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Improved Expansion of Random Cayley Graphs

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Alon and Roichman (1994) proved that for every $\varepsilon > 0$ there is a finite $c(\varepsilon)$ such that for any sufficiently large group G, the expected value of the second largest (in absolute value) eigenvalue of the normalized adjacency matrix of the Cayley graph with respect to $c(\varepsilon)\log|G|$ random elements is less than ε. We reduce the number of elements to $c(\varepsilon)\log D(G)$ (for the same c), where $D(G)$ is the sum of the dimensions of the irreducible representations of G. In sufficiently non-abelian families of groups (as measured by these dimensions), $\log D(G)$ is asymptotically $(1/2)\log|G|$. As is well known, a small eigenvalue implies large graph expansion (and conversely); see Tanner (1984) and Alon and Milman (1984, 1985). For any specified eigenvalue or expansion, therefore, random Cayley graphs (of sufficiently non-abelian groups) require only half as many edges as was previously known.

Keywords: expander graphs, Cayley graphs, second eigenvalue, logarithmic generators

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1 Introduction

All groups considered in this paper are finite.

Definition 1 *Let G be a group, and $S \subset G$ be a multiset. The **Cayley graph** $X(G,S)$ is the multigraph on vertex set G, with n undirected edges connecting g and tg if t appears n times in the multiset union $S \sqcup S^{-1}$, where S^{-1} is the multiset $\{s^{-1} : s \in S\}$. The **normalized adjacency matrix** $A^*_{X(G,S)}$ is $1/(2|S|)$ times the adjacency matrix of $X(G,S)$.*

Definition 2 *Let M be an $n \times n$ matrix with real eigenvalues $x_1, \ldots, x_n$, where $|x_1| \geq \cdots \geq |x_n|$. Define $\lambda(M) = |x_1|$ and $\mu(M) = |x_2|$. Write $\mu(X(G,S))$ for $\mu(A^*_{X(G,S)})$.*

Definition 3 *Let $D(G)$ be the sum of the dimensions of the irreducible representations of G.*

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Observe that $|G|^{1/2} < D(G) \leq |G|$. The upper bound is met only by abelian groups but is approached also by other groups whose irreducible representations are mostly low-dimensional, such as dihedral groups. The lower bound is approached, in the sense that $\log D(G) \to (1/2)\log|G|$, by a variety of families of groups possessing mostly high-dimensional irreducible representations.

Examples:

(a) The affine group A_p over the prime field $GF(p)$. $|A_p| = p(p-1)$, while $D(A_p) = 2p-2$.

(b) The symmetric group S_n. $|S_n| = n!$, hence $\log|S_n| \in n\log n - O(n)$, while $D(S_n) \in e^{O(\sqrt{n})}\sqrt{n!}$, hence $\log D(S_n) \in (1/2)n\log n + O(\sqrt{n})$.

(For the upper bound on $D(S_n)$, take the number of irreducible representations of S_n times the maximum of their dimensions. The first of these is $p(n)$, the number of partitions of n, which has the asymptotic behavior $p(n) \sim \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{2n/3}}$. The second was shown by Vershik and Kerov (1985) to be bounded above by $e^{-k\sqrt{n}}\sqrt{n!}$ for a positive constant k.)

Theorem 1 *For any* $\varepsilon > 0$ *the following holds for every sufficiently large group* G. *Let* S *be a multiset of* $c(\varepsilon)\log D(G)$ *uniformly and independently sampled elements of* G, *for* $c(\varepsilon) = 4e/\varepsilon^2$. *Then we have* $\mathrm{E}[\mu(X(G,S))] < (1+o(1))\varepsilon$.

(Here and throughout $o(1)$ allows for a quantity tending to 0 for large $|G|$.) Russell and Landau (2004) have independently obtained a similar result.

As a detail note that in Alon and Roichman (1994), S is generated by sampling without repetition (i.e., S is a set), while we employ sampling with repetition. The principal benefit of this is to simplify the argument, but it also leads to some sharpening: the value of $c(\varepsilon)$ obtained in Alon and Roichman (1994) is slightly larger than given here, while substituting sampling with repetition into their argument leads to the same $c(\varepsilon)$.

2 Proof

The combinatorial outline of the proof follows that of Alon and Roichman; the heart of the improvement lies in taking a certain union bound over the irreducible representations, rather than over the entire regular representation, of the group.

2.1 Decomposition into irreducible representations

Fix a group G, and let S be a multiset of N elements of G. Let $T = S \sqcup S^{-1}$; let α be the element in the group algebra $\mathbb{C}[G]$ defined by:

$$\alpha = \sum_{t\in T}\frac{1}{|T|}t.$$

Let the operator L be the left-action of α on $\mathbb{C}[G]$. Its matrix representation with respect to the standard basis is the normalized adjacency matrix of $X(G,S)$. The Fourier Transform $\mathcal{F}$ is an algebra isomorphism from $\mathbb{C}[G]$ to $\bigoplus_{r=1}^{R}\mathcal{M}_r$, where R is the number of irreducible representations of G, and $\mathcal{M}_r = \mathrm{Mat}_{d_r\times d_r}(\mathbb{C})$. Hence the eigenvalues of L are the same as the eigenvalues of the left-action of $\mathcal{F}(\alpha)$ on $\bigoplus\mathcal{M}_r$. Explicitly,

$$\mathcal{F}(\alpha) = \bigoplus_{r=1}^{R} \left(\sum_{t \in T} \frac{1}{|T|} \rho_r(t) \right),$$

where $\rho_r : G \to \mathcal{M}_r$ are the (unitary) irreducible representations, expressed with respect to fixed bases. Focus on an arbitrary component r of $\mathcal{F}(\alpha)$: let $\Psi_r = (1/|T|) \sum_{t \in T} \rho_r(t)$.

Since Ψ_r is an average of unitary matrices, its eigenvalues are bounded in absolute value by 1.

Let ρ_1 be the one-dimensional trivial representation $\rho_1 : G \mapsto \mathbb{C}$. Then for any S, $\Psi_1 = 1$. Therefore, $\mu(X(G,S)) = \lambda(A)$, where A is the following block-diagonal matrix:

$$A = \begin{pmatrix} \Psi_2 & 0 & \ldots & 0 \\ 0 & \Psi_3 & \ldots & 0 \\ . & . & \ldots & . \\ 0 & 0 & \ldots & \Psi_R \end{pmatrix}.$$

2.2 From eigenvalues to random walks

Fact 1 *Let M be a square matrix with real eigenvalues. Then for every positive integer m,*

$$\lambda(M) \leq \left(\mathrm{Tr}(M^{2m}) \right)^{1/2m}.$$

Because of the symmetric construction of T, A is Hermitian. By convexity,

$$\mathrm{E}[\mu(X(G,S))] \leq \left(\mathrm{E}[\mathrm{Tr}(A^{2m})] \right)^{1/2m}.$$

Since A is block-diagonal, A^{2m} shares the same block structure, with blocks Ψ_i^{2m} ($2 \leq i \leq R$).

$$\begin{aligned} \mathrm{Tr}(A^{2m}) &= \sum_{r=2}^{R} \mathrm{Tr}(\Psi_r^{2m}) \\ &= \sum_{r=2}^{R} \left(\sum_{t_1,\ldots,t_{2m} \in T} \frac{\chi_r(t_1 \cdots t_{2m})}{|T|^{2m}} \right) \\ &= \sum_{r=2}^{R} \sum_{g \in G} \chi_r(g) \frac{N_g}{|T|^{2m}}, \end{aligned}$$

where χ_r is the character of ρ_r and N_g is the number of ways to produce g as a product of $2m$ (not necessarily distinct) elements of T.

Definition 4 *Let **RW** denote the following random walk process.*

(1) Choose a uniform random word of length $2m$ from the free monoid on the N letters $\{a_1, a_2, \ldots, a_N\}$ (e.g., $a_2 a_5 a_5^{-1} a_1^{-1} a_7 a_3$).

(2) Reduce the word in the free group (e.g., $a_2 a_5 a_5^{-1} a_1^{-1} a_7 a_3 \to a_2 a_1^{-1} a_7 a_3$).

(3) Uniformly and independently assign (not necessarily distinct) group elements to the letters that appear in the remaining word, and evaluate the product in G.

Let $\mathbf{RW}_g$ be the event that the result is g. $\Pr(\mathbf{RW}_g) = N_g/|T|^{2m}$, so

$$\mathrm{E}[\mathrm{Tr}(A^{2m})] \quad = \quad \sum_{g\in G} \Pr(\mathbf{RW}_g) \sum_{r=2}^{R} \mathrm{Re}\,\chi_r(g).$$

2.3 Mixing in the random walk

Definition 5 *Let ω be a reduced word as obtained via step (2) of process* **RW** *(definition 4). Say that ω has a* ***singleton*** *if there is an i such that the number of occurrences of a_i in ω plus the number of occurrences of a_i^{-1} in ω is exactly one.*

Let Ω be the event that the reduced word has a singleton. Now:

$$\begin{aligned} & \sum_{g\in G} \Pr(\mathbf{RW}_g) \sum_{r=2}^{R} \mathrm{Re}\,\chi_r(g) \\ = \quad & \sum_{g\in G} \Pr(\Omega \wedge \mathbf{RW}_g) \sum_{r=2}^{R} \mathrm{Re}\,\chi_r(g) + \sum_{g\in G} \Pr(\overline{\Omega} \wedge \mathbf{RW}_g) \sum_{r=2}^{R} \mathrm{Re}\,\chi_r(g) \\ \leq \quad & \sum_{g\in G} \Pr(\Omega \wedge \mathbf{RW}_g) \sum_{r=2}^{R} \mathrm{Re}\,\chi_r(g) + \Pr(\overline{\Omega}) D(G). \end{aligned} \tag{1}$$

Lemma 1 $\Pr(\boldsymbol{RW}_g|\Omega) = 1/|G|$.

Proof: In step (3) of **RW** (definition 4), assign the singleton element last; then, there will exist a unique group element that makes ω evaluate to g. □

Comment: This lemma replaces an upper bound of $1/|G| + O(m/G^2)$ in Alon and Roichman (1994), the additional term being the result of their requiring distinct assigments in step (3). This additional term leads in turn to an extra summand of e^{-b} in the analogue, in their work, of the center expression in Inequality (2).

By Lemma 1 and the orthogonality of characters, the first term of Bound (1) vanishes. Combining our inequalities:

$$\mathrm{E}[\mu(X(G,S))] \leq (\mathrm{E}[\mathrm{Tr}(A^{2m})])^{1/2m} \leq \Pr(\overline{\Omega})^{1/2m} D(G)^{1/2m}.$$

To bound $\Pr(\overline{\Omega})$, we follow the spirit of Alon and Roichman (1994) and define the following two events in terms of the quantity $M = 2m(1 - \log\log 2m/\log 2m)$:

(A) After step (2) of **RW** (definition 4), the length of the reduced word is less than M.

(B) After step (2) of **RW** (definition 4), the length of the reduced word is at least M, but there are no singletons.

Clearly, $\Pr(\overline{\Omega}) \le \Pr(A) + \Pr(B)$. Alon and Roichman (1994) produced these bounds:

$$\begin{aligned}\Pr(A) &\le 2^{2m}(2/N)^{m\log\log 2m/\log 2m}\\ \Pr(B) &\le 2^{M}(m/N)^{M/2}.\end{aligned}$$

Substituting $N = c(\varepsilon)\log D(G)$ and $2m = (1/b)\log D(G)$, for any constant b, we obtain an expression almost identical to one of Alon and Roichman (1994), except that $|G|$'s are replaced by $D(G)$'s:

$$\Pr(\overline{\Omega})^{1/2m} D(G)^{1/2m} \le (1+o(1))e^{b}\sqrt{\frac{2}{bc(\varepsilon)}} \le (1+o(1))\varepsilon \qquad (2)$$

where we use the choices $c(\varepsilon) = 4e/\varepsilon^2$ and $b = 1/2$. □

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