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The Existence of Planar Hypotraceable Oriented Graphs

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A digraph is *traceable* if it has a path that visits every vertex. A digraph D is *hypotraceable* if D is not traceable but $D - v$ is traceable for every vertex $v \in V(D)$. It is known that there exists a planar hypotraceable digraph of order n for every $n \geq 7$, but no examples of planar hypotraceable oriented graphs (digraphs without 2-cycles) have yet appeared in the literature. We show that there exists a planar hypotraceable oriented graph of order n for every even $n \geq 10$, with the possible exception of $n = 14$.

Keywords: Hypotraceable, hypohamiltonian, planar, oriented graph

1 Introduction and background

We denote the vertex set, the arc set and the order of a digraph D by $V(D)$, $A(D)$ and $n(D)$, respectively. Any (undirected) graph may be viewed as a symmetric digraph (by regarding an edge as being equivalent to two oppositely directed arcs). A vertex v of a digraph is called a sink (source) if it does not have out-neighbours (in-neighbours). A digraph that does not contain any pair of oppositely directed arcs is called an *oriented* graph.

A digraph is *hamiltonian* if it has a Hamilton cycle, *i.e.*, a cycle that visits every vertex. A digraph D is *hypohamiltonian* if D is nonhamiltonian and $D - v$ is hamiltonian for every $v \in V(D)$.

A digraph is *traceable* if it has a Hamilton path, *i.e.*, a path that visits every vertex. A digraph D is *hypotraceable* if D is nontraceable but $D - v$ is traceable for every $v \in V(D)$. For undefined concepts we refer the reader to Bang-Jensen and Gutin (2009).

Hypotraceability in graph theory has an intriguing history. Gallai (1968) asked whether all longest paths in a graph share a common vertex. That was before hypotraceable graphs were discovered. (In a hypotraceable graph of order n the longest paths have $n - 1$ vertices each and they have an empty intersection.) Kapoor et al. (1968) asked whether hypotraceable graphs exist. Also, Kronk (1969) posed a problem in the American Mathematical Monthly entitled "Does there exist a hypotraceable graph?"

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In the discussion of the problem, Kronk states that he "feels strongly" that hypotraceable graphs do not exist. Four years later, Horton (1973) constructed a hypotraceable graph on 40 vertices. Thomassen (1974) presented a procedure by which any four hypohamiltonian graphs with minimum degree 3 may be combined to produce a hypotraceable graph. This resulted in the construction of a hypotraceable graph of order n for every $n \in \{34, 37, 39, 40\}$ and for all $n \geq 42$. A few years later, Thomassen (1976) also provided a hypotraceable graph of order 41.

Chvátal (1973) raised the problem of the existence of planar hypohamiltonian graphs and Grünbaum (1974) conjectured that such graphs do not exist. However, Thomassen (1974) constructed a planar hypohamiltonian graph of order 105 and presented a recursive procedure for constructing infinitely many planar hypohamiltonian graphs. Later, planar hypohamiltonian graphs of smaller order were found by Hatzel (1979) (order 57), Zamfirescu and Zamfirescu (2007) (order 48), Araya and Wiener (2011) (order 42), and Jooyandeh et al. (2016) (order 40). It was also shown in the last mentioned paper that the construction procedures of Thomassen (1976) yield planar hypohamiltonian graphs of all orders greater than 42, and planar hypotraceable graphs of order 154 and all orders greater than or equal to 156.

The importance of hypotraceable graphs was recognised when Grötschel (1980) showed that certain classes of hypotraceable graphs induce facets of the monotone symmetric travelling salesman polytope. Since no good (or even nearly good) characterisation of hypotraceable graphs has yet been found, it is unlikely that an explicit characterisation of these polytopes can ever be given. Grötschel and Wakabayashi (1981) also showed that hypotraceable digraphs contribute considerably to the difficulty of the asymmetric traveling salesman problem.

Thomassen (1978) showed that there exists a planar hypohamiltonian digraph of order n if and only if $n \geq 6$. Hypotraceable digraphs are easily obtained from hypohamiltonian digraphs by the following construction of Grötschel et al. (1980).

Construction 1 (Grötschel et al. (1980)) *Let D be a hypohamiltonian digraph of order n and let $y \in V(D)$. Now split y into two vertices x and z such that all the out-neighbours of y become out-neighbours of x and all the in-neighbours of y become in-neighbours of z. The result is a hypotraceable digraph of order $n+1$. We say that it is obtained from D by splitting the vertex y into a source and a sink.*

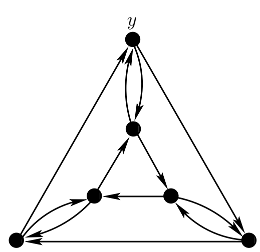

Fig. 1: The smallest planar hypohamiltonian digraph

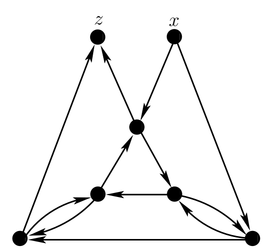

Fig. 2: The smallest planar hypotraceable digraph

The vertex splitting procedure, applied to the planar hypohamiltonian graphs constructed by Thomassen (1978), yields planar hypotraceable oriented graphs of every order from 7 upwards. Figures 1 and 2 depict the smallest planar hypohamiltonian digraph (see Thomassen (1978)) and the smallest planar hypotraceable digraph (see Grötschel et al. (1980)), respectively.

The existence of hypohamiltonian oriented graphs was established by Thomassen (1978). He showed that the Cartesian product $\overrightarrow{C}_k \times \overrightarrow{C}_{mk-1}$ of two directed cycles is a hypohamiltonian oriented graph if $k \geq 3$ and $m \geq 1$, and also that $\overrightarrow{C}_3 \times \overrightarrow{C}_{6k+4}$ is hypohamiltonian for each $k \geq 0$. Penn and Witte (1983) proved that the Cartesian product $\overrightarrow{C}_a \times \overrightarrow{C}_b$ is hypohamiltonian if and only if there is a pair of relatively prime positive integers m and n such that $ma + nb = ab - 1$. Recently, van Aardt et al. (2015) showed, by means of various other constructions, that there exists a hypohamiltonian oriented graph of order n for every $n \geq 9$. They also showed with an exhaustive computer search that there are no hypohamiltonian oriented graphs of order less than 9.

The vertex splitting procedure applied to hypohamiltonian oriented graphs yields hypotraceable oriented graphs of every order greater than 9. van Aardt et al. (2011) also found a hypotraceable oriented graph of order 8. It is obtained from a hypohamiltonian digraph that is not an oriented graph but has a vertex incident with all its 2-cycles, so splitting that vertex into a source and a sink destroys all the 2-cycles. Frick and Katrenič (2008) proved that there are no hypotraceable oriented graphs of order less than 8, and Burger (2013) showed by means of an exhaustive computer search that there does not exist a hypotraceable oriented graph of order 9. Thus there exists a hypotraceable oriented graph of order n if and only if $n = 8$ or $n \geq 10$.

Thomassen (1978) asked whether there exist planar hypohamiltonian oriented graphs. Recently, van Aardt et al. (2013) answered this question in the affirmative by constructing a planar hypohamiltonian oriented graph of order $9 + 12k$ for every $k \geq 0$. By adapting this construction, van Aardt et al. (2015) showed that, in fact, there exists a planar hypohamiltonian oriented graph of order $9 + 6k$ for every $k \geq 0$.

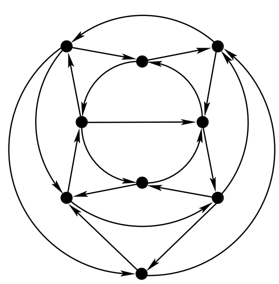

Fig. 3: A planar hypohamiltonian oriented graph of order 9

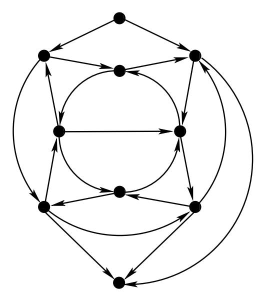

Fig. 4: A planar hypotraceable oriented graph of order 10

The next question to ask is whether there exist planar hypotraceable oriented graphs. Note that if any vertex of the hypohamiltonian oriented graph depicted in Figure 3 is split into a source and a sink, the result is nonplanar. In fact, no planar hypotraceable oriented graph is obtained by applying the vertex splitting procedure to any of the known planar hypohamiltonian oriented graphs. However, in the next section we construct, for each $k \geq 1$, a planar hypotraceable oriented graph of order $6k+4$ having a source and a sink. The smallest one (of order 10) is depicted in Figure 4. We also present a planar hypotraceable oriented graph of order 12 that has a source and a sink. Then, using a method devised by Grötschel et al. (1980), we combine pairs of the constructed planar hypotraceable oriented graphs to produce strong (strongly connected) planar hypotraceable oriented graphs of order $6k$ and $6k + 2$ for every $k \geq 3$. We conclude that there exists a planar hypotraceable oriented graph of order n for every even $n \geq 10$, with the possible exception of $n = 14$.

2 Constructions of planar hypotraceable oriented graphs

As in the case of planar hypohamiltonian oriented graphs (see van Aardt et al. (2013)), the circulant digraphs with jump set $\{1, -2\}$ form the basis of our constructions. In general, for an integer $n \geq 3$ and a jump set S of nonzero integers, the circulant digraph $\overrightarrow{C}_n(S)$ is defined as follows:

$V(\overrightarrow{C}_n(S)) = \{v_0, v_1, \ldots, v_{n-1}\}$,

$A(\overrightarrow{C}_n(S)) = \{(v_i, v_{i+j}) : 0 \leq i \leq n-1 \text{ and } j \in S\}$, where indices are taken modulo n.

For example, the circulant digraph $\overrightarrow{C}_{14}(1, -2)$ is depicted in Figure 5. We note that $\overrightarrow{C}_n(1, -2)$ is planar if and only if $n = 3$ or n is even.

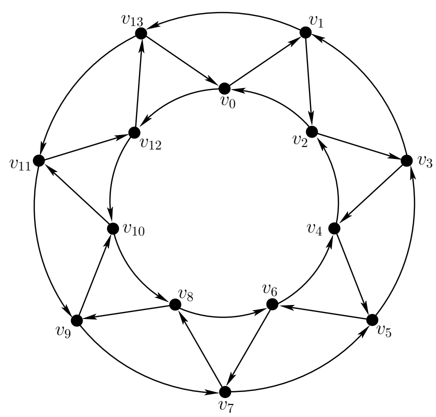

Fig. 5: The circulant digraph $\overrightarrow{C}_{14}(1, -2)$

Construction 2 *For each integer $k \geq 1$, let H_{6k+4} be the oriented graph obtained from the circulant digraph $\overrightarrow{C}_{6k+2}(1, -2)$ by deleting the arc $v_1 v_{6k+1}$ and adding the arc $v_{6k} v_2$, and then adding two new vertices x and z together with the arcs xv_1, xv_{6k+1}, $v_1 z$, $v_3 z$, $v_{6k-1} z$.*

The oriented graphs H_{10} and H_{16} are depicted in Figure 4 and Figure 6, respectively. We shall show that H_{6k+4} is a planar hypotraceable oriented graph for every $k \geq 1$. First, we present some notation and general observations concerning paths in $\overrightarrow{C}_n(1, -2)$.

Consider any pair of distinct vertices v_i, v_j in $\overrightarrow{C}_n(1, -2)$. We denote the $v_i - v_j$ path $v_i v_{i+1} \ldots v_j$ by $v_i \overrightarrow{C} v_j$. We note that $v_3 v_1 v_2 v_0$ is a $v_3 - v_0$ path of length three that use jumps $-2, 1, -2$ with the consecutive vertex set $\{v_0, v_1, v_2, v_3\}$. We can create a longer path with a consecutive vertex set by repeating this jumping pattern. In general, for any positive integer $t < n/3$, there is a $v_{i+3t} - v_i$ path in $\overrightarrow{C}_n(1, -2)$ with vertex set $\{v_i, v_{i+1}, \ldots v_{i+3t}\}$, namely the path

$$v_{i+3t} v_{i+3t-2} v_{i+3t-1} v_{i+3(t-1)} \ldots v_{i+3} v_{i+1} v_{i+2} v_i.$$

We denote this path by $v_{i+3t} \overleftarrow{C} v_i$.

Observation 1 *Let v_i, v_j be two distinct vertices in $\overrightarrow{C}_n(1,-2)$. Then the following hold.*

(a) $v_i \overrightarrow{C} v_j$ is the only $v_i - v_j$ path in $\overrightarrow{C}_n(1,-2)$ with vertex set $\{v_i, v_{i+1}, \ldots, v_j\}$.

(b) If $j-i$ (modulo n) is a multiple of 3, then $v_j \overleftarrow{C} v_i$ is the only $v_j - v_i$ path in $\overrightarrow{C}_n(1,-2)$ with vertex set $\{v_i, v_{i+1}, \ldots, v_j\}$.

(c) If $j-i$ (modulo n) is not a multiple of 3, then there is no $v_j - v_i$ path in $\overrightarrow{C}_n(1,-2)$ with vertex set $\{v_i, v_{i+1}, \ldots, v_j\}$.

We define the *parity of a vertex* in $\overrightarrow{C}_n(1,-2)$ as the parity of its index. We shall use the following result concerning Hamilton paths in $\overrightarrow{C}_n(1,-2)$.

Lemma 1 *Suppose n is even and let P be a Hamilton path in $\overrightarrow{C}_n(1,-2)$ such that its initial and terminal vertex have the same parity. Then any subpath of P containing only vertices of the same parity has length at most two.*

Proof: Let Q be a longest subpath of P that contains only vertices of the same parity. Then Q has less than $n/2$ vertices. Suppose Q is the path $v_i v_{i-2} \ldots v_{i-2j}$, with $j \geq 3$. Then $v_i v_{i+1}$, $v_{i-2j-1} v_{i-2j} \notin A(P)$ and $v_{i-r} v_{i-r+1} \notin A(P)$, for $r = 2, 3, \ldots, 2j-1$. Moreover, by the maximality of Q, $v_{i-2j} v_{i-2(j+1)}$, $v_{i+2} v_i \notin A(P)$.

Suppose v_i is the initial vertex of P. Then $v_{i-1} v_i \notin A(P)$ and v_{i-2j} is not the terminal vertex of P, since Q is not P. Hence $v_{i-2j} v_{i-2j+1} \in A(P)$, so $v_{i-2j+3} v_{i-2j+1} \notin A(P)$ and therefore v_{i-2j+3} is the terminal vertex of P, contradicting our assumption that the initial and terminal vertices of P have the same parity.

Hence v_i is not the initial vertex of P and similarly we can show that v_{i-2j} is not the terminal vertex of P. Hence $v_{i-1} v_i, v_{i-2j} v_{i-2j+1} \in A(P)$ and therefore $v_{i-1} v_{i-3}$, $v_{i-2j+3} v_{i-2j+1} \notin A(P)$. Then v_{i-3} is the initial vertex of P and v_{i-2j+3} is the terminal vertex of P. Thus P is the path $v_{i-3} v_{i-5} \ldots v_{i-2j+3}$, contradicting our assumption that P is a Hamilton path of $\overrightarrow{C}_n(1,-2)$. $\square$

For the particular case $n = 6k+2$ we have the following useful result.

Lemma 2 *For any integer $k \geq 0$ the initial and terminal vertices of any Hamilton path of $\overrightarrow{C}_{6k+2}(1,-2)$ have different parities.*

Proof: Let P be a Hamilton path in $\overrightarrow{C}_{6k+2}(1,-2)$ with initial vertex v_1 and terminal vertex v_ℓ and suppose ℓ is odd.

We now consider the following four cases.

Case 1: P contains the subpath $v_1 v_2 v_3$:
Then $v_3 v_1 \notin A(P)$ and hence $v_3 v_4 \in A(P)$. An inductive argument then shows that P is the path $v_1 v_2 v_3 v_4 \ldots v_{6k+1} v_0$, so in this case $\ell = 0$, contradicting our assumption that ℓ is odd.

Case 2: P contains the subpath $v_1 v_2 v_0$:
Then $v_0 v_1$, $v_1 v_{6k+1}$, $v_{6k+1} v_0 \notin A(P)$ and so $v_0 v_{6k}$, $v_{6k} v_{6k+1} \in A(P)$. Now $v_{6k-1} v_{6k}$, $v_{6k} v_{6k-2} \notin A(P)$. Hence $v_{6k+1} v_{6k-1}$, $v_{6k-1} v_{6k-3} \in A(P)$. Repeated application of this argument together with Observation 1 shows that P is the path $v_1 v_2\ v_0 \overleftarrow{C} v_5 v_3 v_4$, since $0-5 \equiv 6k-3$ mod $(6k+2)$. This again contradicts our assumption that ℓ is odd.

Case 3: P contains the subpath $v_1 v_{6k+1} v_0$:
A similar argument as above shows that P contains the subpath $v_1 \overleftarrow{C} v_3$. But then $v_2 \notin V(P)$, contradicting our assumption that P is a Hamilton path of $\overrightarrow{C}_{6k+2}(1,-2)$.

Case 4: P contains the subpath $v_1 v_{6k+1} v_{6k-1}$:
Then by Lemma 1, $v_{6k-1} v_{6k-3} \notin A(P)$. Also $v_{6k} v_{6k+1}, v_{6k+1} v_{6k+2} \notin A(D)$. Since P is not the path $v_1 v_{6k+1} v_{6k-1}$ it follows that $v_{6k-1} v_{6k}, v_{6k} v_{6k-2} \in A(D)$. A similar argument as above shows that P contains the subpath $v_1 v_{6k+1} v_{6k-1}\ v_{6k} v_{6k-2} \overleftarrow{C} v_4 v_2$. Hence P cannot contain both v_3 and v_{6k+2}, contradicting our assumption that P is a Hamilton path in $\overrightarrow{C}_{6k+2}(1,-2)$.

□

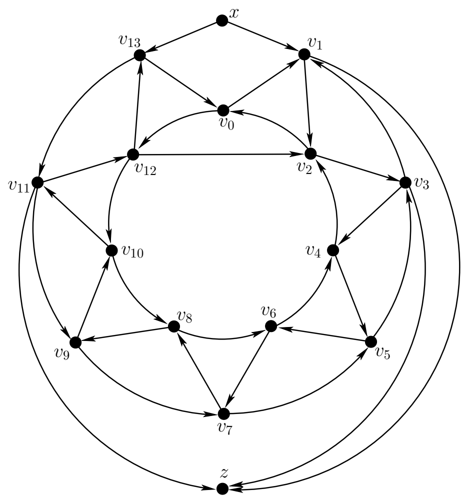

Fig. 6: H_{16}

Theorem 1 *H_{6k+4} is a planar hypotraceable oriented graph of order $6k+4$, for every integer $k \geq 1$.*

Proof: Let k be any positive integer. Then H_{6k+4} is obviously a planar oriented graph - see the planar depiction of H_{16} in Figure 6. We now prove that it is hypotraceable.

Since all the out-neighbours of x as well as all the in-neighbours of z are vertices with odd index, it follows from Lemma 2 that $H_{6k+4} - v_{6k} v_2$ is nontraceable.

Thus, if P is a Hamilton path of H_{6k+4}, then P contains the arc $v_{6k} v_2$. Hence P does not contain the arcs $v_1 v_2$, $v_4 v_2$, $v_{6k} v_{6k-2}$ and $v_{6k} v_{6k+1}$. This implies that $x v_{6k+1}$ and $v_1 z$ are, respectively, the initial and terminal arcs of P. Observe that P contains at most one of the arcs $v_2 v_0$ and $v_0 v_{6k}$ and at most one of the arcs $v_{6k+1} v_0$ and $v_0 v_1$. Hence P contains either the subpath $v_2 v_0 v_1 z$ or the subpath $x v_{6k+1} v_0 v_{6k}$. Suppose the former. Then P does not contain the arcs $v_{6k+1} v_0$ and $v_0 v_{6k}$. But then $v_{6k+1} v_{6k-1}$ and $v_{6k-1} v_{6k}$ are in P. Then P is the path $x v_{6k+1} v_{6k-1} v_{6k} v_2 v_0 v_1 z$, contradicting that H_{6k+4} has at least 10 vertices. By a symmetric argument we obtain a contradiction if $x v_{6k+1} v_0 v_{6k-2}$ is a subpath of P. This proves that H_{6k+4} is nontraceable.

Next we show that $H_{6k+4} - v$ is traceable for any vertex $v \in H_{6k+4}$. Since $H_{6k+4} - \{x, z\}$ is hamiltonian, $H_{6k+4} - x$ and $H_{6k+4} - z$ are both traceable. Using Observation 1 we now present a Hamilton path of the graph $H_{6k+4} - v_j$ for $j = 0, 1, \dots, 6k + 1$.

Subgraph	Hamilton path	values of i
$H_{6k+4} - v_0$	$xv_{6k+1}\overleftarrow{C}v_1z$	
$H_{6k+4} - v_1$	$xv_{6k+1}v_0v_{6k}v_2\overrightarrow{C}v_{6k-1}z$	
$H_{6k+4} - v_{6i+1}$	$xv_1v_2v_0\overleftarrow{C}v_{6i+2}v_{6i}\overleftarrow{C}v_3z$	$i = 1, \dots, k$
$H_{6k+4} - v_{6i+2}$	$xv_{6k+1}v_0v_{6k}\overleftarrow{C}v_{6i+3}v_{6i+1}\overleftarrow{C}v_1z$	$i = 0, \dots, k$
$H_{6k+4} - v_{6i+3}$	$xv_{6k+1}\overleftarrow{C}v_{6i+4}v_{6i+2}\overleftarrow{C}v_2v_0v_1z$	$i = 0, \dots, k$
$H_{6k+4} - v_{6i+4}$	$xv_1v_2v_0\overleftarrow{C}v_{6i+5}v_{6i+3}\overleftarrow{C}v_3z$	$i = 0, \dots, k$
$H_{6k+4} - v_{6i+5}$	$xv_{6k+1}v_0v_{6k}\overleftarrow{C}v_{6i+6}v_{6i+4}\overleftarrow{C}v_1z$	$i = 0, \dots, k$
$H_{6k+4} - v_{6i}$	$xv_{6k+1}\overleftarrow{C}v_{6i+1}v_{6i-1}\overleftarrow{C}v_2v_0v_1z$	$i = 1, \dots, k$

□

A computer search showed that every planar hypotraceable oriented graph of order 10 contains H_{10} as a spanning subdigraph. From the characterisation of hypotraceable oriented graphs of order 8 presented by van Aardt et al. (2011), we note that no hypotraceable oriented graph of order 8 is planar. Burger (2013) showed by means of an exhaustive computer search that there does not exist a hypotraceable oriented graph of order 9. Hence H_{10} is the planar hypotraceable oriented graph of smallest order and size.

For each $k \geq 1$, the graph H_{6k+4} is an arc-minimal hypotraceable oriented graph, *i.e.*, removing any arc destroys the hypotraceability. This follows from the following observations and the fact that a hypotraceable oriented graph does not contain a vertex with in- or out-degree 1:

Any Hamilton path in $H_{6k+4} - v_1$ contains both the arcs $v_{6k}v_2$ and $v_{6k-1}z$,
Any Hamilton path in $H_{6k+4} - v_2$ contains the arc v_3v_1,
Any Hamilton path in $H_{6k+4} - v_4$ contains the arc v_3z.

A computer search (for small k) showed that the digraph obtained from H_{6k+4} by adding any of the arcs $\{v_{2i+1}z : i = 2, \dots, 3k + 1\}$ is also a planar hypotraceable oriented graph. We can prove this analytically in general, but the proof is tedious and therefore omitted.

Figure 7 depicts an arc-minimal planar hypotraceable oriented graph of order 12, which was found by computer.

We now use the following construction of Grötschel and Wakabayashi (1984) to construct strong planar hypotraceable oriented graphs.

Construction 3 (Grötschel and Wakabayashi (1984)) *For $i = 1, 2$ let T_i be a hypotraceable digraph of order n_i, with a source x_i and a sink z_i. Form the disjoint union of T_1 and T_2. Then identify x_1 and z_2 to a single vertex and identify z_1 and x_2 to a single vertex. The result, which we denote by $T_1 * T_2$, is a strong hypotraceable digraph of order $n_1 + n_2 - 2$.*

Note that if, in Construction 3, each of T_1 and T_2 is a planar oriented graph that can be depicted with the source and sink in the same face, then $T_1 * T_2$ is also planar. Thus, if k_1 and k_2 are any two nonnegative integers and $T_i = H_{6k_i+4}$ for $i = 1, 2$, then $T_1 * T_2$ is a strong planar hypotraceable oriented graph of order $6(k_1 + k_2) + 6$. If T_1 is the planar hypotraceable graph of order 12 depicted in Figure 7 and

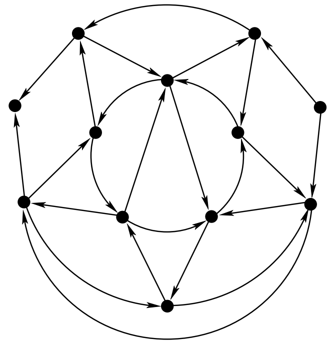

Fig. 7: A planar hypotraceable oriented graph of order 12

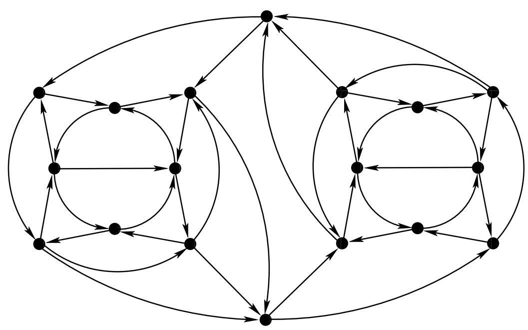

Fig. 8: A strong planar hypotraceable oriented graph of order 18

$T_2 = H_{6k+4}$, $k \geq 1$, then $T_1 * T_2$ is a strong planar hypotraceable oriented graph of order $6k + 14$. Thus we have proved the following.

Theorem 2 *There exists a strong planar hypotraceable oriented graph of order* $6k$ *and of order* $6k + 2$ *for every integer* $k \geq 3$*.*

Theorem 3 *There exists a planar hypotraceable oriented graph of order* n *for all even* $n \geq 10$ *with the possible exception of* $n = 14$*.*

Figure 8 depicts the strong planar hypotraceable oriented graph of order 18 that is obtained by using two copies of H_{10} in Construction 3.

It is still an open question whether there exists a planar hypotraceable oriented graph of order 14 or one of odd order. We also do not know whether there is a strong planar hypotraceable oriented graph of order less than 18.

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