

A New Game Invariant of Graphs: the Game Distinguishing Number

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The distinguishing number of a graph G is a symmetry related graph invariant whose study started two decades ago. The distinguishing number $D(G)$ is the least integer d such that G has a distinguishing d -coloring. A distinguishing d -coloring is a coloring $c : V(G) \rightarrow \{1, \dots, d\}$ invariant only under the trivial automorphism. In this paper, we introduce a game variant of the distinguishing number. The distinguishing game is a game with two players, the Gentle and the Rascal, with antagonist goals. This game is played on a graph G with a set of $d \in \mathbb{N}^*$ colors. Alternately, the two players choose a vertex of G and color it with one of the d colors. The game ends when all the vertices have been colored. Then the Gentle wins if the coloring is distinguishing and the Rascal wins otherwise. This game leads to define two new invariants for a graph G , which are the minimum numbers of colors needed to ensure that the Gentle has a winning strategy, depending on who starts. These invariants could be infinite, thus we start by giving sufficient conditions to have infinite game distinguishing numbers. We also show that for graphs with cyclic automorphism group of prime odd order, both game invariants are finite. After that, we define a class of graphs, the involutive graphs, for which the game distinguishing number can be quadratically bounded above by the classical distinguishing number. The definition of this class is closely related to imprimitive actions whose blocks have size 2. Then, we apply results on involutive graphs to compute the exact value of these invariants for hypercubes and even cycles. Finally, we study odd cycles, for which we are able to compute the exact value when their order is not prime. In the prime order case, we give an upper bound of 3.

Keywords: distinguishing number, graph automorphism, combinatorial game, hypercube

1 Background and definition of the game

The distinguishing number of a graph G is a symmetry related graph invariant. Its study started two decades ago in a work by Albertson and Collins [1]. Given a graph G the distinguishing number $D(G)$ is the least integer d such that G has a distinguishing d -coloring. A distinguishing d -coloring is a coloring $c : V(G) \rightarrow \{1, \dots, d\}$ invariant only under the trivial automorphism. More generally, we say that an automorphism σ of a graph G preserves the coloring c or is a color preserving automorphism, if $c(u) = c(\sigma(u))$ for all $u \in V(G)$. We denote by $\text{Aut}(G)$, the automorphism group of G . Clearly, for each coloring c of the vertex set of G , the set $\text{Aut}_c(G) = \{\sigma \in \text{Aut}(G) : c \circ \sigma = c\}$ is a subgroup of

$\text{Aut}(G)$. A coloring c is distinguishing if $\text{Aut}_c(G)$ is trivial. The group $\text{Aut}(G)$ acts naturally on the vertex set of G and this action induces a partition of the vertex set $V(G)$ into orbits. If H is a subgroup of $\text{Aut}_c(G)$ then the action of H on $V(G)$ is such that each orbit induced by this action is monochromatic. In particular, an automorphism σ preserves the coloring if and only if the orbits under the action of the subgroup $\langle \sigma \rangle$ generated by σ , are all monochromatic.

In the last couple of years the study of this invariant was particularly flourishing. See [10, 11] for the work of Imrich, Jerebic and Klavžar on the distinguishing number of the Cartesian products or [7] for an analog of Brook's theorem. In [7] Collins and Trenk also introduce a distinguishing coloring which must be a proper coloring. The distinguishing number is also studied in a more general context than graphs [6, 13]. Our goal in this paper is to introduce a game variant of this invariant in the spirit of the game chromatic number χ_G introduced by Brahm's in 1981 (see [8]) or of the most recent domination game (see [4]). Even if inventing new game invariants could raise a lot of promising and interesting questions, it might seem artificial at first glance. To defend our approach, we recall that game invariants have already proved useful to give a new insight on the classical invariant they are related. We cite in particular [5], where a greedy like strategy used in the study of the domination game is used to improve several upper bounds on the domination number. See also [12] for an application of the coloring game to the graph packing problem.

The distinguishing game is a game with two players, the Gentle and the Rascal, with antagonist goals. This game is played on a graph G with a set of $d \in \mathbb{N}^*$ colors. Alternately, the two players choose a vertex of G and color it with one of the d colors. The game ends when all the vertices have been colored. Then the Gentle wins if the coloring is distinguishing and the Rascal wins otherwise.

This game leads to two new invariants for a graph G . The \mathcal{G} -game distinguishing number $D_G(G)$ is the minimum of colors needed to ensure that the Gentle has a winning strategy for the game on G , assuming he is playing first. If the Rascal is sure to win whatever the number of colors we allow, then $D_G(G) = \infty$. Similarly, the \mathcal{R} -game distinguishing number $D_{\mathcal{R}}(G)$ is the minimum number of colors needed to ensure that the Gentle has a winning strategy, assuming the Rascal is playing first.

In Section 2, some basic results about the distinguishing games are given. In particular, we study when the distinguishing numbers are finite or not. This question is far from being easy in general. The third section is devoted to introduce the class of involutive graphs, for which we prove that the \mathcal{R} -game distinguishing number is finite and more precisely, quadratically bounded above by the classical distinguishing number of the graph. This class, closely related to imprimitive action of group with complete block system whose blocks have size 2, contains a variety of graphs such as hypercubes, even cycles, spherical graphs and even graphs [9, 15]. In Section 4, the general result on involutive graphs is used to compute the exact values of the game invariants for hypercubes.

Theorem 1.1 *Let Q_n be the hypercube of dimension $n \geq 2$.*

1. *We have $D_G(Q_n) = \infty$.*
2. *If $n \geq 5$, then $D_{\mathcal{R}}(Q_n) = 2$. Moreover $D_{\mathcal{R}}(Q_2) = D_{\mathcal{R}}(Q_3) = 3$.*
3. *We have $2 \leq D_{\mathcal{R}}(Q_4) \leq 3$.*

Finally, in Section 5, we solve the problem for cycles, except when the number of vertices is prime. For even cycles, it is a straightforward application of a result on involutive graphs, but for odd cycles new ideas are needed.

Theorem 1.2 *Let C_n be a cycle of order $n \geq 3$.*

1. *If n is even, then $D_G(C_n) = \infty$. If n is odd, then $D_{\mathcal{R}}(C_n) = \infty$.*
2. *If n is even and $n \geq 8$, then $D_{\mathcal{R}}(C_n) = 2$. Moreover $D_{\mathcal{R}}(C_4) = D_{\mathcal{R}}(C_6) = 3$.*
3. *If n is odd, not prime and $n \geq 9$, then $D_G(C_n) = 2$.*
4. *If n is prime and $n \geq 5$, then $D_G(C_n) \leq 3$. Moreover $D_G(C_3) = \infty$ and $D_G(C_5) = D_G(C_7) = 3$.*

2 Basic results

This section is devoted to basic results about the distinguishing games. Especially, we are interested in determining when the \mathcal{G} -game and \mathcal{R} -game distinguishing numbers are infinite.

It arises directly from the definition that $D(G) \leq D_G(G)$ and $D(G) \leq D_{\mathcal{R}}(G)$, for any graph G . Another straightforward remark is that the game distinguishing number of a graph and its complement are equal. One first natural question is, if having a winning strategy with k colors is a growing property. In other words, has the Gentle a winning strategy if $k \geq D_G$ or $k \geq D_{\mathcal{R}}$ colors are allowed? The answer is not as obvious as it looks at first glance. We recall that for the game chromatic number χ_g , it is not known if there is always a winning strategy if we play with $k \geq \chi_g$ colors. However, for the game distinguishing numbers we easily show they both have this growing property.

Proposition 2.1 *Let G be a graph and k a positive integer. If $k \geq D_G(G)$ (resp. $k \geq D_{\mathcal{R}}(G)$), then the Gentle has a winning strategy with k colors, if he starts (resp. the Rascal starts).*

Proof: In order to win with k colors, the Gentle plays the same winning strategy he would have played with $D_G(G)$ (resp. $D_{\mathcal{R}}(G)$) colors, except when the Rascal chooses a color strictly greater than $D_G(G)$ (resp. $D_{\mathcal{R}}(G)$). In that case, the Gentle plays the winning move he would have played if the Rascal had played the color 1. Let c be the final coloring in the game with k colors with respect to the above strategy. Let \tilde{c} be the coloring defined for all $v \in V(G)$ as follows:

$$\tilde{c}(v) = \begin{cases} c(v) & \text{if } c(v) \leq D_G(G) \text{ (resp. } c(v) \leq D_{\mathcal{R}}(G)) \\ 1 & \text{otherwise} \end{cases}.$$

The coloring \tilde{c} is a distinguishing coloring since it is obtained by following a winning strategy for the game with only $D_G(G)$ (resp. $D_{\mathcal{R}}(G)$) colors. Let σ be an automorphism which preserves c . It is clear that σ preserves \tilde{c} too. Since \tilde{c} is distinguishing, σ must be the identity. This shows that c is a distinguishing k -coloring and that the Gentle has a winning strategy with k colors. \square

As mentioned in the definition, D_G and $D_{\mathcal{R}}$ could possibly be infinite. For example, if the graph has an automorphism of order two, the Rascal can win using a strategy close to the Tweedledee Tweedledum one for the sum of opposite combinatorial games. He uses the involutive automorphism to copy the move the Gentle just made.

Proposition 2.2 *If G is a graph with a nontrivial automorphism of order 2, then:*

1. $D_G(G) = \infty$ if $|V(G)|$ is even,
2. $D_{\mathcal{R}}(G) = \infty$, if $|V(G)|$ is odd.

Proof: Let A be the set of vertices fixed by σ : $A = \{v \in G : \sigma(v) = v\}$. Note that $|V(G)|$ and $|A|$ have the same parity. We denote by r_i and s_i the i -th vertex played by the Rascal and the Gentle respectively.

First, assume $|A|$ is even and the Gentle starts. The winning strategy for the Rascal is as follows. If s_i is in A , then the Rascal plays another vertex r_i in A and does not pay attention to the color. This is always possible since $|A|$ is even. Else the Rascal plays such that $r_i = \sigma(s_i)$ and $c(r_i) = c(s_i)$. Since σ has order 2, the vertices outside A can be grouped in pairs (u, v) with $\sigma(u) = v$ and $\sigma(v) = u$. Moreover the Gentle will be the first to play outside A . Hence, such a move is always available for the Rascal.

Now suppose $|A|$ is odd and the Rascal begins. His first move is to color a vertex in A . The number of uncolored vertices in A are now even and it is the Gentle's turn. Then the Rascal wins with exactly the same strategy as above. \square

For example, for cycles C_n or paths P_n on n vertices, we have $D_G(C_n) = D_G(P_n) = \infty$, if n is even, and $D_{\mathcal{R}}(C_n) = D_{\mathcal{R}}(P_n) = \infty$, if n is odd. Note that Proposition 2.2 tells nothing about the possible values of D_G (resp. $D_{\mathcal{R}}$) when the number of fixed points is odd (resp. even). For cycles the question is rather complicated. We partially solve it in Section 5. But for paths P_n , with $n \geq 2$, it is easy to show that $D_G(P_n) = 2$ (resp. $D_{\mathcal{R}}(P_n) = 2$) if n is odd (resp. even).

There are also graphs with both D_G and $D_{\mathcal{R}}$ infinite, for example graphs with two transpositions moving a common vertex, like the complete graphs on more than 3 vertices. Finding graphs with both invariants finite, is a bit less obvious. Before stating the result, we recall that for a given finite group Γ , there are infinitely many non-isomorphic graphs whose automorphism group is Γ .

Proposition 2.3 *If G is a graph such that $\text{Aut}(G) = \mathbb{Z}/p\mathbb{Z}$, where p is prime and $p \geq 3$, then $D_G(G) = D_{\mathcal{R}}(G) = 2$.*

Proof: Since $|\text{Aut}(G)|$ is prime all the orbits under the action of the whole group have either size 1 or size p . Let $\mathcal{O}_1, \dots, \mathcal{O}_l$, with $l > 0$ be the orbits of size p . Since there are at least three distinct vertices in each orbit \mathcal{O}_i and two colors are allowed, the Gentle can always play, whoever starts, in a manner which ensures that at least one of the \mathcal{O}_i is not monochromatic.

Let σ be a nontrivial automorphism of G . Since $|\text{Aut}(G)|$ is prime, σ generates $\text{Aut}(G)$ and the orbits under $\text{Aut}(G)$ and $\langle \sigma \rangle$ are the same. The automorphism σ cannot preserve the coloring because in that case all the orbits \mathcal{O}_i have to be monochromatic. This shows that the above strategy is winning for the Gentle and that $D_G(G) = D_{\mathcal{R}}(G) = 2$. \square

3 Involution graphs

In this section, we introduce the new class of involutive graphs. This class contains well known graphs such as hypercubes, even cycles, even toroidal grids and even graphs [9, 15]. For graphs in this class, we are going to prove that their \mathcal{R} -game distinguishing number is quadratically bounded above by the distinguishing number.

An *involutive graph* G is a graph together with an involution, $\text{Bar} : V(G) \rightarrow V(G)$, which commutes with all automorphisms and has no fixed point. In others words:

- $\overline{\overline{u}} = u$ and $\overline{u} \neq u$ for every $u \in V(G)$
- $\sigma(\overline{u}) = \overline{\sigma(u)}$ for every $\sigma \in \text{Aut}(G)$ and $u \in V(G)$.

The set $\{u, \bar{u}\}$ will be called a *block* and \bar{u} will be called the *opposite* of u . An important remark is that, since Bar commutes with all automorphisms, the image of a block under an automorphism is also a block. The block terminology comes from imprimitive group action theory. Indeed, if we add the condition that the automorphism group of the graph acts transitively, then an involutive graph is a graph such that this action has a complete block system, whose blocks have size 2.

Let G be an involutive graph on n vertices. We define G' as the graph obtained from G by deleting all the edges and putting a new edge between all vertices and their opposites. The resulting graph is a disjoint union of $n/2$ copies of K_2 . Moreover, it is also an involutive graph and it has the same blocks as G . The next proposition states that these graphs are in a sense the worst involutive graphs for the Gentle.

Proposition 3.1 *If G is an involutive graph then $D(G) \leq D(G')$ and $D_{\mathcal{R}}(G) \leq D_{\mathcal{R}}(G')$.*

Proof: It is enough to prove that any distinguishing coloring c of G' is also a distinguishing coloring of G . Let σ be a nontrivial automorphism of G . Assume first, there are two disjoint blocks $\{u, \bar{u}\}$ and $\{v, \bar{v}\}$, with $u, v \in V(G)$, such that $\sigma(\{u, \bar{u}\}) = \{v, \bar{v}\}$. Since the coloring c is distinguishing for G' , we have $c(\{u, \bar{u}\}) \neq c(\{v, \bar{v}\})$. Hence, σ does not preserve c . Suppose now that there are no two such blocks. Since σ is not trivial, there exists $u \in V(G)$ such that $\sigma(u) = \bar{u}$. But u and \bar{u} must have distinct colors, otherwise the transposition which switch them would be a nontrivial color preserving automorphism of G' . Therefore, σ is not a color preserving automorphism. We conclude that c is also a distinguishing coloring for G . \square

For graphs which are disjoint union of K_2 , we are able to compute the exact value of D_G and $D_{\mathcal{R}}$.

Proposition 3.2 *If G is the disjoint union of $n \geq 1$ copies of K_2 , then $D_G(G) = \infty$ and $D_{\mathcal{R}}(G) = n + 1$.*

Proof: Such a graph G has an automorphism of order 2, with no fixed point, the one which exchanges the two vertices in each copy of K_2 . By Proposition 2.2, we directly conclude that $D_G(G) = \infty$.

We are now going to prove that $D_{\mathcal{R}}(G) = n + 1$. First, assume that only $k < n + 1$ colors are allowed during the game. The Rascal's strategy is to play always the color 1 in a way that at least one vertex of each copy of K_2 is colored with 1. Since there is strictly less than $n + 1$ different colors, whatever the Gentle plays there will be two distinct K_2 colored with the same pair of colors or there will be a monochromatic K_2 . If there is a monochromatic K_2 , the transposition which permutes its two vertices is clearly a color preserving automorphism. In the other case, there are four distinct vertices $u_1, u_2, v_1, v_2 \in V(G)$, such that u_1 and u_2 (resp. v_1 and v_2) are in the same K_2 . Moreover, u_1 and v_1 (resp. u_2 and v_2) share the same color. The automorphism σ , defined by $\sigma(u_i) = v_i$, $\sigma(v_i) = u_i$, with $i \in \{1, 2\}$, and $\sigma(w) = w$ for $w \in V(G) \setminus \{u_1, u_2, v_1, v_2\}$ preserves the coloring. Hence, the strategy described is winning for the Rascal. In conclusion, $D_{\mathcal{R}}(G) \geq n + 1$.

We now allow $n + 1$ colors. We recall that the Rascal plays first. The winning strategy for the Gentle is as follows. He always colors the remaining uncolored vertex in the copy of K_2 , where the Rascal has just played before. He chooses his color such that the pair of colors for this K_2 is different from all the pairs of colors of the previously totally colored copies of K_2 . Moreover, he ensures that no K_2 is monochromatic. Let σ be a nontrivial automorphism of G and let $u \in V(G)$ be such that $\sigma(u) \neq u$. If u and $\sigma(u)$ belong to the same copy of K_2 , σ does not preserve the coloring. Otherwise, the copy of K_2 which contains u is sent by σ to another one. This implies that σ is not a color preserving automorphism. This shows that the Gentle has a winning strategy with $n + 1$ colors. In conclusion, $D_{\mathcal{R}}(G) = n + 1$. \square

It is shown in [7], that if G consists of n disjoint copies of K_2 , then $D(G) = \lceil \frac{1 + \sqrt{8n+1}}{2} \rceil$. Hence, we get examples of graphs for which the \mathcal{R} -game distinguishing number is quadratic in the classical one. The following results are straightforward consequences of Proposition 3.1 and Proposition 3.2.

Proposition 3.3 *If G is an involutive graph on n vertices, then:*

- $D(G) \leq \lceil \frac{1 + \sqrt{4n+1}}{2} \rceil$ and
- $D_{\mathcal{R}}(G) \leq \frac{n}{2} + 1$.

□

This result shows that involutive graphs have finite \mathcal{R} -game distinguishing number. But the bound we obtained does not depend on the classical distinguishing number of the graph. If the Gentle knows this number, having in mind a distinguishing coloring, he can use it to play in a smarter way. In other words, for an involutive graph G , we can bound the game distinguishing number $D_{\mathcal{R}}$ using the classical one. It turns out that $D_{\mathcal{R}}(G)$ is in this case at most of order $D(G)^2$.

Theorem 3.4 *If G is an involutive graph with $D(G) \geq 2$, then $D_{\mathcal{R}}(G) \leq D(G)^2 + D(G) - 2$.*

Proof: We set $d = D(G)$. We are going to give a winning strategy for the Gentle with $d^2 + d - 2$ colors. Note that, since the Rascal starts the game, the Gentle can play in a way he is never the first to color a vertex in a block. When the Rascal colors a vertex $u \in V(G)$, the Gentle's strategy is to always answer by coloring the vertex \bar{u} in the same block.

Before stating how he chooses the color, we need some definitions. Let c be a distinguishing coloring of G with $d \geq 2$ colors. We set $k = d^2 + d - 2$. For all $i, j \in \{1, \dots, d\}$, we define the following sets of vertices: $V_{ij} = \{u \in V(G) \mid c(u) = i \text{ and } c(\bar{u}) = j\}$. There are $\frac{d(d-1)}{2}$ sets V_{ij} , with $1 \leq i < j \leq d$, and d sets V_{ii} , with $i \in \{1, \dots, d\}$. To each set V_{ij} , with $1 \leq i < j \leq d$, we associate a specific number δ_{ij} in $\{1, \dots, \frac{d(d-1)}{2}\}$. For each V_{ij} , with $i, j \in \{1, \dots, d\}$, we define r_{ij} as follows:

$$r_{ij} = \begin{cases} \delta_{ij} & \text{if } 1 \leq i < j \leq d \\ k - r_{ji} & \text{if } 1 \leq j < i \leq d \\ \frac{d(d-1)}{2} + i & \text{if } i = j \text{ and } 1 \leq i \leq d-1 \\ 0 & \text{if } i = j = d \end{cases}$$

This defines only d^2 distinct numbers. However, we need $d^2 + d - 2$ colors to ensure the second of the following properties.

- (i) For every $i, i', j, j' \in \{1, \dots, d\}$, we have $r_{ij} \equiv r_{i'j'} \pmod{k}$ if and only if $i = i'$ and $j = j'$.
- (ii) For every $i, j \in \{1, \dots, d\}$, we have $r_{ij} \equiv -r_{ji} \pmod{k}$ if $i \neq j$.
- (iii) For every $i, j \in \{1, \dots, d\}$, we have $r_{ii} \not\equiv -r_{jj} \pmod{k}$ if $i \neq j$.

Let now $c' : V(G) \rightarrow \{1, \dots, k\}$ be the coloring built during the game. The Gentle chooses the color as follows. When the Rascal colors $u \in V_{ij}$ with $c'(u)$, he colors \bar{u} such that $c'(\bar{u}) - c'(u) \equiv r_{ij} \pmod{k}$.

To prove this strategy is winning, it is enough to show that for any automorphism σ which preserves c' , we have $\sigma(V_{ij}) = V_{ij}$. Since all V_{ij} are monochromatic for the distinguishing coloring c , this will show that σ also preserves c , and σ must therefore be the identity.

Let u be a vertex of G and $i, j \in \{1, \dots, d\}$. Properties (i), (ii), and (iii) easily imply the following facts.

(a) If $i \neq j$, then $u \in V_{ij}$ if and only if $c'(\bar{u}) - c'(u) \equiv r_{ij} \pmod{k}$.

(b) $u \in V_{ii}$ if and only if $c'(\bar{u}) - c'(u) \equiv \pm r_{ii} \pmod{k}$.

Let σ be an automorphism which preserves c' . For any vertex $u \in V_{ij}$, with $i, j \in \{1, \dots, d\}$, we have $c'(\bar{u}) - c'(u) \equiv c'(\sigma(\bar{u})) - c'(\sigma(u)) \pmod{k}$. By (a), we get that $c'(\sigma(\bar{u})) - c'(\sigma(u)) \equiv r_{ik} \pmod{k}$, and by (b), we have that $\sigma(u)$ is in the same set V_{ij} as u . \square

In the above proof, decreasing the number of sets V_{ij} tightens the bound. In particular, we have the following corollary.

Corollary 3.5 *Let G be an involutive graph. If there is a coloring with $d \geq 2$ colors, such that Bar is the only nontrivial automorphism which preserves this coloring, then $D_{\mathcal{R}}(G) \leq 2d - 2$.*

Proof: We use the same notations as in the proof of Theorem 3.4. As in this proof, the Gentle uses a coloring c with d colors which is only preserved by Bar to build his strategy. Since we ask that Bar preserves c , all the sets V_{ij} with $i \neq j$ are empty. So the Gentle needs only $2d - 2$ colors to apply his strategy. Indeed, he plays in a way that for all $u \in V(G)$, $c'(\bar{u}) - c'(u) \equiv i - 1 \pmod{2d - 2}$, where $i \in \{1, \dots, d\}$ is such that $u \in V_{ii}$. As for the proof of the previous theorem, we can prove that an automorphism which preserves the coloring c' must also preserve the coloring c . Thus, it is either the identity or the Bar involution. All blocks whose vertices do not belong to V_{11} are not monochromatic. Hence, the Bar involution cannot preserve the coloring c' . In conclusion, c' is a distinguishing coloring and the Gentle has a winning strategy with $2d - 2$ colors. \square

4 Hypercubes

In this section, we use Corollary 3.5 to study the game distinguishing number of Q_n , the hypercube of dimension n where $n \geq 2$, and prove Theorem 1.1. The vertices of Q_n will be denoted by words of length n on the binary alphabet $\{0, 1\}$. Let $u(i)$ denote its i -th letter of a vertex u . For the classical distinguishing number the question is solved in [2]: $D(Q_3) = D(Q_2) = 3$ and $D(Q_n) = 2$ for $n \geq 4$. As mentioned before, hypercubes are involutive graphs. The Bar involution can be defined as the operation which switches all the letters 0 to 1 and all the letters 1 to 0. Hence, by Proposition 2.2, we know that $D_G(Q_n) = \infty$, for all $n \geq 2$. If we apply directly Theorem 3.4, we obtain that $D_{\mathcal{R}}(Q_n) \leq 4$, for $n \geq 2$. To improve this bound in order to prove the first item of Theorem 1.1, we are going to use Corollary 3.5. We will have to design nice distinguishing coloring of the hypercube with 2 colors. For this purpose, we introduce determining sets.

A subset S of vertices of a graph G is a *determining set* if the identity is the only automorphism of G whose restriction to S is the identity on S . Determining sets are introduced and studied in [3]. In the following lemma, we give a sufficient condition for a subset of Q_n to be determining.

Lemma 4.1 *Let S be a subset of vertices of Q_n , with $n \geq 2$. If for all $i \in \{1, \dots, n\}$, there are two vertices v_0^i and v_1^i in S such that $v_0^i(i) = 0$, $v_1^i(i) = 1$ and all the other letters are the same, then S is a determining set of Q_n .*

Proof: Let $\sigma \in \text{Aut}(G)$, which is the identity on S . Suppose for a contradiction that there exists a vertex u in $Q_n \setminus S$ such that $\sigma(u) \neq u$. Then, there exists $k \in \{1, \dots, n\}$ such that $u(k) \neq \sigma(u)(k)$. Without loss of generality, we suppose that $u(k) = 0$. The vertices u and v_0^k differ on l letters, with $l \in \{1, \dots, n-1\}$. This means that the distance between u and v_0^k is equal to l . Since σ fixes v_0^k the distance between $\sigma(u)$ and v_0^k must be l , too. Therefore, the vertices u and v_1^k have $l+1$ distinct letters, whereas the vertices $\sigma(u)$ and v_1^k have only $l-1$ distinct letters. This is impossible because the distance between u and v_1^k must be the same as the distance between $\sigma(u)$ and v_1^k . Therefore $\sigma(u) = u$ for all $u \in V(Q_n)$. In conclusion, S is a determining set. \square

Proposition 4.2 *Let Q_n be the n -dimensional hypercube with $n \geq 5$. Then $D_{\mathcal{R}}(Q_n) = 2$.*

Proof: For $i \in \{0, \dots, n-1\}$, let v_i be the vertex with the i first letters equal to 1 and the $n-i$ other letters equal to 0. We define also the following vertices in Q_n : $f = 10010\dots 0$, $c_1 = 010\dots 01$ and $c_2 = 110\dots 01$. The subgraph S will be the graph induced by $\{v_i, \bar{v}_i | 0 \leq i < n\} \cup \{f, \bar{f}, c_1, \bar{c}_1, c_2, \bar{c}_2\}$ (see Figure 1). Let c be the coloring with two colors defined by: $c(u) = 1$ if $u \in V(S)$ and $c(u) = 2$ otherwise. We will show that this coloring fits hypothesis of Corollary 3.5. This would imply that $D_{\mathcal{R}}(Q_n) = 2 \times 2 - 2 = 2$.

Clearly, Bar preserves the coloring c and we prove now this is the only nontrivial color preserving automorphism. An automorphism σ which preserves this coloring fixes S setwise : $\sigma(S) = S$. The restriction of σ to S , say $\sigma|_S$, is an automorphism of S . The vertices v_0, \dots, v_{n-1} , and \bar{v}_0 show that the hypothesis of Lemma 4.1 holds for the subgraph S . Hence σ is totally determined by the images of the elements of S . The two vertices f and \bar{f} are the only vertices of degree 1 in S . So, either $\sigma(f) = \bar{f}$ and $\sigma(\bar{f}) = f$ or both are fixed. In the first case, this implies that $\sigma(v_1) = \bar{v}_1$ and $\sigma(\bar{v}_1) = v_1$. In the second case, v_1 and \bar{v}_1 are also fixed by σ . This is because they are respectively the only neighbors of f and \bar{f} in S . The vertices v_0 and v_2 are the two remaining neighbors of v_1 . Since they have not the same degree in S , they cannot be switched by σ . Hence, in the first case, $\sigma(v_i) = \bar{v}_i$ and $\sigma(\bar{v}_i) = v_i$, with $i \in \{0, 2\}$. In the second case, v_i and \bar{v}_i , with $i \in \{0, 2\}$, are fixed by σ . There is exactly one path of size $n-2$ between v_0 and \bar{v}_2 (resp. v_2 and \bar{v}_0). Hence, in the first case these paths are switched, whereas they are fixed pointwise in the second. After that, it is easy to show that $\sigma|_S$ is either the identity or the Bar involution. \square

The above proof fails for Q_4 because the subgraph S will have automorphisms other than the Bar involution (see Figure 2). Nevertheless, we can decrease the upper bound from 4 to 3.

Proposition 4.3 *We have $D_{\mathcal{R}}(Q_4) \leq 3$.*

Proof: Let S be the subgraph of Q_4 induced by the five vertices 0000, 1000, 1100, 1110 and 1011. Let \bar{S} be the subgraph induced by the opposite vertices of those in S . Note that S and \bar{S} are disjoint. The Gentle's strategy is as follows. When the Rascal colors a vertex u in $V(G)$, the Gentle colors \bar{u} . He chooses the coloring c according to these rules:

- if u is in $V(S)$, then $c(\bar{u}) - c(u) \equiv 1 \pmod{3}$,
- if u is in $V(\bar{S})$, then $c(\bar{u}) - c(u) \equiv -1 \pmod{3}$,
- $c(\bar{u}) - c(u) \equiv 0 \pmod{3}$ otherwise.

Let σ be a color preserving automorphism. Since $c(\bar{u}) - c(u) \equiv 1 \pmod{3}$ if and only if $u \in S$, we have that S is fixed setwise by σ . Let $\sigma|_S$ be the restriction of σ to S . The vertices 0000 and 1110

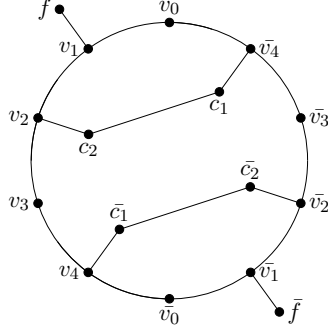


Fig. 1: The induced subgraph S in Q_5

are the only vertices of degree 1 in S . Then either $\sigma|_S$ fixes them or switches them. But in Q_4 , we have $d(0000, 1011) \neq d(1110, 1011)$. Since $\sigma|_S$ comes from an automorphism of Q_n , it must preserve distances in Q_n . But, the vertex 1011 is fixed by $\sigma|_S$ because it is the only isolated vertex of S . Hence, we have that $\sigma|_S$ fixes 0000 and 1110. Moreover, it is clear that all the vertices of S are fixed by $\sigma|_S$ and hence by σ . The vertex 1111 is also fixed by σ , because it is the opposite vertex of 0000. Finally, all the vertices of $V(S) \cup \{1111\}$ are fixed by σ and this subset fits hypothesis of Lemma 4.1. This shows that σ is the identity of Q_4 . \square

To complete the proof of Theorem 1.1, it remains to settle the case of hypercubes of dimension 2 and 3. For the square Q_2 , isomorphic to C_4 , an easy computation shows that $D_{\mathcal{R}}(Q_2) = 3$. For Q_3 , it turns out that $D_{\mathcal{R}}(Q_3)$ is also equal to 3. It comes from the fact that the complementary graph of Q_3 is isomorphic to $K_4 \square K_2$ and from the following proposition.

We will write the vertices of $K_4 \square K_2$ as couples (i, x) , with $i \in \{1, \dots, 4\}$ and $x \in \{l, r\}$. The first coordinate is the one associated to K_4 and the second the one associated to K_2 . We denote by K_2^i the K_2 -fiber whose two vertices have $i \in \{1, \dots, 4\}$ as first coordinate. Let c be a coloring of $K_4 \square K_2$. We say that two distinct K_2 -fibers, K_2^i and K_2^j , with $i, j \in \{1, \dots, 4\}$ are *colored the same* if $c(K_2^i) = c(K_2^j)$. Moreover, if $c((i, l)) = c((j, l))$, we say that the two fibers are *strictly colored the same*.

Proposition 4.4 *We have $D_{\mathcal{R}}(K_4 \square K_2) = 3$.*

Proof: Since $D(K_4 \square K_2) = 3$, we only have to prove that the Gentle has a winning strategy with 3 colors. The Rascal starts by coloring the vertex $(1, l)$ with 1. The Gentle replies by coloring with 2 the vertex $(2, l)$. Now, there are two cases.

Case 1: The Rascal colors the second vertex of K_2^1 or K_2^2 with a colors $\delta \in \{1, 2, 3\}$. Without loss of generality we assume he colors $(1, r)$. There are two subcases.

Subcase 1.1: $\delta \in \{1, 3\}$. The Gentle colors $(2, r)$ with 2. The Gentle's strategy is now to play in the same K_2 -fiber as Rascal. Since the two K_2 -fibers K_2^1 and K_2^2 share no color, the Gentle can ensure that at

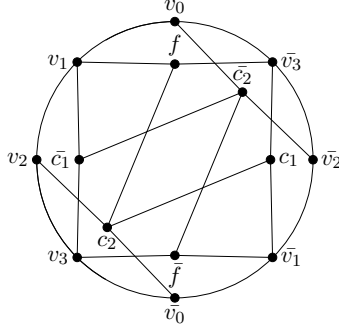


Fig. 2: The induced subgraph S in Q_4

the end of the game no pair of K_2 -fibers are colored the same and K_2^3 is not monochromatic. This shows that a color preserving automorphism σ cannot switch the K_2 -fibers. Hence $\sigma(K_2^3) = K_2^3$ and, since this fiber is not monochromatic, we have that σ cannot switch the K_4 -fibers, too. Finally, σ is the identity.

Subcase 1.2: $\delta = 2$. The Gentle colors $(2, r)$ with color 1. By coloring the vertices in the same K_2 -fiber as the vertices the Rascal will play, the Gentle can ensure that at the end of the game K_2^1 and K_2^2 are the only K_2 -fibers colored the same. Moreover, he can ensure that one K_2 -fiber, say K_2^3 , is not monochromatic. Let σ be a color preserving automorphism. The fibers K_2^3 and K_2^4 are fixed setwise by σ . Since K_2^3 is not monochromatic, σ cannot switch the K_4 -fibers. We conclude that every K_2 -fibers are fixed pointwise by σ , which means σ is trivial.

Case 2: The Rascal colors the vertex $(3, x)$, with $x \in \{l, r\}$. The Gentle answers by coloring the vertex $(4, x)$. The Gentle chooses his colors such that we can assume $(1, l)$ and $(3, x)$ are colored with 1, $(2, l)$ is colored with 2 and $(4, x)$ is colored with 3 (maybe we need to permute colors 1 and 2). We only deal with the case where $x = l$. The case where $x = r$ is very similar. When the Rascal's move is to color $(1, r)$ (resp. $(3, r)$), the Gentle answers by coloring $(3, r)$ (resp. $(1, r)$). When the Rascal's move is to color $(2, r)$ (resp. $(4, r)$), the Gentle answers by coloring $(4, r)$ (resp. $(2, r)$). He can choose the colors he uses, such that no pair of K_2 -fibers is strictly colored the same. Moreover, one of the four K_2 -fibers is not monochromatic and is not colored the same as each of the three other K_2 -fibers. At the end of the game, we are in the same situation as in subcase 1.2. In conclusion, the coloring built in this case is also 3-distinguishing. \square

5 Cycles

In this section, we prove Theorem 1.2 about cycles. We start with even cycles which are involutive graphs. For such graphs, the opposite of a vertex can be defined as the unique vertex at maximal distance. Corollary 3.5 enables us to conclude when the size of the even cycle is greater or equal to 12. For C_4 , it is easy to verify that $D_{\mathcal{R}}(C_4) = 3$. Using the fact that the coloring shown in Figure 3 is the only distinguishing 2-coloring of C_6 , it is easy to see that $D_{\mathcal{R}}(C_6) = 3$. For C_8 and C_{10} , the Gentle's strategies are less obvious. We give them in the below proposition.

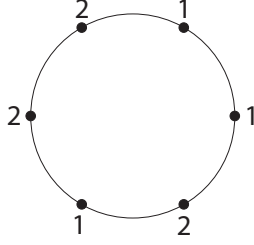


Fig. 3: The unique distinguishing 2-coloring of C_6

Proposition 5.1 We have $D_{\mathcal{R}}(C_8) = D_{\mathcal{R}}(C_{10}) = 2$.

Proof: We start with Gentle’s winning strategy for C_8 . Let (x_1, \dots, x_8) be the cycle of order 8. Let c be a 2-coloring with exactly three vertices colored with color 2. If these three vertices do not induce a stable set or a P_3 , then c is 2-distinguishing. The Rascal starts by coloring x_1 with color 1. The Gentle replies by coloring \bar{x}_1 with the same color. Gentle’s strategy is now to use the other color than the one the Rascal played just before. In this way, the number of vertices colored with color 2 will be exactly three. To avoid that these three vertices induce a stable set or a P_3 , he plays exactly one move in each of the pairs $\{x_2, x_4\}, \{x_6, x_8\}, \{x_3, x_7\}$.

Now, we give Gentle’s strategy for C_{10} . Let (x_1, \dots, x_{10}) be the cycle of order 10. Rascal’s first move is to color x_1 with color 1. The Gentle replies by coloring \bar{x}_1 with the same color. After that there are two cases.

Case 1: The Rascal uses color 1 for his second move. Using symmetries, we obtain only two subcases. In both subcases, the Gentle answers first with color 1. Then, he uses the other color than the one used by the Rascal just before.

Subcase 1.1: The Rascal colors x_2 with 1. The Gentle colors x_7 with 1. Then, he plays using the pairs $\{x_3, x_5\}, \{x_8, x_{10}\}, \{x_4, x_7\}$.

Subcase 1.2: The Rascal colors x_3 with 1. The Gentle answers by coloring x_4 with 1. Then, he plays following the pairs $\{x_2, x_5\}, \{x_7, x_9\}, \{x_8, x_{10}\}$.

Case 2: The Rascal use color 2 for his second move. The Gentle answers by coloring a vertex with color 2, too. By symmetry, we assume that x_2 and x_3 are colored with 2. The Gentle plays according to the pairs $s_1 = \{x_4, x_5\}$, $d = \{x_7, x_8\}$ and $s_2 = \{x_9, x_{10}\}$. If the Rascal plays in s_1 or s_2 , the Gentle copies the color he used, whereas, when the Rascal plays in d , the Gentle plays the other color.

We let the reader check that the strategies for C_8 and C_{10} are indeed winning. □

Proposition 5.2 For $n \geq 6$, we have $D_{\mathcal{R}}(C_{2n}) = 2$.

Proof: Let u, v, w be vertices of C_{2n} such that $d(u, v) = 1$, $d(u, w) = 3$ and $d(v, w) = 2$. We set $S = \{u, \bar{u}, v, \bar{v}, w, \bar{w}\}$ and define the coloring c with 2 colors as follows: $c(x) = 1$ if $x \in S$ and $c(x) = 0$ otherwise. Let σ be a nontrivial color preserving automorphism. The subgroup $\langle \sigma \rangle$ acts on the set of the three blocks contained in S . If σ is an axial reflection then one block must be stable under σ . This

block is the axis of the reflection. Since $2n \geq 12$, we easily check that the two other blocks cannot be map to each other by this reflection. Since σ cannot be a reflection, σ must be a rotation. The only possibility for a rotation to let S stable is to map each vertex to its opposite. Hence σ restricted to S is the Bar involution. But it is straightforward to show that σ must be the Bar involution on the whole cycle. This shows that the coloring c fits Corollary 3.5 hypothesis. We conclude that $D_{\mathcal{R}}(C_{2n}) \leq 2$. \square

We compute now the \mathcal{G} -game distinguishing number for odd, but not prime cycles. Odd cycles are not involutive graphs, but when the number n of vertices is not prime, the action of the automorphism group is not primitive. We are going to use a complete block system, with blocks of size equal to the least prime divisor of n .

Proposition 5.3 *If C_n is an odd cycle such that $n > 9$ and n is not prime, then $D_{\mathcal{G}}(C_n) = 2$.*

Proof: Let p be the least prime divisor of n . Since $n \geq 15$, we have $n = kp$, with $k \geq 5$. Let (x_1, \dots, x_n) be the cycle C_n . We define k disjoint subsets of vertices $V_j = \{x_{j+lk} | l \in \{0, \dots, p-1\}\}$, with $j \in \{1, \dots, k\}$. The set V_1 will play a particular role. We denote by Δ_i , with $i \in \{1, \dots, p\}$, the reflection with fixed point $x_{1+(i-1)k}$. Note that the automorphisms Δ_i , with $i \in \{1, \dots, p\}$, moves the sets V_j , with $j \in \{1, \dots, k\}$, in the following way: $\Delta_i(V_1) = V_1$ and for $j \neq 1$, $\Delta_i(V_j) = V_{k+2-j}$. Since k is odd, $j \neq k+2-j$ and $\Delta_i(V_j) \neq V_j$, if $j \neq 1$.

We describe the Gentle's strategy. He starts by coloring x_1 with color 1. More precisely, he chooses the fixed point of Δ_1 .

- If the Rascal colors a vertex x in V_1 , he answers by coloring the vertex $\Delta_1(x)$ in V_1 with the other color. Note that since the Gentle starts the game by coloring the fixed point of Δ_1 , this is always possible. Therefore, Δ_1 is broken and there will be exactly $1 + \frac{p-1}{2}$ vertices colored with 1 in V_1 at the end of the game.
- If the Rascal plays in V_j , with $j \neq 1$, then he plays in V_{k+2-j} . Except if the Rascal just colors the penultimate vertex of V_j . In that case he plays in V_j too, and colors the last vertex in a way the parity of the number of 1 in V_j is not the same as the parity of $1 + \frac{p-1}{2}$. The vertex he chooses to color in V_{k+2-j} is determined as follows. If at least one reflection Δ_i , with $i \in \{2, \dots, p\}$, is not already broken by the coloring, he chooses one of them, say Δ_{i_0} . We call V_j' the set of already colored vertices in V_j . There is exactly one more colored vertex in V_j than in V_{k+2-j} . Hence $\Delta_{i_0}(V_j')$ has at least an uncolored vertex, say u . The Gentle colors u such as u and $\Delta_{i_0}^{-1}(u)$ are not of the same color. If all the symmetries Δ_i are already broken, he plays randomly in V_{k+2-j} .

At the end of the game, there are exactly $1 + \frac{p-1}{2}$ vertices colored with 1 in V_1 . It is clear that Gentle's strategy leads him to play the last vertex of any set V_j . Hence, in all sets V_j , with $j \neq 1$, the number of 1s has not the same parity as in V_1 . This proves that the only possible color preserving automorphisms are those under which V_1 is stable. The Gentle plays $\frac{(k-1)p}{2}$ moves outside V_1 . There are only $k-1$ moves used to control the parity of 1s. The number of remaining moves to break the $p-1$ symmetries $\Delta_2, \dots, \Delta_p$ is $\frac{(p-2)(k-1)}{2}$. Since $k \geq p$ and $k \geq 5$, this number is greater or equal to $p-1$. This shows that all the symmetries under which V_1 is stable are broken. A color preserving automorphism σ is finally either a rotation under which V_1 is stable or the identity. Assume σ is a rotation. The subgroup $\langle \sigma \rangle$ acts naturally on V_1 . Since $|V_1|$ is prime, $\langle \sigma \rangle$ must be of order p and acts transitively on V_1 . This is impossible because V_1 is not monochromatic. In conclusion, σ must be the identity and the Gentle has a winning strategy with two colors. \square

For C_9 , the general strategy fails but two colors are still enough for Gentle to win.

Proposition 5.4 *We have $D_G(C_9) = 2$.*

Proof: Let (x_1, \dots, x_9) be the cycle of order 9 and c the coloring built throughout the game. The Gentle can play such that after his second move, the game is in one of the following cases.

Case 1: $c(x_1) = c(x_4) = 1$ and $c(x_7) = 2$. The Gentle applies the same strategy as in Proposition 5.3.

For the other cases, the Gentle defines three pairs S, O, O' of uncolored vertices. He plays in a way the pairs O and O' have vertices of different colors, and the pair S is monochromatic.

Case 2: $c(x_1) = c(x_2) = 1$ and $c(x_6) = 2$. The pairs are defined by $O = \{x_3, x_9\}$, $O' = \{x_5, x_7\}$ and $S = \{x_4, x_8\}$.

Case 3: $c(x_1) = c(x_3) = 1$ and $c(x_2) = 2$. He defines $S = \{x_4, x_6\}$, $O = \{x_5, x_8\}$ and $O' = \{x_7, x_9\}$.

Case 4: $c(x_1) = c(x_5) = 1$ and $c(x_3) = 2$. He makes $S = \{x_2, x_4\}$, $O = \{x_6, x_9\}$ and $O' = \{x_7, x_8\}$. \square

To finish the proof of Theorem 1.2, we deal with prime cycles. Note that, in that case, the automorphism group action is primitive. Once more, the small case C_5 does not fit the general strategy. By computation, we get $D_G(C_5) = 3$.

Proposition 5.5 *If C_p is a cycle of prime order $p > 5$, then $D_G(C_p) \leq 3$.*

Proof: Assume that three colors are allowed during the game. We begin the proof by showing the Rascal cannot win if he does not always play the same color as the one used by the Gentle just before. We denote respectively by n_1, n_2 and n_3 , the number of vertices colored with 1, 2 and 3 during the game. As long as the Rascal copies the color played by the Gentle just before, the three numbers n_1, n_2 and n_3 are even at the end of the Rascal's turn. The first time the Rascal plays a different color, two of the three numbers, n_1, n_2 and n_3 are odd at the end of his turn. Without loss of generality, we can say n_1 and n_2 are odd. Then the Gentle colors a vertex with 3. The numbers n_1, n_2 and n_3 are now all odd. Until the end of the game, the Gentle strategy is now to play the same color as the one the Rascal just used before him. In this way, he preserves the parity of n_1, n_2 and n_3 . At the end of the game, the coloring is such that n_1, n_2 and n_3 are odd. We show that this coloring is a distinguishing coloring. Let σ be a color preserving automorphism. The size of an orbit \mathcal{O} under the action of $\langle \sigma \rangle$ must divide $|\langle \sigma \rangle|$. But the cardinality of $\langle \sigma \rangle$ is either 1, 2 or p . Then, this orbit has 1, 2 or p as its size. None of n_1, n_2 and n_3 are null, so the coloring is not monochromatic and then the size of \mathcal{O} cannot be p . The automorphism σ cannot be a rotation because, for prime cycle, a rotation acts transitively and has an orbit of size p . Assume σ is a reflection. Then, there is exactly one fixed point. Without loss of generality, we suppose it is colored 1. All the vertices colored with 2 are in orbits of size 2. Since n_2 is not null, there are $k > 0$ such orbits. We get $n_2 = 2k$, which is a contradiction. Hence σ must be the identity.

We can now assume that the Rascal always copies the color played by the Gentle just before. Playing a different move will actually lead him to defeat, whatever has happened before. The winning strategy for the Gentle is as follows. He starts with 1 and his second move is to color a vertex with 2. The Rascal's second move must also be 2. Let σ_2 be the reflection which switches the two vertices colored with 2 and v the vertex fixed by σ_2 . If v is already colored, the Gentle plays 1 wherever he wants. If not, he colors v with 1. After that, the Gentle always uses the color 1, except for his last turn, where he will play 3. Since $p > 5$, he really has enough turns to play this 3. At the end of the game, the coloring has the following properties: exactly two vertices are colored with 2 and exactly one vertex is colored with 3. Since there

is only one vertex colored with 3 no rotation can preserve the coloring. Since there are only two vertices colored with 2, the only possible colors preserving reflection is σ_2 . But the Gentle played in a manner that the unique 3 is not the fixed point of this reflection. In conclusion, the coloring is distinguishing and the Gentle wins with 3 colors. \square

6 Conclusion and further works

In this article, we have defined two new game invariants of graphs, in the same spirit as the game chromatic number or as the game domination number. Since these invariants could be infinite, we started by giving some sufficient conditions for a graph to have infinite game distinguishing numbers. But a total characterization seems far to be found and sounds like a very challenging open problem. We propose the following conjecture.

Conjecture 6.1 *If G is a graph with no automorphism of order 2, then $D_G(G)$ and $D_{\mathcal{R}}(G)$ are both finite.*

After that, we defined a new class of graphs, the involutive graphs, for which we can bound quadratically the \mathcal{G} -game distinguishing number using informations on the classical one. Results on this class are then applied to compute the exact value of the game distinguishing number of hypercubes and even cycles. For the hypercube of dimension 4, determining if $D_{\mathcal{R}}(Q_4)$ is equal to 2 or 3 remains open.

For odd but not prime cycles, we were also able to compute the exact value of the two game invariants. The Gentle strategy made an intensive use of the imprimitivity of the automorphism group action. When the cycle is prime, we only know that D_G is bounded above by 3. This bound is sharp for small cases. Indeed, $D_G(C_5) = D_G(C_7) = 3$. But, it seems this is more related to the small size of these cycles than to the primality of their order. Supported by computer experimentation, we state the following conjecture.

Conjecture 6.2 *If C_p is a cycle of prime order $p \geq 11$, then $D_G(C_p) = 2$.*

Remark that for prime cycles, the automorphism group action is primitive. It would be interesting to study other graphs for which it is the case. It seems that the question is harder than in the imprimitive case.

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