr}(p) \cap [q]\}_{p \in A \cup G})$ .

The reader will notice that we use the trace form definition of the witness prestructure to define stabilization. The equivalent formulation for the table form of the witness prestructure is of course possible, but unfortunately is more involved. The stabilization is done by first finding the rightmost column with an active process, truncating at this column, and then for each ghost process which appeared in the truncated part we find the rightmost column in which it appears and move it in this column from the active state to the ghost state. An example is shown on Figure 2.3, where  $A = \{0, 4\}$ ,  $G = \{1, 2, 3\}$ , and  $q = 2$ .

[4]	1	0, 3, 4	2, 3	1	1	$\emptyset$	→	[4]	1	0, 3, 4	→	0, 1, 3, 4	$\emptyset$	0, 4
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	3	2	1	→	$\emptyset$	$\emptyset$	$\emptyset$	→	2	1	3

**Fig. 2.3:** Stabilizing a witness prestructure.

**Proposition 2.7** For an arbitrary witness prestructure  $\sigma$ , the witness prestructure  $\text{st}(\sigma)$  is well-defined and stable. These two prestructures have the same support, dimension, ghost and active sets. Furthermore, we have  $\sigma = \text{st}(\sigma)$  if and only if  $\sigma$  is stable.

**Proof:** Both (T1) and (T2) are immediate, since we are simply restricting traces to the set  $[q]$ , where  $q \geq 0$ . The stability condition (TS) follows from the choice of  $q$ .  $\square$

### 2.3.2 Canonical form of a stable witness prestructure

As the next step, any stable witness prestructure can be turned into a witness structure by means of the following operation.

**Definition 2.8** Assume  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  is an arbitrary stable witness prestructure. Set  $q := |\{1 \leq i \leq t \mid W_i \neq \emptyset\}|$ . Pick  $0 = i_0 < i_1 < \dots < i_q = t$ , such that

$$\{i_1, \dots, i_q\} = \{1 \leq i \leq t \mid W_i \neq \emptyset\}.$$

We define the witness structure  $C(\sigma) = ((W_0, G_0), (\widetilde{W}_1, \widetilde{G}_1), \dots, (\widetilde{W}_q, \widetilde{G}_q))$ , which is called the **canonical form** of  $\sigma$ , by setting

$$\widetilde{W}_k := W_{i_k}, \quad \widetilde{G}_k := G_{i_{k-1}+1} \cup \dots \cup G_{i_k}, \text{ for all } k = 1, \dots, q, \quad (2.1)$$

A graphical way of thinking about obtaining the canonical form, is that all the columns with the empty set in the upper box get “merged to the right”. Since we assume that the upper box of the last column is not empty, this is well-defined for the entire table.

The reason for the asymmetry when merging to the right is because the time of the turns increases from left to right, so when some witnesses disappear, their information will be recovered at later turns. The construction in Definition 2.8 is illustrated by Figure 2.4.

[3]	∅	2	∅	1	→	[3]	2	1
4	0	∅	2	3		4	0	2, 3

**Fig. 2.4:** A stable witness prestructure and its canonical form.

**Proposition 2.9** Assume  $\sigma$  is an arbitrary stable witness prestructure.

- (a) The canonical form of  $\sigma$  is a well-defined witness structure.
- (b) We have  $C(\sigma) = \sigma$  if and only if  $\sigma$  is itself a witness structure.
- (c) We have  $\text{supp}(C(\sigma)) = \text{supp}(\sigma)$ ,  $A(\sigma) = A(C(\sigma))$ ,  $G(\sigma) = G(C(\sigma))$ , and finally  $\dim(\sigma) = \dim(C(\sigma))$ .

**Proof:** Assume  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ ,  $q$  and  $i_1, \dots, i_q$  as in the Definition 2.8, and  $C(\sigma) = ((W_0, G_0), (\widetilde{W}_1, \widetilde{G}_1), \dots, (\widetilde{W}_q, \widetilde{G}_q))$ .

To prove (a) note first that all the sets involved are finite subsets of  $\mathbb{Z}_+$ . Conditions (P1) and (P2) for  $C(\sigma)$  follow immediately from the corresponding conditions on  $\sigma$ . To see (P3), pick some  $p \in \widetilde{G}_k$ . Then there exists a unique  $j$ , such that  $i_{k-1} < j \leq i_k$  and  $p \in G_j$ . Then  $p \notin W_j \cup \dots \cup W_t$ , but  $W_j \cup \dots \cup W_t = W_{i_k} \cup \dots \cup W_{i_q}$ , hence  $p \notin \widetilde{W}_k \cup \dots \cup \widetilde{W}_q$ . Finally, to see (W) note that  $W_{i_k} \neq \emptyset$  for all  $k = 1, \dots, q$ , hence  $\widetilde{W}_k \neq \emptyset$ .

To prove (b) note that if  $\sigma$  is a witness structure, then  $W_1, \dots, W_t \neq \emptyset$ , hence  $q = t$ ,  $i_k = k$ , for  $k = 1, \dots, t$ . It follows that  $\widetilde{W}_k = W_k$ ,  $\widetilde{G}_k = G_k$ , for all  $k = 1, \dots, t$ . Reversely, assume  $C(\sigma) = \sigma$ , then  $q = t$ , hence  $i_k = k$ , for all  $k = 1, \dots, t$ , implying  $W_1, \dots, W_t \neq \emptyset$ .



To prove (c) note that the first pair of sets in  $\sigma$  and in  $C(\sigma)$  is the same, hence  $\text{supp}(C(\sigma)) = \text{supp}(\sigma)$ . By (2.1) we have  $\widetilde{W}_1 \cup \dots \cup \widetilde{W}_q = W_1 \cup \dots \cup W_t$ , and  $\widetilde{G}_1 \cup \dots \cup \widetilde{G}_q = G_1 \cup \dots \cup G_t$ , hence  $A(\sigma) = A(C(\sigma))$ . The other two equalities follow.  $\square$

### 2.3.3 Stabilization of witness prestructures modulo a set of processes

The Definition 2.6 can be generalized as follows.

**Definition 2.10** *Let  $\sigma = (A, G, \{\text{Tr}(p)\}_{p \in A \cup G})$  be a witness prestructure, and  $S \subseteq A$ . If  $S \subset A$ , set  $q := \max \bigcup_{p \in A \setminus S} \text{Tr}(p)$ , else set  $q := 0$ . The **stabilization of  $\sigma$  modulo  $S$**  is the witness prestructure  $\text{st}_S(\sigma)$  whose trace form is  $(A \setminus S, G \cup S, \{\text{Tr}(p) \cap [q]\}_{p \in A \cup G})$ .*

Intuitively speaking, stabilization modulo  $S$  is just like the usual stabilization defined above, with the addition that the processes from  $S$  are not considered to be active, so the truncation happens at the rightmost column which contains an active process which does not belong to  $S$ .

The following three properties can be taken as a recursive alternative to Definition 2.10.

(1) If  $t = 0$ , then  $\text{st}_S(\sigma) = ((W_0 \setminus S, G_0 \cup S))$ .

(2) If  $t \geq 1$  and  $W_t \subseteq S$ , then

$$\text{st}_S(\sigma) = \text{st}_{S \cup G_t}(((W_0, G_0), \dots, (W_{t-1}, G_{t-1}))).$$

(3) If  $t \geq 1$  and  $W_t \not\subseteq S$ , then the trace form of  $\text{st}_S(\sigma)$  is  $(A(\sigma) \setminus S, G(\sigma) \cup S, \{\text{Tr}(p)\}_{p \in A \cup G})$ .

Assume now that  $\text{st}_S(\sigma) = ((\widetilde{W}_0, \widetilde{G}_0), \dots, (\widetilde{W}_q, \widetilde{G}_q))$ . By Definition 2.10 we have  $\widetilde{W}_i \cup \widetilde{G}_i = W_i \cup G_i$ , and  $W_i \supseteq \widetilde{W}_i$ , for all  $0 \leq i \leq q$ . Hence, for some sets  $J_0, \dots, J_q$  we have

$$\text{st}_S(\sigma) = ((W_0 \setminus J_0, G_0 \cup J_0), \dots, (W_q \setminus J_q, G_q \cup J_q)). \quad (2.2)$$

The sets  $J_i$  can be explicitly described by the following formula:

$$J_i := (S \cup G(\sigma)) \cap (W_i \setminus (W_{i+1} \cup \dots \cup W_q \cup G_{i+1} \cup \dots \cup G_q)).$$

Note that unlike the usual stabilization, the witness prestructure  $\text{st}_S(\sigma)$  might be different from  $\sigma$  even when  $\sigma$  is a witness structure.

**Proposition 2.11** *Assume as before that we are given a witness prestructure  $\sigma$ , and  $S \subset A(\sigma)$ . Then, the witness prestructure  $\text{st}_S(\sigma)$  is well-defined and stable. It satisfies the following properties:*

- (1)  $\text{supp}(\text{st}_S(\sigma)) = \text{supp} \sigma$ ;
- (2)  $G(\text{st}_S(\sigma)) = G(\sigma) \cup S$ ;
- (3)  $A(\text{st}_S(\sigma)) = A(\sigma) \setminus S$ ;
- (4)  $\dim \text{st}_S(\sigma) = \dim \sigma - |S|$ .

**Proof:** Clearly, the conditions (T1) and (T2) are still satisfied, so the witness prestructure  $\text{st}_S(\sigma)$  is well-defined. Using an argument verbatim to the proof of the Proposition 2.7, we conclude that it is stable due to the choice of  $q$ . The identities (2) and (3) are integral parts of Definition 2.10, and (1) and (4) are direct consequences.  $\square$

The following property of the stabilization will be very useful later on.

**Proposition 2.12** *Assume  $\sigma$  is a witness prestructure, and  $S, T \subseteq A(\sigma)$ , such that  $S \cap T = \emptyset$ . Then we have*

$$\text{st}_T(\text{st}_S(\sigma)) = \text{st}_{S \cup T}(\sigma). \quad (2.3)$$

**Proof:** Assume  $\sigma = (A, G, \{\text{Tr}(p)\}_{p \in A \cup G})$ , and set  $\sigma' := \text{st}_T(\text{st}_S(\sigma))$ ,  $\sigma'' := \text{st}_{S \cup T}(\sigma)$ . To show that  $\sigma' = \sigma''$  we compare their trace forms. To start with, by Definition 2.10 we have  $\text{supp } \sigma' = \text{supp } \sigma$  and  $\text{supp } \sigma'' = \text{supp } \sigma$ . Furthermore,  $A(\sigma'') = A \setminus (S \cup T)$ , and  $A(\sigma') = A(\text{st}_S(\sigma)) \setminus T = (A \setminus S) \setminus T$ , hence  $A(\sigma') = A(\sigma'')$  and  $G(\sigma') = G(\sigma'')$ .

Finally, both in  $\sigma'$  as well as in  $\sigma''$  the traces of elements from  $A \cup G$  are truncated at the index  $\max \bigcup_{p \in A \setminus (S \cup T)} \text{Tr}(p)$ .  $\square$

### 2.3.4 Ghosting operation on the witness structures

We are now ready to define the main operation on witness structures.

**Definition 2.13** *For an arbitrary witness structure  $\sigma$ , and an arbitrary  $S \subseteq A(\sigma)$ , we define  $\Gamma_S(\sigma) := C(\text{st}_S(\sigma))$ . We say that  $\Gamma_S(\sigma)$  is obtained from  $\sigma$  by ghosting  $S$ .*

The ghosting operation is illustrated on Figure 2.5. In the simplicial context, this figure shows how to find one of the end vertices of the edge indexed by the witness structure on the left.

$$\begin{array}{|c|c|c|c|c|} \hline [3] & 3 & 1, 2 & 3 & 3 \\ \hline \emptyset & \emptyset & \emptyset & 0 & 1 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline 1, 2, 3 & \emptyset & 2 \\ \hline 0 & 3 & 1 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \hline 1, 2, 3 & 2 \\ \hline 0 & 1, 3 \\ \hline \end{array}$$

**Fig. 2.5:** Ghosting a witness structure with respect to  $S = \{3\}$ ; here the first arrow represents stabilizing this structure with respect to  $S$ .

Clearly, we have  $\Gamma_\emptyset(\sigma) = \sigma$ . If  $S = \{p\}$ , we shall simply write  $\Gamma_p(\sigma)$ . In this case we are ghosting a single element, and though the situation is not quite straightforward, several special cases can be formulated in a simpler manner.

Let  $l := \text{last}(p)$ . If  $|W_l| \geq 2$ , then the situation is much simpler indeed. In this case  $J_i = \emptyset$ , for all  $i \neq l$ , while  $J_l = \{p\}$ . Accordingly, we get

$$\Gamma_p(\sigma) = ((W_0, G_0), \dots, (W_{l-1}, G_{l-1}), (W_l \setminus \{p\}, G_l \cup \{p\}), (W_{l+1}, G_{l+1}), \dots, (W_t, G_t)).$$

The situation is slightly more complex if  $|W_l| = 1$ , i.e.,  $W_l = \{p\}$ . Assume that  $l \leq t - 1$ . Then, we still have  $J_i = \emptyset$ , for all  $i \neq l$ , and  $J_l = \{p\}$ . The difference now is that

$$\text{st}_S(\sigma) = ((W_0, G_0), \dots, (W_{l-1}, G_{l-1}), (\emptyset, G_l \cup \{p\}), (W_{l+1}, G_{l+1}), \dots, (W_t, G_t))$$

is now only a stable witness prestructure, so in this case we get

$$\Gamma_p(\sigma) = ((W_0, G_0), \dots, (W_{l-1}, G_{l-1}), (W_{l+1}, G_l \cup \{p\} \cup G_{l+1}), \dots, (W_t, G_t)).$$

Once  $l = t$ , i.e.,  $W_t = \{p\}$ , we will need the full generality of the Definition 2.13.

The situation is similar if  $|S| \geq 2$ . For each element  $s \in S$  we set  $l(s) := \text{last}(s)$ . As long as each  $W_{l(s)}$  contains elements outside of  $S$ , all that happens is that each element  $s \in S$  gets moved from  $W_{l(s)}$  to  $G_{l(s)}$ . Once this is not true, a more complex construction is needed.

**Proposition 2.14** *Assume we are given a witness structure  $\sigma = (A, G, \{\text{Tr}(p)\}_{p \in A \cup G})$ , and an arbitrary  $S \subseteq A$ . The construction in Definition 2.13 is well-defined, and yields a witness structure  $\Gamma_S(\sigma)$ , satisfying the following properties:*

- (1)  $\text{supp}(\Gamma_S(\sigma)) = A \cup G$ ;
- (2)  $G(\Gamma_S(\sigma)) = G \cup S$ ;
- (3)  $A(\Gamma_S(\sigma)) = A \setminus S$ ;
- (4)  $\dim \Gamma_S(\sigma) = \dim \sigma - |S|$ .

**Proof:** All equalities follow from the Propositions 2.9 and 2.11.  $\square$

**Remark 2.15** *For future reference we make the following observation. Let  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  be a witness structure, and assume  $p, q \in \text{supp} \sigma$ ,  $p \neq q$ . Clearly, we have  $|\text{Tr}(q, \Gamma_p(\sigma))| \leq |\text{Tr}(q, \sigma)|$ . Furthermore, if  $W_t \neq \{p\}$  we get an equality  $|\text{Tr}(q, \Gamma_p(\sigma))| = |\text{Tr}(q, \sigma)|$ , for all  $q$ .*

**Lemma 2.16** *Assume  $\sigma$  is a stable prestructure, and  $S \subseteq A(\sigma)$ , then we have  $C(\text{st}_S(\sigma)) = C(\text{st}_S(C(\sigma)))$ , or expressed functorially  $C \circ \text{st}_S \circ C = C \circ \text{st}_S$ .*

**Proof:** Assume  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ . We first describe the witness structure  $C(\text{st}_S(C(\sigma)))$ .

By Definition 2.6, we have  $C(\sigma) = ((W_{i_0}, \tilde{G}_{i_0}), \dots, (W_{i_q}, \tilde{G}_{i_q}))$ , where  $\tilde{G}_{i_k} = \bigcup_{\alpha=i_{k-1}+1}^{i_k} G_\alpha$ , for all  $k = 0, \dots, q$ , and the indices  $q$  and  $0 = i_0 < i_1 < \dots < i_q = t$  are chosen appropriately.

Set  $\tilde{S} := S \cup G(\sigma)$ , set  $r := \max\{0 \leq k \leq q \mid W_{i_k} \not\subseteq \tilde{S}\}$ , and set  $J_k := \tilde{S} \cap (W_{i_k} \setminus \bigcup_{\alpha=k+1}^q (W_{i_\alpha} \cup \tilde{G}_{i_\alpha}))$ , for  $0 \leq k \leq r$ . Then

$$\text{st}_S(C(\sigma)) = ((W_{i_0} \setminus J_0, \tilde{G}_{i_0} \cup J_0), \dots, (W_{i_r} \setminus J_r, \tilde{G}_{i_r} \cup J_r)).$$

On the other hand, we have  $i_r = \max\{0 \leq j \leq t \mid W_j \not\subseteq \tilde{S}\}$  and  $J_k = \tilde{S} \cap (W_{i_k} \setminus \bigcup_{\alpha=i_k+1}^t (W_\alpha \cup G_\alpha))$ ,

for  $0 \leq k \leq r$ , since  $W_j = \emptyset$  whenever  $j \notin \{i_0, \dots, i_q\}$ , and  $\tilde{G}_{i_k} = \bigcup_{\alpha=i_{k-1}+1}^{i_k} G_\alpha$ , for all  $k = 0, \dots, q$ .

It follows, that we have  $\text{st}_S(\sigma) = ((W'_0, G'_0), (W'_1, G'_1), \dots, (W'_{i_r}, G'_{i_r}))$ , where

$$(W'_j, G'_j) = \begin{cases} (W_{i_k} \setminus J_k, G_{i_k} \cup J_k), & \text{if } j = i_k, \text{ for some } 0 \leq k \leq r, \\ (\emptyset, G_j), & \text{otherwise.} \end{cases} \quad (2.4)$$

Set  $d := |\{0 \leq k \leq r \mid W_{i_k} \setminus J_k \neq \emptyset\}|$ , then (2.4) implies that we also have

$$d = |\{0 \leq k \leq i_r \mid W'_k \neq \emptyset\}|.$$

This means that  $C(\text{st}_S(\sigma))$  and  $C(\text{st}_S(C(\sigma)))$  have the same length.

For the appropriate choice of  $0 = a(0) < a(1) < \dots < a(d) = r$  we have

$$\{a(0), \dots, a(d)\} = \{0 \leq k \leq r \mid W_{i_k} \setminus J_k \neq \emptyset\}.$$

Assume  $C(\text{st}_S(C(\sigma))) = ((V_0, H_0), \dots, (V_d, H_d))$ , then we have  $V_k = W_{i_{a(k)}} \setminus J_{a(k)}$ ,

$$H_k = \bigcup_{\alpha=a(k-1)+1}^{a(k)} (\tilde{G}_{i_\alpha} \cup J_\alpha) = \bigcup_{\alpha=a(k-1)+1}^{a(k)} \tilde{G}_{i_\alpha} \cup \bigcup_{\alpha=a(k-1)+1}^{a(k)} J_\alpha, \quad (2.5)$$

for  $0 \leq k \leq d$ .

Assume now that  $C(\text{st}_S(\sigma)) = ((V'_0, H'_0), \dots, (V'_d, H'_d))$ . Note that

$$\{i_{a(0)}, \dots, i_{a(d)}\} = \{0 \leq k \leq i_r \mid W'_k \neq \emptyset\},$$

hence, for  $0 \leq k \leq d$ , we get  $V'_k = W'_{i_{a(k)}} = W_{i_{a(k)}} \setminus J_{a(k)}$ , and

$$H'_k = \bigcup_{\alpha=i_{a(k-1)}+1}^{i_{a(k)}} G'_\alpha = \bigcup_{\alpha=i_{a(k-1)}+1}^{i_{a(k)}} G_\alpha \cup \bigcup_{\alpha=a(k-1)+1}^{a(k)} J_\alpha,$$

where the last equality is a consequence of (2.4). Combining the identity

$$\bigcup_{\alpha=a(k-1)+1}^{a(k)} \tilde{G}_{i_\alpha} = \bigcup_{\alpha=a(k-1)+1}^{a(k)} \bigcup_{\beta=i_{\alpha-1}+1}^{i_\alpha} G_\beta = \bigcup_{\beta=i_{a(k-1)}+1}^{i_{a(k)}} G_\beta$$

with (2.5), we see that  $H_k = H'_k$ , for all  $0 \leq k \leq d$ .  $\square$

The next theorem is the pinnacle of our efforts. It allows us to ghost processes in an arbitrary order. A direct proof of this fact would be possible, but technically involved and unilluminating. This is reason, why we choose to define stable prestructures, stabilization and canonical form. Having worked out the necessary properties of these, we can now give an extremely short and purely symbolic argument.

**Theorem 2.17** *Assume  $\sigma$  is a witness structure, and  $S, T \subseteq A(\sigma)$ , such that  $S \cap T = \emptyset$ . Then we have  $\Gamma_T(\Gamma_S(\sigma)) = \Gamma_{S \cup T}(\sigma)$ , expressed functorially we have  $\Gamma_T \circ \Gamma_S = \Gamma_{S \cup T}$ .*

**Proof:** We have

$$\Gamma_T \circ \Gamma_S = C \circ \text{st}_T \circ C \circ \text{st}_S = C \circ \text{st}_T \circ \text{st}_S = C \circ \text{st}_{S \cup T} = \Gamma_{S \cup T},$$

where the first and the fourth equalities follow from Definition 2.13, the second equality follows from Lemma 2.16, and the third equality follows from Proposition 2.12.  $\square$

### 3 Immediate snapshot complexes

#### 3.1 Round counters

Our main objects of study, the immediate snapshot complexes, are indexed by finite tuples of nonnegative integers.

**Definition 3.1** Given a function  $\bar{r} : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+ \cup \{\perp\}$ , we consider the set

$$\text{supp } \bar{r} := \{i \in \mathbb{Z}_+ \mid \bar{r}(i) \neq \perp\}.$$

This set is called the **support set** of  $\bar{r}$ .

A **round counter** is a function  $\bar{r} : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+ \cup \{\perp\}$  with a finite support set.

Obviously, a round counter can be thought of as an infinite sequence  $\bar{r} = (\bar{r}(0), \bar{r}(1), \dots)$ , where, for all  $i \in \mathbb{Z}_+$ , either  $\bar{r}(i)$  is a nonnegative integer, or  $\bar{r}(i) = \perp$ , such that only finitely many entries of  $\bar{r}$  are nonnegative integers. We shall frequently use a short-hand notation  $\bar{r} = (r_0, \dots, r_n)$  to denote the round counter given by

$$\bar{r}(i) = \begin{cases} r_i, & \text{for } 0 \leq i \leq n; \\ \perp, & \text{for } i > n. \end{cases}$$

Operationally, the round counter models the number of rounds taken by each process in the execution of the corresponding distributed protocol:  $r_k$  counts the number of times process  $k$  becomes active, where  $r_k = \perp$  means that the process  $k$  does not participate in the computation.

**Definition 3.2** Given a round counter  $\bar{r}$ , the number  $\sum_{i \in \text{supp } \bar{r}} \bar{r}(i)$  is called the **cardinality** of  $\bar{r}$ , and is denoted  $|\bar{r}|$ . The sets

$$\text{act } \bar{r} := \{i \in \text{supp } \bar{r} \mid \bar{r}(i) \geq 1\} \text{ and } \text{pass } \bar{r} := \{i \in \text{supp } \bar{r} \mid \bar{r}(i) = 0\}$$

are called the **active** and the **passive** sets of  $\bar{r}$ .

Assume now we are given a round counter  $\bar{r}$ , and let  $\varphi : \text{supp } \bar{r} \rightarrow [|\text{supp } \bar{r}| - 1]$  denote the unique order-preserving bijection. The round counter  $c(\bar{r})$  is defined by

$$c(\bar{r})(i) := \begin{cases} \bar{r}(\varphi^{-1}(i)), & \text{for } 0 \leq i \leq |\text{supp } \bar{r}| - 1; \\ \perp, & \text{for } i \geq |\text{supp } \bar{r}|. \end{cases}$$

We call  $c(\bar{r})$  the *canonical form* of  $\bar{r}$ . Note that  $\text{supp } c(\bar{r}) = [|\text{supp } \bar{r}| - 1]$ ,  $|\text{act } (c(\bar{r}))| = |\text{act } \bar{r}|$ , and  $|\text{pass } (c(\bar{r}))| = |\text{pass } \bar{r}|$ .

Let  $\mathcal{S}_{\mathbb{Z}_+}$  denote the group of bijections  $\pi : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ , such that  $\pi(i) \neq i$  for only finitely many  $i$ . This group acts on the set of all round counters, namely for  $\pi \in \mathcal{S}_{\mathbb{Z}_+}$ , and a round counter  $\bar{r}$  we set  $\pi(\bar{r})(i) := \bar{r}(\pi(i))$ .

If desired by the reader, it is sufficient to think about round counters as finite sequences. However, it is still necessary for some entries to be allowed to take the value  $\perp$ . This is because the corresponding immediate snapshot complexes have canonical decompositions, whose descriptions are easier if we allow this generality. For the same reason it would not be opportune to sort the entries of  $\bar{r}$ . On the other hand, permuting the entries of  $\bar{r}$  obviously induces simplicial isomorphism. Considering the infinite counters (with only a finite relevant part) and considering the action by the group  $\mathcal{S}_{\mathbb{Z}_+}$  provides a convenient formal framework to deal with all the cases at once.

### 3.2 Combinatorial definition

We now describe our main objects of study. Our definition proceeds as follows. We first describe the set of labels of the simplices, distinguishing the set of labels of the vertices. Then for each simplex we describe its set of vertices. Clearly, several conditions will have to be checked later. We have to see that any subset of a set of vertices of a simplex corresponds to some simplex, and crucially, that any two simplices with the same set of vertices must coincide.

**Definition 3.3** *Assume  $\bar{r}$  is a round counter. We define an abstract simplicial complex  $P(\bar{r})$ , called the **immediate snapshot complex** associated to the round counter  $\bar{r}$ , as follows. The vertices of  $P(\bar{r})$  are indexed by all witness structures  $\sigma = (\{p\}, G, \{\text{Tr}(q)\}_{q \in \{p\} \cup G})$ , satisfying these three conditions:*

1.  $\{p\} \cup G = \text{supp } \bar{r}$ ;
2.  $|\text{Tr}(p)| = \bar{r}(p) + 1$ ;
3.  $|\text{Tr}(q)| \leq \bar{r}(q) + 1$ , for all  $q \in G$ .

*We say that such a vertex has **color**  $p$ . In general, the simplices of  $P(\bar{r})$  are indexed by all witness structures  $\sigma = (A, G, \{\text{Tr}(q)\}_{q \in A \cup G})$ , satisfying:*

1.  $A \cup G = \text{supp } \bar{r}$ ;
2.  $|\text{Tr}(q)| = \bar{r}(q) + 1$ , for all  $q \in A$ ;
3.  $|\text{Tr}(q)| \leq \bar{r}(q) + 1$ , for all  $q \in G$ .

*The empty witness structure  $(\{\emptyset\}, \text{supp } \bar{r})$  indexes the empty simplex of  $P(\bar{r})$ . When convenient, we identify the simplices of  $P(\bar{r})$  with the witness structures which index them.*

*Let  $\sigma$  be a non-empty witness structure satisfying the conditions above. The set of vertices  $V(\sigma)$  of the simplex  $\sigma$  is given by  $\{\Gamma_{A(\sigma) \setminus \{p\}}(\sigma) \mid p \in A(\sigma)\}$ .*

Note, that for an arbitrary witness structure  $\sigma$  and an arbitrary  $p \in A(\sigma)$  we have the equality  $A(\Gamma_{A(\sigma) \setminus \{p\}}(\sigma)) = \{p\}$ . Hence we have

$$A(\sigma) = \{A(v) \mid v \in V(\sigma)\}, \quad (3.1)$$

in particular, the set of vertices of  $\sigma$  uniquely determines  $A(\sigma)$ .

Assume  $\bar{r}$  is a round counter, such that  $\bar{r}(i) = \perp$  for all  $i \geq n + 1$ . In line with our short-hand notation for the round counters, and in addition skipping a pair of brackets, we shall use an alternative notation  $P(\bar{r}(0), \dots, \bar{r}(n))$  instead of  $P(\bar{r})$ . An example of an immediate snapshot complex  $P(0, 1, 1)$  is shown on Figure 5.1, and a more sophisticated example  $P(2, 1, 1)$  is shown on Figure 3.1.

### 3.3 The Reconstruction Lemma.

From the point of view of distributed computing, the vertices of  $P(\bar{r})$  should be thought of as *local views* of specific processors. In this intuitive picture, the next Reconstruction Lemma 3.4 says that any set of local views corresponds to at most one global view.

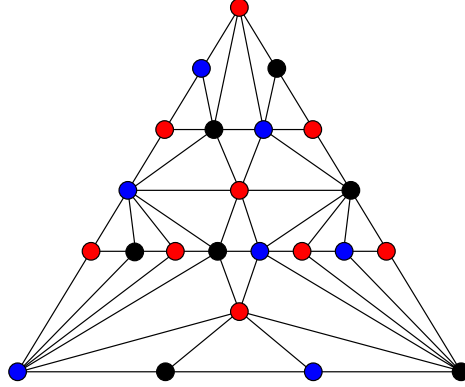


Fig. 3.1: The immediate snapshot complex  $P(2, 1, 1)$ .

**Lemma 3.4** (Reconstruction Lemma).

Assume  $\sigma$  and  $\tau$  are witness structures of dimension  $d$ , such that the corresponding  $d$ -simplices of  $P(\bar{r})$  have the same set of vertices, then we must have  $\sigma = \tau$ .

**Proof:** Assume the statement of lemma is not satisfied, and pick a pair of  $d$ -dimensional simplices  $\sigma \neq \tau$ , such that  $V(\sigma) = V(\tau)$ , and  $d$  is minimal possible. Obviously, we must have  $d \geq 1$ .

By (3.1), we have  $A(\sigma) = A(\tau)$ . Since  $\text{supp } \sigma = \text{supp } \tau = \text{supp } \bar{r}$ , we also have  $G(\sigma) = G(\tau)$ . For brevity, we set  $\Sigma := A(\sigma)$ , and for each  $p \in \Sigma$ , we set  $v_p := \Gamma_{\Sigma \setminus \{p\}}(\sigma) = \Gamma_{\Sigma \setminus \{p\}}(\tau)$ . For each  $p \in \Sigma$ , it follows from the definition of the ghosting operation that  $|\text{Tr}(p, \sigma)| = |\text{Tr}(p, v_p)|$ . This implies that the  $\Sigma$ -tuples  $(|\text{Tr}(p, \sigma)|)_{p \in \Sigma}$  and  $(|\text{Tr}(p, \tau)|)_{p \in \Sigma}$  are equal.

Now pick an arbitrary  $p \in \Sigma$ . For every  $q \in \Sigma$ , such that  $q \neq p$ , we have

$$\Gamma_{(\Sigma \setminus \{p\}) \setminus \{q\}}(\Gamma_p(\sigma)) = \Gamma_{\Sigma \setminus \{q\}}(\sigma) = v_q = \Gamma_{\Sigma \setminus \{q\}}(\tau) = \Gamma_{(\Sigma \setminus \{p\}) \setminus \{q\}}(\Gamma_p(\tau)),$$

hence  $V(\Gamma_p(\sigma)) = V(\Gamma_p(\tau))$ . By the minimality of  $\sigma$ , this implies  $\Gamma_p(\sigma) = \Gamma_p(\tau)$ .

Let  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ . Assume there exists  $0 \leq k \leq t$ , and  $p, q \in \Sigma$ ,  $p \neq q$ , such that  $\text{last}(p, \sigma) = \text{last}(q, \sigma)$ . Then, we have

$$\Gamma_p(\sigma) = \begin{array}{|c|c|c|c|c|c|c|} \hline W_0 & \dots & W_{k-1} & W_k \setminus \{p\} & W_{k+1} & \dots & W_t \\ \hline G_0 & \dots & G_{k-1} & G_k \cup \{p\} & G_{k+1} & \dots & G_t \\ \hline \end{array},$$

since  $\Gamma_p(\sigma) = \Gamma_p(\tau)$ , but  $\sigma \neq \tau$ , we get

$$\tau = \begin{array}{|c|c|c|c|c|c|c|} \hline W_0 & \dots & W_{k-1} & p & W_k \setminus \{p\} & W_{k+1} & \dots & W_t \\ \hline G_0 & \dots & G_{k-1} & A_p & B_p & G_{k+1} & \dots & G_t \\ \hline \end{array}, \quad (3.2)$$

for some  $A_p, B_p$  such that  $A_p \cup B_p = G_k$ . Repeating the same argument with  $q$  instead of  $p$  we get

$$\tau = \begin{array}{|c|c|c|c|c|c|c|} \hline W_0 & \dots & W_{k-1} & q & W_k \setminus \{q\} & W_{k+1} & \dots & W_t \\ \hline G_0 & \dots & G_{k-1} & A_q & B_q & G_{k+1} & \dots & G_t \\ \hline \end{array}, \quad (3.3)$$

for some  $A_q, B_q$  such that  $A_q \cup B_q = G_k$ . The equations (3.2) and (3.3) contradict each other. It is thus safe to assume that for every  $0 \leq k \leq t$ , there exists at most one  $p \in \Sigma$ , such that  $\text{last}(p, \sigma) = k$ .

Set  $F := \{p \in \Sigma \mid |\text{Tr}(p, \sigma)| = |\text{Tr}(p, \Gamma_p(\sigma))|\}$ . Note that

$$F = \{p \in \Sigma \mid |\text{Tr}(p, \tau)| = |\text{Tr}(p, \Gamma_p(\tau))|\}.$$

Using Remark 2.15, the previous observation  $|\text{Tr}(p, \sigma)| \leq |\text{Tr}(p, \Gamma_p(\sigma))|$  can be strengthened as follows: we know that  $F = \Sigma \setminus \{l\}$ , for some  $l \in \Sigma$ . Specifically,  $W_t = \{l\}$ , and the last pair of sets in  $\tau$  is also  $(\{l\}, H)$ , for some  $H \subseteq G(\tau)$ .

Pick  $p \in F$  such that  $\text{last}(p) = \max_{q \in F} \text{last}(q)$ . Assume

$$\Gamma_p(\sigma) = \begin{array}{|c|c|c|c|c|c|c|} \hline W_0 & \dots & W_{k-1} & W_k & W_{k+1} & \dots & W_t \\ \hline G_0 & \dots & G_{k-1} & G_k \cup \{p\} & G_{k+1} & \dots & G_t \\ \hline \end{array}.$$

We observe, that  $p$  was chosen so that  $(W_k \cup \dots \cup W_t) \cap F = \emptyset$ . We can easily describe the set  $\Lambda$  of all  $d$ -simplices  $\gamma$ , for which  $p \in \text{supp } \gamma$  and  $\Gamma_p(\gamma) = \Gamma_p(\sigma)$ . Set

$$\gamma^p := \begin{array}{|c|c|c|c|c|c|c|} \hline W_0 & \dots & W_{k-1} & W_k \cup \{p\} & W_{k+1} & \dots & W_t \\ \hline G_0 & \dots & G_{k-1} & G_k & G_{k+1} & \dots & G_t \\ \hline \end{array},$$

and

$$\gamma_{A,B} := \begin{array}{|c|c|c|c|c|c|c|} \hline W_0 & \dots & W_{k-1} & p & W_k & W_{k+1} & \dots & W_t \\ \hline G_0 & \dots & G_{k-1} & A & B & G_{k+1} & \dots & G_t \\ \hline \end{array},$$

where  $A \cup B = G_k$ . Then  $\Lambda = \{\gamma_{A,B} \mid A \cup B = G_k\} \cup \{\gamma^p\}$ . Clearly,  $\sigma, \tau \in \Lambda$ . We shall show that  $\Gamma_l(\sigma) \neq \Gamma_l(\tau)$ .

Assume  $A \cup B = G_k$ , and pick  $\alpha \in W_k$ . Then

$$|\text{Tr}(\alpha, \Gamma_l(\gamma^p))| = \sum_{i=0}^{k-1} \chi(\alpha, R_i) + 1 \neq \sum_{i=0}^{k-1} \chi(\alpha, R_i) = |\text{Tr}(\alpha, \Gamma_l(\gamma_{A,B}))|,$$

hence  $\Gamma_l(\gamma^p) \neq \Gamma_l(\gamma_{A,B})$ .

Assume now we have further sets  $A'$  and  $B'$ , such that  $A' \cup B' = G_k$ ,  $A \neq A'$ . Without loss of generality, we can assume that  $A \not\subseteq A'$ . Pick now  $\alpha \in A \setminus A'$ . Then

$$|\text{Tr}(\alpha, \Gamma_l(\gamma_{A,B}))| = \sum_{i=0}^{k-1} \chi(\alpha, R_i) + 1 \neq \sum_{i=0}^{k-1} \chi(\alpha, R_i) = |\text{Tr}(\alpha, \Gamma_l(\gamma_{A',B'}))|,$$

hence  $\Gamma_l(\gamma_{A,B}) \neq \Gamma_l(\gamma_{A',B'})$ .

We have thus proved that  $\Gamma_l(\sigma) \neq \Gamma_l(\tau)$ , contradicting the choice of  $\sigma$  and  $\tau$ .  $\square$

### 3.4 Immediate snapshot complexes are well-defined

We are now in a position to check that the Definition 3.3 yields a well-defined simplicial complex, and to show that the ghosting operation provides the right combinatorial language to describe boundaries in  $P(\bar{r})$ .



**Proposition 3.5** *Assume  $\bar{r}$  is the round counter.*

- (1) *The associated immediate snapshot complex  $P(\bar{r})$  is a well-defined simplicial complex. In this complex the dimension of the simplex indexed by  $\sigma$  is equal to  $\dim \sigma$ ,*
- (2) *Assume  $\sigma$  and  $\tau$  are simplices of  $P(\bar{r})$ . Then  $\tau \subseteq \sigma$  if and only if there exists  $S \subseteq A(\sigma)$ , such that  $\tau = \Gamma_S(\sigma)$ .*

**Proof:** Assume that the witness structure  $\sigma$  indexes a simplex of  $P(\bar{r})$ . Set  $d := \dim \sigma$ , implying that  $A(\sigma) = \{p_0, \dots, p_d\}$  for  $p_0 < \dots < p_d$ ,  $p_i \in \mathbb{Z}_+$ . For  $0 \leq i \leq d$ , we set  $v_i := \Gamma_{A(\sigma) \setminus \{p_i\}}(\sigma)$ . We see that the  $d$ -dimensional simplex  $\sigma$  has  $d + 1$  vertices, which are all distinct, since  $A(v_i) = p_i$ , for  $0 \leq i \leq d$ . Furthermore, it follows from the Reconstruction Lemma 3.4 that any two simplices with the same set of vertices are equal.

Assume now that  $\tau = \Gamma_S(\sigma)$ , for some  $S \subseteq A(\sigma)$ . By Proposition 2.14 we have  $A(\tau) = A(\sigma) \setminus S$ . It follows from Theorem 2.17 that for every  $p \in A(\tau)$  we have

$$\Gamma_{A(\tau) \setminus \{p\}}(\tau) = \Gamma_{A(\tau) \setminus \{p\}}(\Gamma_S(\sigma)) = \Gamma_{A(\tau) \cup S \setminus \{p\}}(\sigma) = \Gamma_{A(\sigma) \setminus \{p\}}(\sigma), \quad (3.4)$$

hence the set of vertices of  $\tau$  is a subset of the set of vertices of  $\sigma$ .

On the other hand, pick an arbitrary  $W \subseteq V(\sigma)$ , for some witness structure  $\sigma$ . By definition, there exists  $T \subseteq A(\sigma)$ , such that  $W = \{\Gamma_{A(\sigma) \setminus \{p\}}(\sigma) \mid p \in T\}$ . Set  $\tau := \Gamma_{A(\sigma) \setminus T}(\sigma)$ . The same computation as in (3.4), shows that  $V(\tau) = W$ ; that is, for any subset of the set of the vertices of  $\sigma$ , there exists a simplex, which has this subset as its set of vertices.

Finally, assume  $V(\tau) \subseteq V(\sigma)$ , for some simplices  $\sigma$  and  $\tau$ . Again, the same computation as in (3.4), shows that  $V(\Gamma_{\text{supp } \sigma \setminus \text{supp } \tau}(\sigma)) = \text{supp } \tau$ , i.e.,  $\tau$  and  $\Gamma_{\text{supp } \sigma \setminus \text{supp } \tau}(\sigma)$  have the same set of vertices. It follows from the Reconstruction Lemma 3.4 that  $\tau = \Gamma_{\text{supp } \sigma \setminus \text{supp } \tau}(\sigma)$ , and so both statements (1) and (2) are proved.  $\square$

## 4 Some observations on immediate snapshot complexes

### 4.1 Elementary properties and examples

We start by listing a few simple but useful properties of the immediate snapshot complexes  $P(\bar{r})$ .

**Proposition 4.1** *For an arbitrary point counter  $\bar{r}$ , we have*

$$P(\bar{r}) \simeq P(c(\bar{r})), \quad (4.1)$$

where  $\simeq$  denotes an isomorphism of simplicial complexes.

**Proof:** Consider the map

$$\Phi : ((W_0, G_0), \dots, (W_t, G_t)) \mapsto ((\varphi(W_0), \varphi(G_0)), \dots, (\varphi(W_t), \varphi(G_t))),$$

where  $\varphi$  is the unique order-preserving bijection  $\varphi : \text{supp } \bar{r} \rightarrow [|\text{supp } \bar{r}| - 1]$ . This gives a bijection between simplices of  $P(\bar{r})$  and simplices of  $P(c(\bar{r}))$ . Since  $\varphi$  is just a renaming bijection, we conclude that  $\Phi$  is a simplicial isomorphism.  $\square$

In particular, if round counters  $\bar{r}$  and  $\bar{q}$  have the same canonical form, then the corresponding immediate snapshot complexes are isomorphic. In other words, the  $\perp$  entries do not matter for the simplicial structure. This can be generalized as follows.

**Proposition 4.2** *For any round counter  $\bar{r}$ , and any permutation  $\pi \in \mathcal{S}_{\mathbb{Z}_+}$ , the simplicial complex  $P(\pi(\bar{r}))$  is isomorphic to the simplicial complex  $P(\bar{r})$ .*

**Proof:** Consider the map

$$\Phi : ((W_0, G_0), \dots, (W_t, G_t)) \mapsto ((\pi(W_0), \pi(G_0)), \dots, (\pi(W_t), \pi(G_t))).$$

This map is a simplicial isomorphism for the same reasons as in the proof of Proposition 4.1.  $\square$

Let us now look at special round counters. If  $\bar{r} = (r)$ , then the simplicial complex  $P(\bar{r})$  is just a point indexed by the witness structure  $\underbrace{((0, \emptyset), \dots, (0, \emptyset))}_{r+1}$ . Recall, that the empty simplex of  $P(r)$  is indexed by the witness structure  $((\emptyset, 0))$ .

**Proposition 4.3** *The immediate snapshot complex  $P(\underbrace{0, \dots, 0}_{n+1})$  is isomorphic as a simplicial complex to the  $n$ -simplex  $\Delta^n$ . More generally, if  $\bar{r}$  is a round counter such that  $r(i) \in \{\perp, 0\}$ , for all  $i \in \mathbb{Z}_+$ , the simplicial complex  $P(\bar{r})$  is isomorphic with  $\Delta^{\text{supp } \bar{r}}$ .*

**Proof:** The simplices of  $P(\bar{r})$  are indexed by all  $((A, B))$  such that  $A \cap B = \emptyset$  and  $A \cup B = [n]$ . The simplicial isomorphism between  $P(\bar{r})$  and  $\Delta^n$  is given by  $((A, B)) \mapsto A$ . The second statement follows from Proposition 4.1.  $\square$

**Proposition 4.4** *Assume  $\bar{r} = (r(0), \dots, r(n))$  and  $\bar{r}(n) = 0$ . Let  $\bar{q}$  denote the truncated round counter  $(r(0), \dots, r(n-1))$ . Consider a cone over  $P(\bar{q})$ , which we denote  $P(\bar{q}) * \{a\}$ , where  $a$  is the apex of the cone. Then we have*

$$P(\bar{r}) \simeq P(\bar{q}) * \{a\}. \quad (4.2)$$

**Proof:** Let  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  be a simplex of  $P(\bar{r})$  and consider the map

$$\Phi : \sigma \mapsto \begin{cases} ((W_0 \setminus \{n\}, G_0), \dots, (W_t, G_t)) * \{a\}, & \text{if } n \in W_0; \\ ((W_0, G_0 \setminus \{n\}), \dots, (W_t, G_t)), & \text{if } n \in G_0. \end{cases}$$

Since  $W_0 \cup G_0 = [n]$ , and  $W_0 \cap G_0 = \emptyset$ , we either have  $n \in W_0$  or  $n \in G_0$ . If  $n \in G_0$ , then  $n \notin W_0 \cup \dots \cup W_t \cup G_1 \cup \dots \cup G_t$ . If  $n \in W_0$ , then  $n \notin G_0$ . Furthermore, since  $\bar{r}(n) = 0$ , we have  $|\text{Tr}(n, \sigma)| \leq 1$ , hence  $\text{Tr}(n, \sigma) = \{0\}$ , and  $n \notin W_1 \cup \dots \cup W_t \cup G_0 \cup \dots \cup G_t$ . In any case,  $\Phi$  is well-defined. Its inverse is also clear, so it is a bijection between simplices of  $P(\bar{r})$  and  $P(\bar{q}) * \{a\}$ .

Under this bijection, the vertex  $((n, [n-1]))$  of  $P(\bar{r})$  corresponds to the apex  $a$ . The map  $\Phi$  is simplicial, since ghosting other elements than  $n$  will be for both complexes; while ghosting the element  $n$  will simply move it from  $W_0$  to  $G_0$  in a simplex from  $P(\bar{r})$ , which corresponds to the deletion of the apex  $a$  in a simplex from  $P(\bar{q}) * \{a\}$ .  $\square$

Clearly, the applications of Proposition 4.4 can be iterated, until each 0 entry in  $\bar{r}$  is replaced with a cone construction.

The Propositions 4.1, 4.2, 4.3, and 4.4, can intuitively be summarized as telling us that if we are interested in understanding the simplicial structure of the complex  $P(\bar{r})$ , we may ignore the entries  $\perp$  and 0, and permute the remaining entries as we see fit.

## 4.2 The purity of the immediate snapshot complexes

Assume  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  is a witness structure which indexes a simplex of  $P(\bar{r})$ . Clearly, we have  $|A(\sigma)| \leq |\text{supp } \bar{r}|$ , hence  $\dim \sigma \leq |\text{supp } \bar{r}| - 1$ . It turns out that every simplex is contained in a simplex of dimension  $|\text{supp } \bar{r}| - 1$ , which is the same as to say that immediate snapshot complexes are always pure (that is all maximal simplices have the same dimension).

**Proposition 4.5** *The simplicial complex  $P(\bar{r})$  is pure of dimension  $|\text{supp } \bar{r}| - 1$ .*

**Proof:** Assume  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  is a witness structure which indexes a simplex of  $P(\bar{r})$ . For each  $p \in G(\sigma)$  we set  $m(p) := r(p) + 1 - |\text{Tr}(p, \sigma)|$ . By construction, we have  $m(p) \geq 0$ . Set furthermore  $q := \max_{p \in G(\sigma)} m(p)$ ,

$$V_i := \{p \in G(\sigma) \mid m(p) \geq i\}, \text{ for } i = 1, \dots, q,$$

and

$$\tilde{\sigma} := (W_0 \cup G_0, W_1 \cup G_1, \dots, W_t \cup G_t, V_1, \dots, V_q).$$

We see that  $\tilde{\sigma}$  is a witness structure: the condition (P1) says that  $V_i \subseteq W_0 \cup G_0$ , which is clear, the conditions (P2) and (P3) are immediate, and condition (W) says that  $V_i \neq \emptyset$ , which is also clear. Furthermore, we have  $\text{supp } \tilde{\sigma} = \text{supp } \sigma$ ,  $G(\tilde{\sigma}) = \emptyset$ , and  $A(\tilde{\sigma}) = \text{supp } \sigma = A(\sigma) \cup G(\sigma)$ . For all  $\sigma \in A(\sigma)$  we have

$$|\text{Tr}(p, \tilde{\sigma})| = |\text{Tr}(p, \sigma)| = r(p) + 1,$$

while for all  $\sigma \in G(\sigma)$  we have

$$|\text{Tr}(p, \tilde{\sigma})| = |\text{Tr}(p, \sigma)| + m(p) = r(p) + 1.$$

We conclude that  $\tilde{\sigma}$  indexes a simplex of  $P(\bar{r})$ . Clearly,  $\dim \tilde{\sigma} = |\text{supp } \sigma| - 1$ . Finally, we have  $\Gamma(\tilde{\sigma}, G(\sigma)) = \sigma$ , so, by Proposition 3.5(2),  $\tilde{\sigma} \subseteq \sigma$  and hence  $P(\bar{r})$  is pure of dimension  $|\text{supp } \bar{r}| - 1$ .  $\square$

## 4.3 Immediate snapshot complexes of dimension 1

It follows from the above, that  $\dim P(\bar{r}) = 0$  if and only if  $|\text{supp } \bar{r}| = 1$ , in which case the simplicial complex  $P(\bar{r})$  is a point. Assume now  $\dim P(\bar{r}) = 1$ . In this case, we have  $|\text{supp } \bar{r}| = 2$ . By (4.1), up to the simplicial isomorphism, we can assume that  $\bar{r} = (m, n)$ ,  $m, n \geq 0$ .

For brevity of notations, when talking about edges of  $P(m, n)$ , we shall skip  $W_0 = [1]$ , and index the edges by the tuples  $(W_1, \dots, W_t)$  of subsets  $W_i \subseteq [1]$ , such that  $\sum_{i=1}^t \chi(0, W_i) = m$  and  $\sum_{i=1}^t \chi(1, W_i) = n$ . We shall make no abbreviations when indexing the vertices.

**Proposition 4.6** *For any integers  $m, n \geq 0$ , the simplicial complex  $P(m, n)$  is a subdivided interval, whose endpoints are indexed by*

$$v_{m,n}^0 := ((0, 1), \underbrace{(0, \emptyset), \dots, (0, \emptyset)}_m), \text{ and } v_{m,n}^1 := ((1, 0), \underbrace{(1, \emptyset), \dots, (1, \emptyset)}_n).$$

**Proof:** To start with, we know that the simplicial complex  $P(m, n)$  is a pure 1-dimensional complex, and that without loss of generality, we can assume  $m \geq n$ . The simplicial complex  $P(m, 0)$  is just a 1-simplex, indexed by  $\underbrace{(0, \dots, 0)}_m$ , and the claim of proposition is obviously true in this case. Our proof now makes use of induction on  $m + n$ .

When proving the induction step, we are free to confine ourselves to the case  $m \geq n \geq 1$ . For the sets  $S = \{0\}$ ,  $\{1\}$ , and  $\{0, 1\}$ , we let  $A_S$  denote the pure 1-dimensional subcomplex of  $P(m, n)$ , obtained by taking the union of all 1-simplices of the form  $(S, S_1, \dots, S_t)$ . For brevity, we shall simply write  $A_0$ ,  $A_1$ , and  $A_{01}$ . Obviously, each 1-simplex of  $P(m, n)$  belongs precisely to one of these three sets, so we have  $A_0 \cup A_1 \cup A_{01} = P(m, n)$ , and we shall now see how the three subcomplexes fit together.

It is easy to see, either directly, or as a special case of Proposition 5.4, that we have simplicial isomorphisms  $A_0 \cong P(m-1, n)$ ,  $A_1 \cong P(m, n-1)$ , and  $A_{01} \cong P(m-1, n-1)$ . Consider now two special vertices of  $P(m, n)$

$$w_0 = (([1], \emptyset), (0, 1), \underbrace{(0, \emptyset), \dots, (0, \emptyset)}_{m-1}), \text{ and } w_1 = (([1], \emptyset), (1, 0), \underbrace{(1, \emptyset), \dots, (1, \emptyset)}_{n-1}).$$

By the induction assumption, each one of the subcomplexes  $A_0$ ,  $A_1$ , and  $A_{01}$ , is a subdivided interval, and by the same assumption, combined with the simplicial isomorphism from Proposition 5.4, we know what the endpoints are. Namely,  $A_0$  has endpoints  $v_{m,n}^0$  and  $w_1$ ,  $A_1$  has endpoints  $v_{m,n}^1$  and  $w_0$ , and  $A_{01}$  has endpoints  $w_0$  and  $w_1$ . Obviously, that means that these three subcomplexes piece together to form a new subdivided interval with endpoints  $v_{m,n}^0$  and  $v_{m,n}^1$ .  $\square$

Let  $f(m, n)$  denote the number of 1-simplices in  $P(m, n)$ . By Proposition 4.6 this number completely describes the complex  $P(m, n)$ . We do not have a closed formula for these numbers, however, we can explicitly describe the corresponding generating function.

**Proposition 4.7** *The numbers  $f(m, n)$  satisfy the recursive relation*

$$f(m, n) = f(m, n-1) + f(m-1, n) + f(m-1, n-1), \quad \forall m, n \geq 1, \quad (4.3)$$

with the boundary conditions  $f(m, 0) = f(0, m) = 1$ . The corresponding generating function

$$F(x, y) = \sum_{m,n=0}^{\infty} f(m, n)x^m y^n$$

is given by the following explicit formula:

$$F(x, y) = \frac{1}{1 - x - y - xy}. \quad (4.4)$$

**Proof:** The fact that  $f(m, 0) = f(0, m) = 1$ , as well as that  $f(m, n) = f(n, m)$ , are both immediate. Assume now that  $m, n \geq 1$ . The number of edges of  $P(m, n)$  for which  $W_1 = \{0\}$  is  $f(m-1, n)$ , the number of edges of  $P(m, n)$  for which  $W_1 = [1]$  is  $f(m-1, n-1)$ , finally, the number of edges of  $P(m, n)$  for which  $W_1 = \{1\}$  is  $f(m, n-1)$ . Summing up we get the recursive formula (4.3).

Multiply (4.3) with  $x^m y^n$  and sum over all  $m, n \geq 1$ . We get

$$\sum_{m,n \geq 1} f(m, n) x^m y^n = \sum_{m \geq 1, n \geq 0} f(m, n) x^m y^{n+1} + \sum_{m \geq 0, n \geq 1} f(m, n) x^{m+1} y^n + \sum_{m, n \geq 0} f(m, n) x^{m+1} y^{n+1}. \quad (4.5)$$

On the left hand side we have

$$\sum_{m,n \geq 1} f(m, n) x^m y^n = F(m, n) - 1 - \sum_{m \geq 1} x^m - \sum_{n \geq 1} y^n = F(x, y) - \frac{1}{1-x} - \frac{1}{1-y} + 1.$$

On the right hand side we have

$$\sum_{m \geq 1, n \geq 0} f(m, n) x^m y^{n+1} = y \cdot \sum_{m \geq 1, n \geq 0} f(m, n) x^m y^n = y \left( F(x, y) - \sum_{n \geq 0} y^n \right) = y \left( F(x, y) - \frac{1}{1-y} \right).$$

Transforming the other terms on the right hand side of (4.5) in a similar way, we get

$$F(x, y) - \frac{1}{1-x} - \frac{1}{1-y} + 1 = xF(x, y) - \frac{x}{1-x} + yF(x, y) - \frac{y}{1-y} + xyF(x, y),$$

which simplifies to  $F(x, y)(1-x-y-xy) = 1$  yielding the formula (4.4).  $\square$

#### 4.4 Number of simplices of maximal dimension in an immediate snapshot complex.

For arbitrary nonnegative integers  $m_0, \dots, m_n$  we let  $f(m_0, \dots, m_n)$  denote the number of top-dimensional simplices in  $P(m_0, \dots, m_n)$ . Note, that the top-dimensional simplices of  $P(m_0, \dots, m_n)$  are indexed by sequences  $(W_1, \dots, W_t)$  of non-empty subsets  $W_i \subseteq [n]$ , such that  $\sum_{i=1}^t \chi(p, W_i) = m_p$ , for all  $p \in [n]$ .

**Proposition 4.8** *We have*

$$f(m_0, \dots, m_{n-1}, 0) = f(m_0, \dots, m_{n-1}),$$

and also

$$f(m_0, \dots, m_n) = f(m_{\pi(0)}, \dots, m_{\pi(n)}),$$

for any  $\pi \in \mathcal{S}_{[n]}$ .

In general, consider a round counter  $\bar{r} = (m_0, \dots, m_n)$ , then we have

$$f(m_0, \dots, m_n) = \sum_{\emptyset \neq S \subseteq \text{act } \bar{r}} f(m_0^S, \dots, m_n^S), \quad (4.6)$$

where

$$m_k^S = \begin{cases} m_k - 1, & \text{if } k \in S; \\ m_k, & \text{if } k \notin S. \end{cases}$$

The corresponding generating function in  $n + 1$  variables is

$$F(x_0, \dots, x_n) = \sum_{m_0, \dots, m_n=0}^{\infty} f(m_0, \dots, m_n) x_0^{m_0} \dots x_n^{m_n}.$$

It is given by the following explicit formula:

$$F(x_0, \dots, x_n) = 1 / \left( 1 - \sum_{\emptyset \neq S \subseteq \text{act } \bar{r}} \prod_{j \in S} x_j \right). \quad (4.7)$$

**Proof:** The first two equalities are immediate. To prove the equality (4.6) simply sum over the top-dimensional simplices grouping them according to the subset  $W_1$ . The formula (4.7) can either be derived same way as we derived the formula (4.4), or by a term-by-term calculation of the product

$$F(x_0, \dots, x_n) \cdot \left( 1 - \sum_{\emptyset \neq S \subseteq \text{act } \bar{r}} \prod_{j \in S} x_j \right)$$

using the recursive formula (4.6). □

#### 4.5 Standard chromatic subdivision as immediate snapshot complex

The *standard chromatic subdivision* of an  $n$ -simplex, denoted  $\chi(\Delta^n)$ , is a prominent and much studied structure in distributed computing. We refer to [HKR14, HS99] for distributed computing background, and to [Ko12, Ko15a] for the analysis of its simplicial structure, where, in particular, the following combinatorial description of  $\chi(\Delta^n)$  has been given.

**Definition 4.9** *Let  $n$  be a natural number. The simplicial complex  $\chi(\Delta^n)$  is defined as follows.*

- *The vertices of  $\chi(\Delta^n)$  are indexed by all pairs  $(p, V)$ , such that  $V \subseteq [n]$ , and  $p \in V$ .*
- *The simplices of  $\chi(\Delta^n)$  are indexed by pairs of tuples of non-empty sets  $((B_1, \dots, B_t), (C_1, \dots, C_t))$ , such that  $B_i$ 's are disjoint subsets of  $[n]$ , and  $C_i \subseteq B_i$  for all  $1 \leq i \leq t$ .*

*Given a simplex  $\tau = ((B_1, \dots, B_t), (C_1, \dots, C_t))$ , its vertices are indexed by all pairs  $(c, B)$ , where  $c \in C_i$ , and  $B = B_i$ , for some  $1 \leq i \leq t$ .*

In particular, the dimension of the simplex  $\tau$  indexed by  $((B_1, \dots, B_t), (C_1, \dots, C_t))$  is equal to  $|C_1| + \dots + |C_t| - 1$ . To describe the boundary relations in  $\chi(\Delta^n)$  pick  $p \in C_1 \cup \dots \cup C_t$ , and let  $\sigma_p$  be the simplex obtained from  $\tau$  by deleting  $p$ . Assume  $p \in C_k$ . If  $|C_k| \geq 2$ , then we have

$$\sigma_p = ((B_1, \dots, B_t), (C_1, \dots, C_k \setminus \{p\}, \dots, C_t)). \quad (4.8)$$

Otherwise, we have  $|C_k| = 1$ , i.e.,  $C_k = \{p\}$ . If  $k < t$ , then we have

$$\sigma_p = ((B_1, \dots, B_{k-1}, B_k \cup B_{k+1}, B_{k+2}, \dots, B_t), (C_1, \dots, C_{k-1}, C_{k+1}, C_{k+2}, \dots, C_t)), \quad (4.9)$$

else  $k = t$ , and we have

$$\sigma_p = ((B_1, \dots, B_{t-1}), (C_1, \dots, C_{t-1})). \quad (4.10)$$

For brevity, we set  $P_n := P(\underbrace{1, \dots, 1}_{n+1})$ .

**Proposition 4.10** *The immediate snapshot complex  $P_n$  and the standard chromatic subdivision of an  $n$ -simplex  $\chi(\Delta^n)$  are isomorphic as simplicial complexes. Explicitly, the isomorphism can be given by*

$$\Phi : ((B_1, \dots, B_t)(C_1, \dots, C_t)) \mapsto \begin{array}{|c|c|c|c|c|} \hline W_0 & C_1 & C_2 & \dots & C_t \\ \hline [n] \setminus W_0 & B_1 \setminus C_1 & B_2 \setminus C_2 & \dots & B_t \setminus C_t \\ \hline \end{array}, \quad (4.11)$$

where  $W_0 = B_1 \cup \dots \cup B_t$ .

**Proof:** Let  $\tau = ((B_1, \dots, B_t)(C_1, \dots, C_t))$  be a simplex of  $\chi(\Delta^n)$ . We can verify that  $\Phi(\tau)$  is a well-defined witness structure: (P1) is true since  $W_0 = B_1 \cup \dots \cup B_t$ , (P2) and (P3) are true since the sets  $B_i$  are disjoint, while (W) is true, since the sets  $C_i$  are non-empty. We have  $\text{supp}(\Phi(\tau)) = [n]$ , and  $A(\Phi(\tau)) = C_1 \cup \dots \cup C_t$ . Furthermore, to see that the witness structure  $\Phi(\tau)$  indexes a simplex of  $P_n$ , we notice that  $|\text{Tr}(p)| \leq 2$ , for all  $p \in [n]$ , follows from the disjointness of the sets  $B_i$ , and that  $|\text{Tr}(p)| = 2$  if and only if  $p \in C_1 \cup \dots \cup C_t$ . We have

$$\dim(\tau) = |C_1| + \dots + |C_t| - 1 = \dim(\Phi(\tau)).$$

Finally, the case-by-case comparison of the equations (4.8), (4.9), and (4.10), with the rules of the ghosting operations shows that the map  $\Phi$  is simplicial.

Let now  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  be a simplex of  $P_n$ . Define

$$\Psi : \sigma \mapsto ((W_1 \cup G_1, \dots, W_t \cup G_t), (W_1, \dots, W_t)). \quad (4.12)$$

Set  $((B_1, \dots, B_t)(C_1, \dots, C_t)) := \Psi(\sigma)$ . Clearly,  $C_i \subseteq B_i$ , for all  $i$ , and the sets  $C_i$  are non-empty, since  $\sigma$  is a witness structure. The disjointness of the sets  $B_i$  is immediate consequence of the inequality  $|\text{Tr}(p)| \leq 2$ , for all  $p \in [n]$ . It follows that  $\Psi(\sigma)$  is a simplex of  $\chi(\Delta^n)$ . Obviously,  $\Psi$  is an inverse of  $\Phi$ , hence  $\Phi$  is a simplicial isomorphism between  $\chi(\Delta^n)$  and  $P_n$ .  $\square$

We note the following direct description of the simplicial structure of  $P_n$ : simplices of  $P_n$  are indexed by all witness structures  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  satisfying the following three conditions:

- (1)  $W_0 \cup G_0 = [n]$ ;
- (2)  $W_0 = W_1 \cup \dots \cup W_t \cup G_1 \cup \dots \cup G_t$ ;
- (3) the sets  $W_1, \dots, W_t, G_1, \dots, G_t$  are disjoint.

## 5 Topology of the immediate snapshot complexes

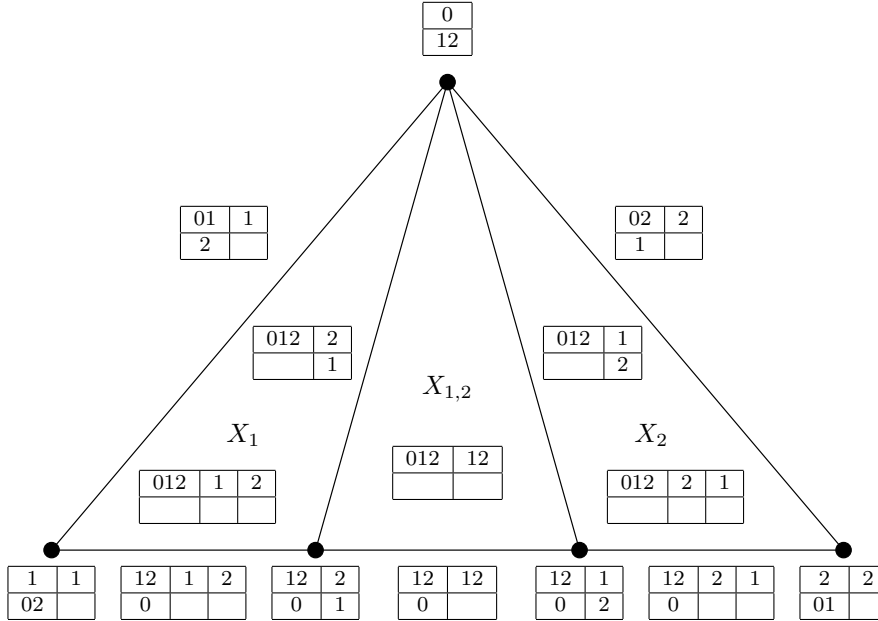
### 5.1 A canonical decomposition of the immediate snapshot complexes

We shall now describe how to decompose the immediate snapshot complex  $P(\bar{r})$  into pieces in a natural way, which we call the canonical decompositions. Intuitively, these pieces correspond to the protocol complexes, for the sets of executions where the first execution step is fixed.

**Definition 5.1** Assume  $\bar{r}$  is a round counter. For every subset  $S \subseteq \text{act } \bar{r}$ , let  $X_S(\bar{r})$  denote the set of all simplices  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  of  $P(\bar{r})$ , such that one of the following three conditions is fulfilled:

- $t = 0$ ;
- $S \subseteq G_1$ ;
- $W_1 \cup G_1 = S$ .

In particular, for  $S = \emptyset$  the condition  $S \subseteq G_1$  is always satisfied, so  $X_\emptyset(\bar{r})$  is the set of all simplices of  $P(\bar{r})$ . An example of a canonical decomposition is given on Figure 5.1.



**Fig. 5.1:** The immediate snapshot complex  $P(0, 1, 1)$  and its canonical decomposition.

**Proposition 5.2** For every round counter  $\bar{r}$ , and for every subset  $S \subseteq \text{act } \bar{r}$ , the set  $X_S(\bar{r})$  is closed under taking boundary, hence forms a simplicial subcomplex of  $P(\bar{r})$ .



**Proof:** Let  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  be a simplex in  $X_S(\bar{r})$ , and assume  $\tau \subset \sigma$ . By Proposition 3.5 there exists  $T \subseteq A(\sigma)$ , such that  $\tau = \Gamma_T(\sigma)$ . By Theorem 2.17 it is enough to consider the case  $|T| = 1$ , so assume  $T = \{p\}$ , and let  $\tau = ((\tilde{W}_0, \tilde{G}_0), \dots, (\tilde{W}_{\tilde{t}}, \tilde{G}_{\tilde{t}}))$ .

If  $\tilde{t} = 0$ , then  $\tau \in X_S(\bar{r})$ , and we are done. So assume  $\tilde{t} \geq 1$ , hence also  $t \geq 1$ . In this case, by definition of  $X_S(\bar{r})$ , we have either  $S \subseteq G_1$  or  $W_1 \cup G_1 = S$ . On the other hand, by the definition of  $\Gamma_p(\sigma)$ , if  $\tilde{t} \geq 1$ , then either  $W_1 \cup G_1 \subseteq \tilde{G}_1$  (if all of  $W_1$  is moved to  $G_1$ ) or  $W_1 \cup G_1 = \tilde{W}_1 \cup \tilde{G}_1$  and  $\tilde{G}_1 \supseteq G_1$  (if only part or none of  $W_1$  is moved to  $G_1$ ).

First, if  $W_1 \cup G_1 \subseteq \tilde{G}_1$ , then in any case  $S \subseteq \tilde{G}_1$ , so  $\tau \in X_S(\bar{r})$ , and we are done. Finally, assume  $W_1 \cup G_1 = \tilde{W}_1 \cup \tilde{G}_1$  and  $\tilde{G}_1 \supseteq G_1$ . If  $W_1 \cup G_1 = S$ , then also  $\tilde{W}_1 \cup \tilde{G}_1 = S$ , and  $\tau \in X_S(\bar{r})$ . If, instead,  $S \subseteq G_1$ , then  $S \subseteq \tilde{G}_1$ , so again  $\tau \in X_S(\bar{r})$ .  $\square$

We shall abuse notations and use  $X_S(\bar{r})$  to denote this simplicial complex as well. Next we prove that the subcomplexes  $X_S(\bar{r})$  can themselves be viewed as immediate snapshot complexes. To formulate this result we need additional terminology.

**Definition 5.3** Assume  $\bar{r}$  is an arbitrary round counter and  $S \subseteq \text{act } \bar{r}$ . We let  $\bar{r} \downarrow S$  denote the round counter defined by

$$(\bar{r} \downarrow S)(i) = \begin{cases} \bar{r}(i), & \text{if } i \notin S; \\ \bar{r}(i) - 1, & \text{if } i \in S. \end{cases}$$

We say that the round counter  $\bar{r} \downarrow S$  is obtained from  $\bar{r}$  by the *execution* of  $S$ . Note that  $\text{supp } (\bar{r} \downarrow S) = \text{supp } \bar{r}$ ,  $\text{act } (\bar{r} \downarrow S) = \{i \in \text{act } \bar{r} \mid i \notin S, \text{ or } \bar{r}(i) \geq 2\}$ , and  $\text{pass } (\bar{r} \downarrow S) = \text{pass } (\bar{r}) \cup \{i \in S \mid \bar{r}(i) = 1\}$ .

**Proposition 5.4** Assume  $\bar{r}$  is an arbitrary round counter and  $S \subseteq \text{act } \bar{r}$ , then there exists a simplicial isomorphism

$$\gamma_S(\bar{r}) : X_S(\bar{r}) \rightarrow P(\bar{r} \downarrow S).$$

**Proof:** Pick an arbitrary simplex  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  belonging to  $X_S(\bar{r})$ . If  $t = 0$ , then we set  $\gamma_S(\sigma) := \sigma$ . Note, that since  $S \subseteq \text{act } \bar{r}$ , we have  $S \subseteq G_0$  in this case. Else, by the construction of  $X_S$ , we either have  $W_1 \cup G_1 = S$ , or  $S \subseteq G_1$ . If  $W_1 \cup G_1 = S$ , then set

$$\gamma_S(\sigma) := \begin{array}{|c|c|c|c|} \hline W_0 \setminus G_1 & W_2 & \dots & W_t \\ \hline G_0 \cup G_1 & G_2 & \dots & G_t \\ \hline \end{array},$$

else  $S \subseteq G_1$ , in which case we set

$$\gamma_S(\sigma) := \begin{array}{|c|c|c|c|c|} \hline W_0 \setminus S & W_1 & W_2 & \dots & W_t \\ \hline G_0 \cup S & G_1 \setminus S & G_2 & \dots & G_t \\ \hline \end{array}.$$

Reversely, assume  $\tau = ((V_0, H_0), \dots, (V_t, H_t))$  is a simplex of  $P(\bar{r} \downarrow S)$ . Since  $\text{supp } \bar{r} = \text{supp } (\bar{r} \downarrow S)$ , we have  $S \subseteq V_0 \cup H_0$ . If  $V_0 \cap S \neq \emptyset$ , we set

$$\rho_S(\tau) := \begin{array}{|c|c|c|c|c|} \hline V_0 \cup (H_0 \cap S) & V_0 \cap S & V_1 & \dots & V_t \\ \hline H_0 \setminus (H_0 \cap S) & H_0 \cap S & H_1 & \dots & H_t \\ \hline \end{array}.$$

Else we have  $S \subseteq H_0$ . If  $t \geq 1$ , we set

$$\rho_S(\tau) := \begin{array}{|c|c|c|c|c|} \hline V_0 \cup S & V_1 & V_2 & \dots & V_t \\ \hline H_0 \setminus S & H_1 \cup S & H_2 & \dots & H_t \\ \hline \end{array},$$

else  $t = 0$ , and we set  $\rho_S(\tau) := \tau$ .

A direct case-by-case verification shows that the maps  $\gamma_S$  and  $\rho_S$  are well-defined simplicial maps, which preserve supports,  $A(-)$ ,  $G(-)$ , and hence also the dimension. Furthermore, they are inverses of each other, hence are simplicial isomorphisms.  $\square$

## 5.2 Immediate snapshot complexes are pseudomanifolds with boundary

In this section we show that immediate snapshot complexes are pseudomanifolds with boundary. We start by showing that  $P(\bar{r})$  is strongly connected.

**Definition 5.5** *Let  $K$  be a pure simplicial complex of dimension  $n$ . Two  $n$ -simplices of  $K$  are said to be **strongly connected** if there is a sequence of  $n$ -simplices so that each pair of consecutive simplices has a common  $(n - 1)$ -dimensional face. The complex  $K$  is said to be **strongly connected** if any two  $n$ -simplices of  $K$  are strongly connected.*

Clearly, being strongly connected is an equivalence relation on the set of all  $n$ -simplices.

**Proposition 5.6** *For an arbitrary round counter  $\bar{r}$ , the immediate snapshot complex  $P(\bar{r})$  is strongly connected.*

**Proof:** Set  $n := |\text{supp } \bar{r}| - 1$ . Proposition 4.5 says that  $P(\bar{r})$  is a pure simplicial complex of dimension  $n$ . We now use induction on  $|\bar{r}|$ . If  $|\bar{r}| = 0$ , or more generally, if  $|\text{act } \bar{r}| \leq 1$ , then  $P(\bar{r})$  is just a single simplex, so it is trivially strongly connected.

Assume  $|\text{act } \bar{r}| \geq 2$ , and consider the canonical decomposition of  $P(\bar{r})$ . By Proposition 5.4, the simplicial complex  $X_S(\bar{r})$  is isomorphic to  $P(\bar{r} \downarrow S)$ , for all  $S \subseteq \text{act } \bar{r}$ . Since  $|\bar{r} \downarrow S| = |\bar{r}| - |S| < |\bar{r}|$ , and  $\text{supp } \bar{r} \downarrow S = \text{supp } \bar{r}$ , we conclude that  $X_S(\bar{r})$  is a pure simplicial complex of dimension  $n$ , which is strongly connected by the induction assumption. Thus, any pair of  $n$ -simplices belonging to the same subcomplex  $X_S(\bar{r})$  is strongly connected.

Pick now any  $p \in \text{act } \bar{r}$ , and any  $S \subseteq \text{act } \bar{r}$ , such that  $p \in S$ ,  $\{p\} \neq S$ , and consider any  $(n - 1)$ -simplex  $\tau = ((W_0, G_0), \dots, (W_t, G_t))$ , such that  $(W_1, G_1) = (S \setminus \{p\}, \{p\})$ . Obviously, such  $\tau$  exists, and  $\tau \in X_S(\bar{r}) \cap X_p(\bar{r})$ . By induction assumptions for  $X_S(\bar{r})$  and  $X_p(\bar{r})$ , there exist  $n$ -simplices  $\sigma_1 \in X_S(\bar{r})$ , and  $\sigma_2 \in X_p(\bar{r})$ , such that  $\tau \in \partial\sigma_1$  and  $\tau \in \partial\sigma_2$ . This means, that  $\sigma_1$  and  $\sigma_2$  are strongly connected. Since being strongly connected is an equivalence relation, any two  $n$ -simplices from  $X_S(\bar{r})$  and  $X_p(\bar{r})$  are strongly connected. This includes the case  $S = \text{act } \bar{r}$ , implying that any pair of  $n$ -simplices in  $P(\bar{r})$  is strongly connected, so  $P(\bar{r})$  itself is strongly connected.  $\square$

**Definition 5.7** *We say that a strongly connected pure simplicial complex  $K$  is a **pseudomanifold** if each  $(n - 1)$ -simplex of  $K$  is a face of precisely one or two  $n$ -simplices of  $K$ . The  $(n - 1)$ -simplices of  $K$  which are faces of precisely one  $n$ -simplex of  $K$  form a (possibly empty) simplicial subcomplex of  $K$ , called the **boundary** of  $K$ , and denoted  $\partial K$ .*

To describe the boundary subcomplex of  $P(\bar{r})$ , we need the following definition.

**Definition 5.8** Let  $\bar{r}$  be an arbitrary round counter, and assume  $V \subset \text{supp } \bar{r}$ . We define  $B_V(\bar{r})$  to be the simplicial subcomplex of  $P(\bar{r})$  consisting of all simplices  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ , satisfying  $V \subseteq G_0$ .

**Proposition 5.9** For an arbitrary round counter  $\bar{r}$ , the simplicial complex  $P(\bar{r})$  is a pseudomanifold, and the subcomplex  $\partial P(\bar{r})$  consists of all simplices  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ , such that  $G_0 \neq \emptyset$ .

**Proof:** By Proposition 5.6 we already know that  $P(\bar{r})$  is strongly connected. Set again  $n := |\text{supp } \bar{r}| - 1$ , and let  $\tau = ((W_0, G_0), \dots, (W_t, G_t))$  be an arbitrary  $(n - 1)$ -simplex of  $P(\bar{r})$ . Note that  $\text{codim } \tau = |G_0| + \dots + |G_t|$ , hence  $\text{codim } \tau = 1$  implies that there exist  $0 \leq k \leq t$ , and  $p \in \text{supp } \bar{r}$ , such that

$$G_i = \begin{cases} \{p\}, & \text{if } i = k; \\ \emptyset, & \text{if } i \neq k. \end{cases}$$

Set  $m := r(p) + 1 - |\text{Tr}(p, \sigma)|$ . Consider

$$\sigma_1 = (W_0, \dots, W_{k-1}, W_k \cup \{p\}, W_{k+1}, \dots, W_t, \underbrace{p, \dots, p}_m),$$

and if  $k \geq 1$ , consider also

$$\sigma_2 = (W_0, \dots, W_{k-1}, p, W_k, \dots, W_t, \underbrace{p, \dots, p}_m).$$

Obviously,  $\Gamma(\sigma_1, p) = \Gamma(\sigma_2, p) = \tau$ , so  $\tau \in \partial\sigma_1$  and  $\tau \in \partial\sigma_2$ . Furthermore, the definition of the ghosting construction implies that these are the only options to find  $\sigma$ , such that  $\Gamma(\sigma, p) = \tau$ .

We conclude that  $P(\bar{r})$  is a pseudomanifold, whose boundary is a union of the  $(n - 1)$ -simplices  $\tau = ((W_0, G_0), \dots, (W_t, G_t))$ , such that  $W_0 \neq \emptyset$ , so then the subcomplex  $\partial P(\bar{r})$  consists of all simplices  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ , such that  $G_0 \neq \emptyset$ .  $\square$

## 6 Immediate snapshot complexes as protocol complexes

### 6.1 The protocol complexes of a standard full-information protocol

This section will provide a bridge between the mathematical and the theoretical distributed computing contexts. Specifically, we shall explain why immediate snapshot complexes provide a correct combinatorial model for the protocol complexes in the immediate snapshot read/write computational model.

As in Section 1, assume that we have  $n + 1$  processes indexed  $0, \dots, n$ , together with a round counter  $\bar{r} = (r_0, \dots, r_n)$ . We consider the standard protocol associated to this data. In this protocol, each process  $p$  starts with some input value  $\alpha_p$ , and then executes  $r_p$  rounds. In each round, the process  $p$  first writes its current state into the register, which is assigned to that process (full-information protocol), and then the process reads the entire memory in one atomic step (snapshot read).

In the topological approach to distributed computing, once the computational model is fixed, one associates a simplicial complex to each protocol. That complex is called a *protocol complex*. We refer to [HKR14] and the citations therein for the further specifics of that construction. In general, the protocol

complex is defined as follows. The maximal simplices are indexed by all possible executions of the protocol. The vertices of the protocol complexes are the *local views* of individual processes. Two maximal simplices, corresponding to executions  $\sigma$  and  $\tau$ , share the simplex consisting of those local views, which are the same in  $\sigma$  and in  $\tau$ .

As was said above, the executions in the immediate snapshot read/write computational model are shaped in layers. In each layer, a group of processes atomically writes to their respective registers, and then takes an atomic snapshot of the entire memory. In other words, that executions can be indexed by tuples  $(W_1, \dots, W_t)$  of sets of processes, where  $W_1$  is the first group of processes which gets activated, followed by  $W_2$ , and so on.

Let  $Q(\bar{r})$  denote the protocol complex associated to the standard full-information protocol for the round counter  $\bar{r}$ . In this case, we have an additional condition  $\sum_{i=1}^n \chi(p, W_i) = r_p$ , for all  $p \in [n]$ . Obviously, we have a one-to-one correspondence between all executions of the protocol and the top-dimensional simplices of the immediate snapshot complex  $P(\bar{r})$ . To summarize, both  $P(\bar{r})$  and  $Q(\bar{r})$  are pure of dimension  $|\text{supp } \bar{r}| - 1$ , and we have a natural bijection between the sets of their top-dimensional simplices. Before proceeding with extending this bijection, we need to analyze the structure of information the processes write into the memory during an execution of the standard full-information protocol.

## 6.2 Witness posets

When a process is activated for the first time, the only information that it has is its input value, so it will simply write its input value into the assigned register. Later on, it will see the information which other processes have written, and write that newly acquired information, as a part of his state, once it is activated next time. To describe this knowledge structure formally, let  $z_{p,k}$  denote the information which process  $p$  has *after* its  $k$ th step (we cannot know for sure in which layer this step takes place). Clearly, it is the same information as the one which process  $p$  will *write* into the memory during its  $(k + 1)$ th step. For ease of notations, we set  $z_{p,0} := \alpha_p$ . Accordingly,  $z_{p,r_p}$  denotes the information which the process  $p$  has once it has executed the entire protocol. In general, we shall write

$$z_{p,i} > z_{q,j} \tag{6.1}$$

to express the fact that *the process  $p$  after its  $i$ th step knows what the process  $q$  knew after its  $j$ th step*. Since all what processes learn during the execution of the protocol is what other processes knew at various stages of the execution (here we are thinking about the input values as the knowledge of processes after the 0th step) the entire knowledge structure generated by an execution  $\sigma$  of the protocol is a poset, which we denote  $Z(\sigma)$ . This poset has elements  $z_{p,i}$ , where  $p \in [n]$  and  $i \in [r_p]$ , with the order relation given by (6.1).

**Definition 6.1** *Assume  $\bar{r} = (r_0, \dots, r_n)$  is a round counter, and  $Z$  is a poset, whose set of elements is  $\{z_{p,i} \mid p \in [n], i \in [k_p]\}$ , for some nonnegative integers  $k_p \leq r_p$ , for  $p \in [n]$ . For all  $p \in [n]$ ,  $i \in [k_p]$ , set  $U(p, i) := Z_{< z_{p,i}}$ , and set furthermore  $A(Z) := \{p \mid k_p = r_p\}$ . The poset  $Z$  is called a **witness poset** with parameter  $\bar{r}$  if its order relation satisfies the following conditions:*

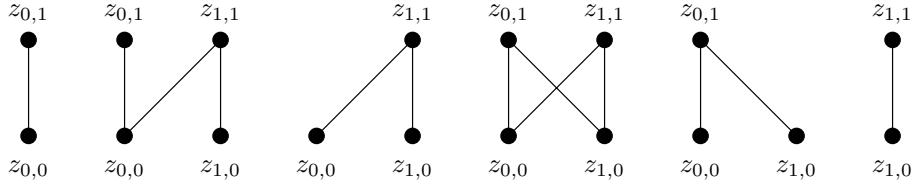
- (1)  $z_{p,i+1} > z_{p,i}$ , for all  $p \in [n]$ ,  $i \in [k_p - 1]$ ;
- (2) assume  $p, q \in [n]$ ,  $1 \leq i \leq k_p$ , and  $1 \leq j \leq k_q$ , then one of the following three conditions is satisfied:

- $U(q, j) \supset U(p, i)$ ,  $z_{q,j} > z_{p,i-1}$ , and  $z_{p,i} \not> z_{q,j-1}$ ;
- $U(p, i) \supset U(q, j)$ ,  $z_{p,i} > z_{q,j-1}$ , and  $z_{q,j} \not> z_{p,i-1}$ ;
- $U(q, j) = U(p, i)$ ,  $z_{p,i} > z_{q,j-1}$ , and  $z_{q,j} > z_{p,i-1}$ .

(3) the set of maximal elements of  $Z$  is given by  $\{z_{p,r_p} \mid p \in A(Z)\}$ .

We call  $Z$  a **complete witness poset** if  $A(Z) = [n]$ .

Note, that since a witness poset  $Z$  has to have some maximal elements, there must exist  $p$  such that  $k_p = r_p$ . Some examples of witness posets are shown on Figure 6.1.



**Fig. 6.1:** All witness posets for  $\bar{r} = (1, 1)$ .

**Definition 6.2** Assume we are given a round counter  $\bar{r}$ . We defined the simplicial complex  $C(\bar{r})$  as follows:

- the set of vertices  $V(C(\bar{r}))$  consists of all witness posets  $Z$  with parameter  $\bar{r}$ , such that  $|A(Z)| = 1$ ;
- a subset  $\{V_0, \dots, V_k\} \subseteq V(C(\bar{r}))$  of vertices forms a simplex if and only if there exists a witness poset  $Z$  with parameter  $\bar{r}$ , such that

$$\{I(Z, v) \mid v \in A(Z)\} = \{V_0, \dots, V_k\},$$

where  $I(Z, v)$  was defined in subsection 2.1.

Note, that for a witness poset  $Z$  and  $\emptyset \neq B \subseteq A \subseteq A(Z)$ , we have

$$I(I(Z, A), B) = I(Z, B),$$

hence  $C(\bar{r})$  is well-defined as a simplicial complex. It is pure, and has dimension  $n$ . The set of all simplices of  $C(\bar{r})$  coincides with the set of all witness posets with parameter  $\bar{r}$ . The maximal simplices of  $C(\bar{r})$  are indexed by the complete witness posets.

### 6.3 Protocol complexes vs witness posets

To proceed, we need additional notation. Assume we have a sequence of sets  $\sigma = (W_1, \dots, W_t)$ ,  $p \in \cup_{i=1}^t W_i$ , and  $k \in [t]$ , we set  $M_\sigma(p, k) := \sum_{i=1}^k \chi(p, W_i)$ . Furthermore, we let  $\rho_\sigma(p, k)$  denote the index such that  $p$  occurs in  $W_{\rho_\sigma(p, k)}$  for the  $k$ th time. In other words,  $\rho_\sigma(p, k)$  is uniquely defined by the following two conditions:  $p \in W_{\rho_\sigma(p, k)}$  and

$$M_\sigma(p, \rho_\sigma(p, k)) = k. \quad (6.2)$$

Note, that

$$\rho_\sigma(p, j) > \rho_\sigma(p, i) \text{ if } j > i, \quad (6.3)$$

and

$$M_\sigma(p, k) \geq M_\sigma(p, l) \text{ if } k > l. \quad (6.4)$$

Furthermore,

$$\text{if } p \in W_k, \text{ then } \rho_\sigma(p, M_\sigma(p, k)) = k. \quad (6.5)$$

When  $\sigma$  is clear from the context, we will skip it from the notations, and simply write  $M(p, k)$  and  $\rho(p, k)$ .

**Proposition 6.3** *For any round counter  $\bar{r} = (r_0, \dots, r_n)$ , there is a simplicial isomorphism between the complexes  $Q(\bar{r})$  and  $C(\bar{r})$ .*

**Proof:** First, we define the map  $\Phi$  which takes an execution  $\sigma = (W_1, \dots, W_t)$  of the protocol to a complete witness poset  $Z = \Phi(\sigma)$ . The set of the elements of  $Z$  is taken to be  $\{z_{p,i} \mid p \in [n], i \in [r_p]\}$ . The order relation is given by the rule: for  $p, q \in [n]$ ,  $1 \leq i \leq r_p$ ,  $0 \leq j \leq r_q$ , we have

$$z_{p,i} > z_{q,j} \text{ if and only if } \rho(p, i) \geq \rho(q, j + 1). \quad (6.6)$$

In words, the inequality (6.6) simply says that  $q$  occurs at least  $j + 1$  times in  $W_1, \dots, W_k$ , where  $p$  occurs for the  $i$ th time in  $W_k$ .

We check that  $\Phi$  is well-defined. First, we check that  $Z$  is actually a poset. Assume  $z_{p,i} > z_{q,j}$  and  $z_{q,j} > z_{p,i}$ . Then, (6.6) implies that  $\rho(p, i) \geq \rho(q, j + 1)$  and  $\rho(q, j) \geq \rho(p, i + 1)$ . This gives a contradiction with (6.3). Assume furthermore that  $z_{p,i} > z_{q,j}$  and  $z_{q,j} > z_{s,k}$ . Here, (6.6) implies that  $\rho(p, i) \geq \rho(q, j + 1)$  and  $\rho(q, j) \geq \rho(s, k + 1)$ . Using (6.3) we then conclude that  $\rho(p, i) > \rho(s, k + 1)$ , and hence  $z_{p,i} > z_{s,k}$ .

Second, we want to check that  $Z$  is a complete witness poset, by verifying the conditions in Definition 6.1. Condition (1) says that  $z_{p,i+1} > z_{p,i}$ , which (6.6) translates to  $\rho(p, i + 1) \geq \rho(p, i + 1)$ , which is a tautology.

Next, we check Condition (2). We pick  $p, q \in [n]$ ,  $1 \leq i \leq r_p$ ,  $1 \leq j \leq r_p$ , and compare  $\rho(p, i)$  with  $\rho(q, j)$ . Without loss of generality, we can assume that  $\rho(p, i) \geq \rho(q, j)$ . This implies  $z_{p,i} > z_{q,j-1}$ . In addition, we can show that  $U(p, i) \supseteq U(q, j)$ . Indeed, take  $z_{s,k} < z_{q,j}$ . By (6.6), we have  $\rho(q, j) \geq \rho(s, k + 1)$ . Since  $\rho(p, i) \geq \rho(q, j)$ , we get  $\rho(p, i) \geq \rho(s, k + 1)$ , and so  $z_{s,k} < z_{p,i}$ . In particular, if  $\rho(p, i) = \rho(q, j)$  then repeating this argument gives  $U(p, i) = U(q, j)$ ,  $z_{p,i} > z_{q,j-1}$ , and  $z_{q,j} > z_{p,i-1}$ . On the other hand, if we have a strict inequality  $\rho(p, i) > \rho(q, j)$  then  $z_{p,i} > z_{q,j-1}$ , and  $z_{q,j} \not> z_{p,i-1}$ , which in turn implies that we have a strict inclusion  $U(p, i) \supset U(q, j)$ . In any case, Condition (2) is satisfied.

Finally, to check Condition (3), as well the completeness, we need to see that one cannot have  $z_{p,r_p} > z_{q,r_q}$ . This is so, since otherwise we would have  $M(q, \rho(p, r_p)) \geq r_q + 1$ , which is impossible. We can therefore conclude that  $\Phi(\sigma)$  is a well-defined complete witness poset.

Now, we define a map  $\Psi$ , which takes an arbitrary complete witness poset  $Z$  with parameter  $\bar{r}$  to the protocol execution  $\Psi(Z)$ . The condition that  $U(p, i)$  is comparable with  $U(q, j)$  for all  $p, q \in [n]$ ,  $i \in [r_p]$ , and  $j \in [r_q]$ , means that we can order all  $U(p, i)$ 's by inclusion. So assume that for all  $k = 1, \dots, t$ , we have sets  $S_k = \{(p_1^k, i_1^k), \dots, (p_{v_k}^k, i_{v_k}^k)\}$ , such that the following two conditions are true:

- $U(p_1^k, i_1^k) = \dots = U(p_{v_k}^k, i_{v_k}^k)$ ,

- $U(p_1^k, i_1^k) \subseteq U(p_1^{k+1}, i_1^{k+1})$ , for all  $k = 1, \dots, t-1$ .

We set  $W_k := \{p_1^k, \dots, p_{v_k}^k\}$ , for all  $k = 1, \dots, t$ . Clearly,  $(W_1, \dots, W_t)$  is a well-defined execution, since all indices in each  $W_k$  are different, and the number of occurrences of each  $p$  is  $r_p$ .

Let now  $\sigma = (W_1, \dots, W_t)$  be an execution of the protocol and let us show that  $\Psi \circ \Phi(\sigma) = \sigma$ . First, pick  $p, q \in [n]$ , and assume that  $p, q \in W_k$  for some  $1 \leq k \leq t$ . By (6.5), we have

$$\rho(p, M(p, k)) = \rho(q, M(q, k)) = k.$$

Assume  $z_{s,i} < z_{p, M(p,k)}$ , then by (6.6), this is equivalent to  $\rho(p, M(p, k)) \geq \rho(s, i+1)$ , which in turn is equivalent to  $\rho(q, M(q, k)) \geq \rho(s, i+1)$ , and hence to  $z_{s,i} < z_{q, M(q,k)}$ . This means that

$$U(p, M(p, k)) = U(q, M(q, k)).$$

Now, let us pick  $p, q \in [n]$ , and  $l < k$ , such that  $p \in W_k, q \in W_l$ . Take  $z_{s,i} < z_{q, M(q,l)}$ , then  $l = \rho(q, M(q, l)) \geq \rho(s, i+1)$ . Since  $\rho(p, M(p, k)) = k > l$ , it follows that  $\rho(p, M(p, k)) \geq \rho(s, i+1)$ , hence  $z_{s,i} < z_{p, M(p,k)}$ . Thus we see that in this case

$$U(p, M(p, k)) \supset U(q, M(q, k)),$$

where the inclusion is strict, since  $z_{p, M(p,k)-1} \in U(p, M(p, k)) \setminus U(q, M(q, k))$ . Together these two calculations show that  $\Psi \circ \Phi(\sigma) = \sigma$ .

On the other hand, take an arbitrary complete witness poset  $Z$  with parameter  $\bar{r}$ . Set  $\tilde{Z} := \Phi \circ \Psi(Z)$ . Note, that both  $Z$  and  $\tilde{Z}$  have the same sets of elements  $z_{p,i}$ , for  $p \in [n], i \in [r_p]$ , and we have  $z_{p,i+1} > z_{p,i}$  for all  $p, i$  in both  $Z$  and  $\tilde{Z}$ . Assume now  $z_{p,i} > z_{q,j}$  in  $Z$ . By Definition 6.1 this is equivalent to  $U(p, i) \supseteq U(q, j+1)$ , which in turn is equivalent to  $z_{p,i} > z_{q,j}$  in  $\tilde{Z}$ .

This shows that both  $\Phi$  and  $\Psi$  are well-defined and are inverses of each other. Furthermore, since the information which the process  $p$  has after its  $i$ th run is precisely  $U(p, i)$ , the poset  $Z_{\leq z_{p,r_p}}$  is the local view of the process  $p$ , and taking lower ideals corresponds to taking a set of local views, which are compatible in some execution. This means that  $\Phi$  and  $\Psi$  are actually simplicial isomorphisms.  $\square$

## 6.4 Witness posets vs witness structures

As a next step we show that witness posets and witness structures encode identical simplicial information.

**Proposition 6.4** *For any round counter  $\bar{r} = (r_0, \dots, r_n)$ , we have a simplicial isomorphism between complexes  $C(\bar{r})$  and  $P(\bar{r})$ .*

**Proof:** We describe maps  $\tilde{\Phi} : P(\bar{r}) \rightarrow C(\bar{r})$  and  $\tilde{\Psi} : C(\bar{r}) \rightarrow P(\bar{r})$ , which will generalize maps  $\Phi$  and  $\Psi$  from the proof of Proposition 6.3. Consider a witness structure  $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$  indexing a simplex of  $P(\bar{r})$ . Set  $k_p := \sum_{i=0}^t \chi(p, W_i) - 1$ . Let the set of elements of  $\tilde{\Phi}(\sigma)$  be  $\{z_{p,i} \mid p \in [n], i \in [k_p]\}$ . For  $p \in W_0, k \in [k_p]$ , we let  $\tilde{\rho}(p, k)$  denote the index  $\rho$ , such that  $p \in W_\rho \cup G_\rho$  and  $\sum_{i=1}^\rho \chi(p, W_i \cup G_i) = k$ . The order relation in  $\tilde{\Phi}(\sigma)$  is then given by:

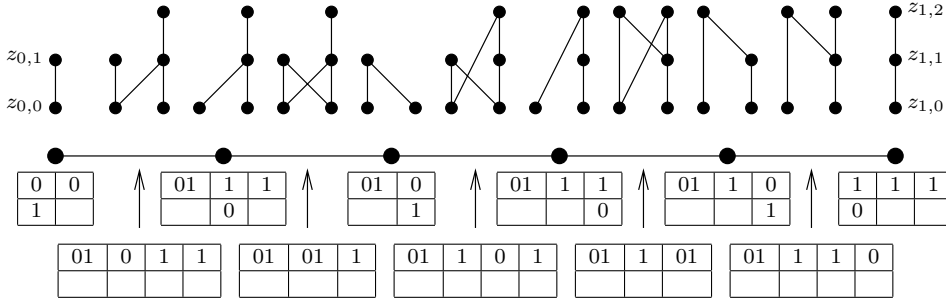
$$z_{p,i} > z_{q,j} \text{ if and only if } \tilde{\rho}(p, i) \geq \tilde{\rho}(q, j+1).$$

The verification that  $\tilde{\Phi}$  is well-defined is verbatim to that in Proposition 6.4. One also sees easily that  $\tilde{\Phi}(\sigma) = I(A(\sigma), \Phi(W_0 \cup G_0, \dots, W_t \cup G_t))$ .

Next, we define  $\tilde{\Psi} : C(\bar{r}) \rightarrow P(\bar{r})$ . Take  $Z \in C(\bar{r})$ , and denote  $\tilde{\Psi}(Z) = ((W_0, G_0), \dots, (W_t, G_t))$ . We let  $W_0$  be given by the identity  $\min Z = \{z_{p,0} \mid p \in W_0\}$ , and set  $G_0 := [n] \setminus W_0$ . Assume the set  $S_1, \dots, S_t$  are chosen in the same way as when we defined  $\Psi$  in the proof of Proposition 6.4, and set  $U_k := U(p_1^k, i_1^k)$ , for all  $1 \leq k \leq t$ . We have  $U_1 \subset \dots \subset U_t$ , and we set  $W_k := \{p_1^k, \dots, p_{v_k}^k\}$ , for all  $1 \leq k \leq t$ . Assume we have  $p$  such that  $k_p < r_p$ . Let  $m$  be the smallest index such that  $z_{p,k_p} \in U_m$ , then  $p \in G_m$ ; this index is well-defined, since  $z_{p,k_p}$  is not a maximal element. This rule defines uniquely the sequence of sets  $G_1, \dots, G_t$ . The verification that  $\tilde{\Psi}$  is well-defined is a straightforward extension of the argument in the proof of Proposition 6.3.

Same way as above, we can see that  $\tilde{\Phi}$  and  $\tilde{\Psi}$  are inverses of each other. Furthermore, the map  $\tilde{\Phi}$  takes the operation of taking lower ideals under a subset of maximal elements to the ghosting operation on the witness structures. It follows that  $\tilde{\Phi}$  and  $\tilde{\Psi}$  are simplicial isomorphisms.  $\square$

Even though the information contained in simplicial complexes  $C(\bar{r})$  and  $P(\bar{r})$  is the same, in various situations it can be more convenient to use one or the other. We feel that taking lower ideals is simpler to grasp than the ghosting operation. On the other hand, the entire witness poset structure is a bit awkward to describe, when we want to work with specific simplices, here, witness structures provide a more succinct description. Figure 6.2 shows the parallel combinatorial encodings of the simplices in the simple case  $\bar{r} = (2, 1)$ .



**Fig. 6.2:** The immediate snapshot complex for  $\bar{r} = (2, 1)$ , with simplex names given as witness posets, as well as witness structures.

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