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Steiner distance in product networks

Yaping Mao[1,4*] Eddie Cheng[2,4] Zhao Wang[3†]

[1] *School of Mathematics and Statistics, Qinghai Normal University, Xining, Qinghai 810008, China*
[2] *Department of Mathematics and Statistics, Oakland University, Rochester, MI USA 48309*
[3] *College of Science, China Jiliang University, Hangzhou 310018, China*
[4] *Center for Mathematics and Interdisciplinary Sciences of Qinghai Province, Xining, Qinghai 810008, China*

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For a connected graph G of order at least 2 and $S \subseteq V(G)$, the *Steiner distance* $d_G(S)$ among the vertices of S is the minimum size among all connected subgraphs whose vertex sets contain S. Let n and k be two integers with $2 \leq k \leq n$. Then the *Steiner k-eccentricity* $e_k(v)$ of a vertex v of G is defined by $e_k(v) = \max\{d_G(S) \mid S \subseteq V(G),\ |S| = k,\ and\ v \in S\}$. Furthermore, the *Steiner k-diameter* of G is $sdiam_k(G) = \max\{e_k(v) \mid v \in V(G)\}$. In this paper, we investigate the Steiner distance and Steiner k-diameter of Cartesian and lexicographical product graphs. Also, we study the Steiner k-diameter of some networks.

Keywords: Distance; diameter; Steiner tree; Steiner distance; Steiner k-diameter; Cartesian product, lexicographical product.

1 Introduction

In this paper, we consider graphs that are undirected, finite and simple. We refer the readers to Bondy and Murty (2008) for graph theoretical notations and terminology that are not defined here. For a graph G, let $V(G)$, $E(G)$, and $\delta(G)$ denote the set of vertices, the set of edges and the minimum degree of G, respectively. We refer to $|V(G)|$ the order of the graph and $|E(G)|$ the size of the graph. The degree of a vertex v in G is denoted by $deg_G(v)$. In this paper, K_n, P_n, $K_{1,n-1}$ and C_n correspond to the complete graph of order n, the path of order n, the star of order n, and the cycle of order n, respectively. If $X \subseteq V(G)$, we use $G[X]$ to denote the subgraph induced by X. Similarly, if $F \subseteq E(G)$, let $G[F]$ denote the subgraph induced by F. If $X \subseteq V(G) \cup E(G)$, we use $G - X$ to denote the subgraph of G obtained from G by removing all the elements of X and the edges incident to vertices that are in X. If $X = \{x\}$, we write $G - x$ for notational simplicity. For $X, Y \subseteq V(G)$, we use $E_G[X, Y]$ to denote the set of edges of G with one end in X and the other end in Y. If $X = \{x\}$, we simply write $E_G[x, Y]$ for $E_G[\{x\}, Y]$. We divide our introduction into subsections to state the motivations of this paper.

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†Corresponding author

1.1 Distance and its generalizations

Distance is a fundamental concept in graph theory. Let G be a connected graph. The *distance* between two vertices u and v in G is the length of a shortest path between them, and it is denoted by $d_G(u, v)$. The *eccentricity* of v in G, denoted by $e_G(v)$ (or simply $e(v)$ if it is clear from the context), is $\max\{d_G(u, v) \mid u \in V(G)\}$. In addition, we define the *radius* $rad(G)$ and the *diameter* $diam(G)$ of G to be $rad(G) = \min\{e(v) \mid v \in V(G)\}$ and $diam(G) = \max\{e(v) \mid v \in V(G)\}$. It is a standard exercise to check that $rad(G) \leq diam(G) \leq 2rad(G)$. The *center* $C(G)$ of G is the subgraph induced by the vertices with eccentricity equal to the radius. For more details on distance, we refer to Buckley and Harary (1990); Goddard and Oellermann (2011).

We observe that the distance between two vertices u and v in G is equal to the minimum size of a connected subgraph of G containing both u and v. This suggests a generalization of the concept of distance. The Steiner distance of a graph, introduced by Chartrand et al. (1989) in 1989, is such a natural and nice generalization. Let S be a set of vertices in a graph $G(V, E)$ where $|S| \geq 2$. We define *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) to be a subgraph $T(V', E')$ of G that is a tree with $S \subseteq V'$. Moreover, the *Steiner distance* $d_G(S)$ of S in G (or simply the distance of S) is the minimum size among all connected subgraphs whose vertex sets contain S. (Set $d_G(S) = \infty$ when there is no S-Steiner tree in G.) We remark that if H is a connected subgraph of G such that $S \subseteq V(H)$ and $|E(H)| = d_G(S)$, then H is a tree. We further remark that $d_G(S) = \min\{e(T) \mid S \subseteq V(T)\}$, where T is subtree of G. Finally, if $S = \{u, v\}$, then $d_G(S) = d(u, v)$ is the classical distance between u and v. The following observation is obvious.

Observation 1.1 *Let G be a graph of order n and k be an integer with $2 \leq k \leq n$. If $S \subseteq V(G)$ and $|S| = k$, then $d_G(S) \geq k - 1$.*

Let n and k be two integers with $2 \leq k \leq n$. We define the *Steiner k-eccentricity* $e_k(v)$ of a vertex v of G to be $e_k(v) = \max\{d(S) \mid S \subseteq V(G), |S| = k, \ and \ v \in S\}$, the *Steiner k-radius* of G to be $srad_k(G) = \min\{e_k(v) \mid v \in V(G)\}$, and the *Steiner k-diameter* of G is $sdiam_k(G) = \max\{e_k(v) \mid v \in V(G)\}$. We remark that for every connected graph G that $e_2(v) = e(v)$ for all vertices v of G and that $srad_2(G) = rad(G)$ and $sdiam_2(G) = diam(G)$. It is not difficult to see the following observation.

Observation 1.2 *Let k, n be two integers with $2 \leq k \leq n$.*
(1) *If H is a spanning subgraph of G, then $sdiam_k(G) \leq sdiam_k(H)$.*
(2) *For a connected graph G, $sdiam_k(G) \leq sdiam_{k+1}(G)$.*

Chartrand et al. (2010) obtained the following upper and lower bounds of $sdiam_k(G)$.

Theorem 1.3 Chartrand et al. (2010) *Let k, n be two integers with $2 \leq k \leq n$, and let G be a connected graph of order n. Then $k - 1 \leq sdiam_k(G) \leq n - 1$. Moreover, the upper and lower bounds are sharp.*

Dankelmann et al. (1999) showed that $sdiam_k(G) \leq \frac{3|V(G)|}{\delta(G)+1} + 3k$. Ali et al. (2012) improved the bound and showed that $sdiam_k(G) \leq \frac{3|V(G)|}{\delta(G)+1} + 2k - 5$ where G is connected. Moreover, they showed that these bounds are asymptotically best possible via a construction.

1.2 Related concepts

Although we will not consider these related concepts in this paper, they provide a context of problems related to Steiner distance. As a generalization of the center of a graph, one defines the *Steiner k-center*

$C_k(G)$ $(k \geq 2)$ of a connected graph G to be the subgraph induced by the vertices v of G where $e_k(v) = srad_k(G)$. Oellermann and Tian (1990) showed that every graph is the k-center of some graph. Moreover, they showed that the k-center of a tree is a tree and they characterized those trees that are k-centers of trees. The *Steiner k-median* of G is the subgraph of G induced by the vertices of G of minimum Steiner k-distance. The papers Oellermann (1995, 1999); Oellermann and Tian (1990) contain important results for Steiner centers and Steiner medians. For more details on the Steiner distance parameters, we refer to the survey paper Mao and papers Ali (2013); Cáceresa et al. (2008); D'Atri and Moscarini (1988); Dankelmann and Entringer (2000); Dankelmann et al. (1999); Day et al. (1994); Goddard and Oellrmann (1994); Mao et al. (2018).

Let G be a k-connected graph and u, v be a pair of vertices of G. Let $P_k(u,v) = \{P_1, P_2, \cdots, P_k\}$ be a family of k internally vertex-disjoint paths between u and v and $l(P_k(u,v))$ be the length of the longest path in $P_k(u,v)$. Then the *k-distance* $d_k(u,v)$ between vertices u and v is the smallest $l(P_k(u,v))$ among all $P_k(u,v)$'s and the *k-diameter* $d_k(G)$ of G is the maximum k-distance $d_k(u,v)$ over all pairs u, v of vertices of G. The concept of k-diameter has its origin in the analysis of routings in networks as described by Chung (1987); Du et al. (1993); Hsu (1994); Hsu and Łuczak (1994); Meyer and Pradhan (1987).

Perhaps the most famous Steiner type problem is the Steiner tree problem. The original Steiner tree problem was stated for the Euclidean plane: Given a set of points on the plane, the goal is to connect these points, and possibly additional points, by line segments between some pairs of these points such that the total length of these line segments is minimized. The graph theoretical version Hakimi (1971); Levi (1971) is as follows: Given a graph and a set of vertices S, find a connected subgraph with minimum number of edges that contains S. This is, in general, an NP-hard problem Hwang et al. (1992). There is also a corresponding weighted version. Obviously, this has applications in computer science and electrical engineering. For example, a graph can be a computer network with vertices being computers and edges being links between them. Here the Steiner tree problem is to find a subnetwork containing these computers with the least number of links. We can replace processors by electrical stations for applications in electrical networks.

Li et al. (2016) gave such a concept. They defined the *k-center Steiner Wiener index* $SW_k(G)$ of the graph G to be

$$SW_k(G) = \sum_{S \subseteq V(G),\ |S|=k} d(S)\,.$$

For $k = 2$, it coincides with the ordinary Wiener index. One usually considers SW_k for $2 \leq k \leq n-1$. However, the above definition can be extended to $k = 1$ and $k = n$ as well where $SW_1(G) = 0$ and $SW_n(G) = n - 1$. There are other related concepts such as the Steiner Harary index. Both indices have chemical applications Furtula et al. (2016); Gutman et al. (2015). In addition, Gutman (2016) gave a generalization of the concept of degree distance, and then Mao and Das (2018) gave a generalization of the concept of Gutman index. We refer the readers to Furtula et al. (2016); Gutman et al. (2015); Gutman (2016); Li et al. (2016, 2017); Mao and Das (2018); Mao et al. (2016, 2017a,b) for details.

1.3 Products of graphs

The main focus of this paper is Steiner k-diameter of two products of graphs, namely, the Cartesian product and the lexicographic product. These are well-known products. See Hammack et al. (2011).

• The *Cartesian product* of two graphs G and H, written as $G \Box H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (g, h) and (g', h') are adjacent if and only if $g = g'$ and $(h, h') \in E(H)$, or

$h = h'$ and $(g, g') \in E(G)$.

• The *lexicographic product* of two graphs G and H, written as $G \circ H$, is defined as follows: $V(G \circ H) = V(G) \times V(H)$, and two distinct vertices (g, h) and (g', h') of $G \circ H$ are adjacent if and only if either $(g, g') \in E(G)$ or $g = g'$ and $(h, h') \in E(H)$.

It is easy to see that the Cartesian product is commutative, that is, $G \Box H$ is isomorphic to $H \Box G$. However, the lexicographic product is non-commutative.

Product networks are important as often the resulting graph inherits properties from its factors. Both the lexicographical product and the Cartesian product are important concepts. See Bao et al. (1998); Day and Al-Ayyoub (1997); Hammack et al. (2011); Ku et al. (2003).

Gologranc (2018) obtained a sharp lower bound for Steiner distance of Cartesian product graphs. We continue this study in Section 2 by obtaining a sharp upper bound for Steiner distance. In addition, we will also present sharp upper and lower bounds for Steiner k-diameter of Cartesian product graphs. In Section 3, we derive the results for Steiner distance and Steiner k-diameter of lexicographic product graphs, which strengthen a result given by Anand et al. (2012). In Section 4, we give some applications of our main results, and study the Steiner diameter of some important networks.

2 Results for Cartesian product

In this paper, let G and H be two graphs with $V(G) = \{g_1, g_2, \ldots, g_n\}$ and $V(H) = \{h_1, h_2, \ldots, h_m\}$, respectively. Then $V(G * H) = \{(g_i, h_j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$, where $*$ denotes the Cartesian product operation or lexicographical product operation. For $h \in V(H)$, we use $G(h)$ to denote the subgraph of $G * H$ induced by the vertex set $\{(g_i, h) \mid 1 \leq i \leq n\}$. Similarly, for $g \in V(G)$, we use $H(g)$ to denote the subgraph of $G \circ H$ induced by the vertex set $\{(g, h_j) \mid 1 \leq j \leq m\}$.

The following observation can be easily seen.

Observation 2.1 *Let G be a connected graph, and let $S \subseteq V(G)$ and $|S| = 3$. Let T be a minimal S-Steiner tree in G. Then the tree T satisfies one of the following conditions.*

• *T is a path;*

• *T is a subdivision of $K_{1,3}$.*

We start with the following basic result.

Lemma 2.2 Hammack et al. (2011) *Let G and H be two graphs, and let (g, h) and (g', h') be two vertices of $G \Box H$. Then*

$$d_{G \Box H}((g, h), (g', h')) = d_G(g, g') + d_H(h, h').$$

2.1 Steiner distance of Cartesian product graphs

Gologranc (2018) obtained the following lower bound for Steiner distance.

Lemma 2.3 Gologranc (2018) *Let $k \geq 2$ be an integer, and let G, H be two connected graphs. Let $S = \{(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_2}), \ldots, (g_{i_k}, h_{j_k})\}$ be a set of distinct vertices of $G \Box H$. Let $S_G = \{g_{i_1}, g_{i_2}, \ldots, g_{i_k}\}$ and $S_H = \{h_{j_1}, h_{j_2}, \ldots, h_{j_k}\}$. Then*

$$d_{G \Box H}(S) \geq d_G(S_G) + d_H(S_H).$$

We will show that the inequality in Lemma 2.3 can be equality if $k = 3$; shown in following Corollary 2.6. But, for general k ($k \geq 4$), from Lemma 2.3 and Corollary 2.6, one may conjecture that for two connected graphs G, H, $d_{G \Box H}(S) = d_G(S_G) + d_H(S_H)$, where $S = \{(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_2}), \ldots, (g_{i_k}, h_{j_k})\} \subseteq$

$V(G\Box H)$, $S_G=\{g_{i_1},g_{i_2},\ldots,g_{i_k}\}\subseteq V(G)$ and $S_H=\{h_{j_1},h_{j_2},\ldots,h_{j_k}\}\subseteq V(H)$.

Remark 1: Actually, the equality $d_{G\Box H}(S)=d_G(S_G)+d_H(S_H)$ is not true for $|S|\geq 4$. For example, let G be a tree with degree sequence $(3,2,1,1,1)$ and H be a path of order 5. Let $S=\{(g_1,h_1),(g_2,h_2),(g_3,h_3),(g_4,h_4)\}$ be a vertex set of $G\Box H$ shown in Fig.1. Then $d_G(S_G)=4$ for $S_G=\{g_1,g_2,g_3,g_4\}$, and $d_H(S_H)=4$ for $S_H=\{h_1,h_2,h_3,h_4\}$. One can check that there is no S-Steiner tree of size 8 in $G\Box H$, which implies $d_{G\Box H}(S)\geq 9$.

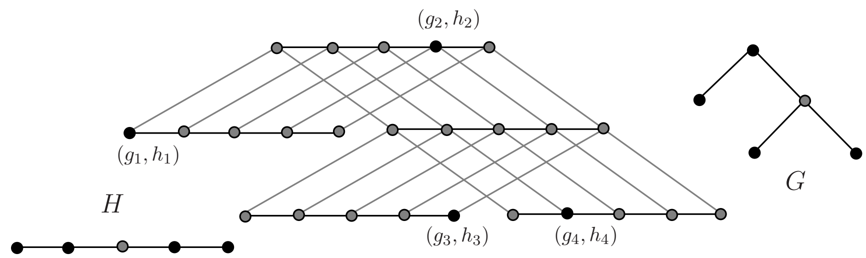

Fig. 1: Graphs for Remark 1.

Although the conjecture of such an ideal formula is not correct, it is possible to give a strong upper bound for general k $(k\geq 3)$. Remark 1 also indicates that obtaining a nice formula for the general case may be difficult. We now give such an upper bound of $d_{G\Box H}(S)$ for $S\subseteq V(G\Box H)$ and $|S|=k$.

Theorem 2.4 *Let k,m,n be three integers with $3\leq k\leq mn$, and let G,H be two connected graphs with $V(G)=\{g_1,g_2,\ldots,g_n\}$ and $V(H)=\{h_1,h_2,\ldots,h_m\}$. Let $S=\{(g_{i_1},h_{j_1}),(g_{i_2},h_{j_2}),\ldots,(g_{i_k},h_{j_k})\}$ be a set of distinct vertices of $G\Box H$, $S_G=\{g_{i_1},g_{i_2},\ldots,g_{i_k}\}$, and $S_H=\{h_{j_1},h_{j_2},\ldots,h_{j_k}\}$, where $S_G\subseteq V(G)$, $S_H\subseteq V(H)$ (S_G,S_H are both multi-sets). Then*

$$\begin{aligned} d_G(S_G)+d_H(S_H) &\leq d_{G\Box H}(S)\\ &\leq \min\{d_G(S_G)+(r+1)d_H(S_H),d_H(S_H)+(t+1)d_G(S_G)\}, \end{aligned}$$

where r,t $(0\leq r,t\leq k-3)$ are defined as follows.

- *Let X_G^i $(1\leq i\leq \binom{k}{3})$ be all the $(k-3)$-multi-subsets of $\{g_{i_1},g_{i_2},,\ldots,g_{i_k}\}$ in G, and let r_i be the numbers of distinct vertices in X_G^i $(1\leq i\leq \binom{k}{3})$, and let $r=\min\{r_i\,|\,1\leq i\leq \binom{k}{3}\}$.*
- *Let Y_H^j $(1\leq j\leq \binom{k}{3})$ be all the $(k-3)$-multi-subsets of $\{h_{j_1},h_{j_2},\ldots,h_{j_k}\}$ in H, and let t_j be the numbers of distinct vertices in Y_H^j $(1\leq j\leq \binom{k}{3})$, and let $t=\min\{t_j\,|\,1\leq j\leq \binom{k}{3}\}$.*

Proof. From Lemma 2.3, we have $d_{G\Box H}(S)\geq d_G(S_G)+d_H(S_H)$. By symmetry, we only need to show $d_{G\Box H}(S)\leq d_G(S_G)+(r+1)d_H(S_H)$. Recall that $V(G)=\{g_1,g_2,\ldots,g_n\}$ and $V(H)=\{h_1,h_2,\ldots,h_m\}$. Without loss of generality, we assume that $H(g_1),H(g_2),\ldots,H(g_a)$ be the H copies such that $|V(H(g_i))\cap S|\neq 0$, $1\leq i\leq a$. Then $(g_{i_1},h_{j_1}),(g_{i_2},h_{j_2}),\ldots,(g_{i_k},h_{j_k})\in\bigcup_{i=1}^a V(H(g_i))$, and hence we have the following cases to consider.

Case 1. For each $H(g_i)$ $(1\leq i\leq a)$, $|V(H(g_i))\cap S|\geq 2$.

Without loss of generality, let $V(H(g_1))\cap S=\{(g_{i_1},h_{j_1}),(g_{i_2},h_{j_2}),\ldots,(g_{i_s},h_{j_s})\}$, where $s\geq 2$. Thus, we have $(g_{i_p},h_{j_p})=(g_1,h_{j_p})$ for each p $(1\leq p\leq s)$, and $(g_{i_{s+1}},h_{j_{s+1}}),(g_{i_{s+2}},h_{j_{s+2}}),\ldots,$

$(g_{i_k}, h_{j_k}) \in \bigcup_{i=2}^{a} V(H(g_i))$. Note that $(g_1, h_{j_1}), (g_1, h_{j_2}), \ldots, (g_1, h_{j_s}) \in V(H(g_1))$. On one hand, since there is an S_H-Steiner tree of size $d_H(S_H)$ in H, it follows that there exists an Steiner tree of size $d_H(S_H)$ connecting

$$\begin{aligned} & \{(g_1, h_{j_1}), (g_1, h_{j_2}), \ldots, (g_1, h_{j_s})\} \cup \{(g_1, h_{j_{s+1}}), (g_1, h_{j_{s+2}}), \ldots, (g_1, h_{j_k})\} \\ = \quad & \{(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_2}), \ldots, (g_{i_s}, h_{j_s})\} \cup \{(g_1, h_{j_{s+1}}), (g_1, h_{j_{s+2}}), \ldots, (g_1, h_{j_k})\} \end{aligned}$$

in $H(g_1)$, say $T(g_1)$. For each i $(2 \leq i \leq k)$, let $T(g_i)$ be the Steiner tree in $H(g_i)$ corresponding to $T(g_1)$ in $H(g_1)$. Note that $T(g_i)$ $(1 \leq i \leq k)$ is the Steiner tree of size $d_H(S_H)$ connecting $\{(g_i, h_{j_1}), (g_i, h_{j_2}), \ldots, (g_i, h_{j_s}), (g_i, h_{j_{s+1}}), (g_i, h_{j_{s+2}}), \ldots, (g_i, h_{j_k})\}$ in $H(g_i)$. One can see that $(g_{i_{s+1}}, h_{j_{s+1}}), \ldots, (g_{i_k}, h_{j_k}) \in \bigcup_{i=2}^{a} V(T(g_i))$. On the other hand, since there is an S_G-Steiner tree of size $d_G(S_G)$ in G, it follows that there exists an Steiner tree of size $d_G(S_G)$ connecting $\{(g_1, h_{j_1}), (g_2, h_{j_1}), \ldots, (g_a, h_{j_1})\}$ in $G(h_{j_1})$, say $T(h_{j_1})$. Furthermore, the subgraph induced by the edges in $(\bigcup_{i=1}^{a} E(T(g_i))) \cup E(T(h_{j_1}))$ is an S-Steiner tree in $G\Box H$ (see Fig.2 (a)), and hence $d_{G\Box H}(S) \leq d_G(S_G) + ad_H(S_H)$.

From the definition of r, if $|V(H(g_i)) \cap S| \geq 4$ for each $H(g_i)$ $(1 \leq i \leq a)$, then $r = a$ and $d_{G\Box H}(S) \leq d_G(S_G) + rd_H(S_H)$. If there exists some $H(g_i)$ $(1 \leq i \leq a)$ such that $2 \leq |V(H(g_i)) \cap S| \leq 3$ for $H(g_i)$ $(1 \leq i \leq a)$, then $r = a - 1$ and $d_{G\Box H}(S) \leq d_G(S_G) + (r+1)d_H(S_H)$.

Case 2. There exists some $H(g_i)$ such that $|V(H(g_i)) \cap S| = 1$, where $1 \leq i \leq a$.

Without loss of generality, we assume that $|V(H(g_i)) \cap S| = 1$ for each i $(1 \leq i \leq x)$, where $1 \leq x \leq a$. For $x \neq a$, we have $|V(H(g_i)) \cap S| \geq 2$ for each i $(x+1 \leq i \leq a)$. One can see that

$$x = |\{H(g_i) \,|\, |V(H(g_i)) \cap S| = 1, \ 1 \leq i \leq a\}|.$$

Subcase 2.1. $x \geq 3$.

If $|\{h_{j_1}, h_{j_2}, \cdots, h_{j_x}\}| = 1$, then $h_{j_1} = h_{j_2} = \cdots = h_{j_x}$. Since there is an S_G-Steiner tree of size $d_G(S_G)$ in G, it follows that there exists an Steiner tree of size $d_G(S_G)$ connecting $\{(g_1, h_{j_1}), (g_2, h_{j_1}), \ldots, (g_a, h_{j_1})\}$ in $G(h_{j_1})$, say $T(h_{j_1})$. Since there is an S_H-Steiner tree of size $d_H(S_H)$ in H, it follows that there exists an Steiner tree of size $d_H(S_H)$ connecting $\{(g_{x+1}, h_{j_1})\} \cup \{(g_{x+1}, h_{j_{x+1}}), (g_{x+1}, h_{j_{x+2}}), \ldots, (g_{x+1}, h_{j_k})\}$ in $H(g_{x+1})$, say $T(g_{x+1})$. For each i $(x+2 \leq i \leq a)$, let $T(g_i)$ be the Steiner tree in $H(g_i)$ corresponding to $T(g_{x+1})$ in $H(g_{x+1})$. Note that $T(g_i)$ $(x+1 \leq i \leq a)$ is the Steiner tree of size $d_H(S_H)$ connecting $\{(g_i, h_{j_{x+1}}), (g_i, h_{j_{x+2}}), \ldots, (g_i, h_{j_k})\}$ in $H(g_i)$. Furthermore, the subgraph induced by the edges in $\left(\bigcup_{i=x+1}^{a} E(T(g_i))\right) \cup E(T(h_{j_1}))$ is an S-Steiner tree (see Fig.2 (b)), and hence $d_{G\Box H}(S) \leq d_G(S_G) + (a-x)d_H(S_H) \leq d_G(S_G) + (a-3)d_H(S_H)$. From the definition of r, we have $r = a - 3$, and hence $d_{G\Box H}(S) \leq d_G(S_G) + rd_H(S_H) \leq d_G(S_G) + (r+1)d_H(S_H)$, as desired.

If $|\{h_{j_1}, h_{j_2}, \cdots, h_{j_x}\}| = 2$, then we can assume that $h_{j_1} = h_{j_2} = \ldots = h_{j_s}$, $h_{j_{s+1}} = h_{j_{s+2}} = \ldots = h_{j_x}$, and $h_{j_1} \neq h_{j_x}$. Furthermore, we can assume that $s \geq 2$. Since there is an S_H-Steiner tree of size $d_H(S_H)$ in H, it follows that there is a Steiner tree of size $d_H(S_H)$ connecting $\{(g_{s+1}, h_{j_{s+1}}), (g_{s+1}, h_{j_{x+2}}), \ldots, (g_{s+1}, h_{j_k})\}$ in $H(g_{s+1})$, say $T(g_{s+1})$. For each i $(s+2 \leq i \leq a)$, let $T(g_i)$ be the Steiner tree in $H(g_i)$ corresponding to $T(g_{s+1})$ in $H(g_{s+1})$. Since there is an S_G-Steiner tree of size $d_G(S_G)$ in G, it follows that there exists an Steiner tree of size $d_G(S_G)$ connecting $\{(g_1, h_{j_1}), (g_2, h_{j_1}), \ldots, (g_a, h_{j_1})\}$ in $G(h_{j_1})$, say $T(h_{j_1})$. Then the subgraph induced by the edges in

$$\left(\bigcup_{i=s+1}^{a} E(H(g_i))\right) \cup E(T(h_{j_1}))$$

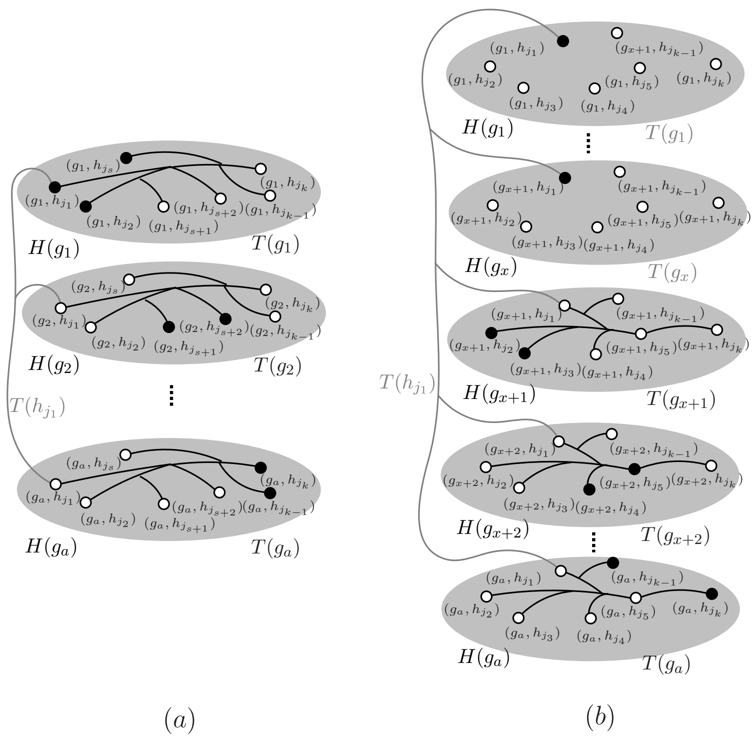

Fig. 2: Graphs for Cases 1 and 2 in the proof of Theorem 2.4.

is an S-Steiner tree in $G\Box H$, and hence $d_{G\Box H}(S)\leq d_G(S_G)+(a-s)d_H(S_H)\leq d_G(S_G)+(a-2)d_H(S_H)$. Since $r=a-3$, it follows that $d_{G\Box H}(S)\leq d_G(S_G)+(a-2)d_H(S_H)=d_G(S_G)+(r+1)d_H(S_H)$.

From now on, we assume $|\{h_{j_1},h_{j_2},\cdots,h_{j_x}\}|\geq 3$. Note that there is an S_H-Steiner tree of size $d_H(S_H)$ in H, say T. Without loss of generality, let $h_{j_1}\neq h_{j_2}\neq h_{j_3}$. Since $h_{j_1},h_{j_2},h_{j_3}\in V(T)$, it follows that there is a minimal subtree T' connecting $\{h_{j_1},h_{j_2},h_{j_3}\}$ in T. From Observation 2.1, T' is a path or T' is a subdivision of $K_{1,3}$. If T' is a path, then without loss of generality, we can assume h_{j_2} is the interval vertex of T'. Therefore, there are a unique (h_{j_1},h_{j_2})-path, say P^1, and a unique (h_{j_2},h_{j_3})-path, say P^2, in T'. If T' is a subdivision of $K_{1,3}$, then there exists a vertex in T', say $h^*\in V(H)\setminus\{h_{j_1},h_{j_2},h_{j_3}\}$, such that there are three paths Q^1,Q^2,Q^3 connecting h^* and h_{j_1},h_{j_2},h_{j_3}, respectively, in T'.

We first consider the case that T' is a path. On one hand, for each i $(1\leq i\leq k)$, let $T(g_i)$ be the Steiner tree in $H(g_i)$ corresponding to T in H. Note that $T(g_i)$ is the Steiner tree of size $d_H(S_H)$ connecting $\{(g_i,h_{j_1}),(g_i,h_{j_2}),\ldots,(g_i,h_{j_k})\}$ in $H(g_i)$. For each i $(1\leq i\leq 3)$, let $P^1(g_i)$ be the path in $H(g_i)$ corresponding to P^1 in H, and let $P^2(g_i)$ be the path in $H(g_i)$ corresponding to P^2 in H. On the other hand, since there is an S_G-Steiner tree of size $d_G(S_G)$ in G, it follows that there exists an Steiner tree of size $d_G(S_G)$ connecting $\{(g_1,h_{j_2}),(g_2,h_{j_2}),\ldots,(g_k,h_{j_2})\}$ in $G(h_{j_2})$, say $T(h_{j_2})$. Furthermore, the subgraph induced by the edges in

$$\left(\bigcup_{i=4}^{a}E(T(g_i))\right)\cup E(P^1(g_1))\cup E(P^2(g_3))\cup E(T(h_{j_2}))$$

is an S-Steiner tree in $G\Box H$ (see Fig.3 (a)), and hence $d_{G\Box H}(S)\leq d_G(S_G)+(a-2)d_H(S_H)$. Since $r=a-3$, it follows that $d_{G\Box H}(S)\leq d_G(S_G)+(r+1)d_H(S_H)$.

Next, we consider the case that T' is a subdivision of $K_{1,3}$. On one hand, for each i $(1\leq i\leq k)$, let $T(g_i)$ be the tree in $H(g_i)$ corresponding to T in H. Note that $T(g_i)$ is the Steiner tree of size $d_H(S_H)$ connecting $\{(g_i,h_{j_1}),(g_i,h_{j_2}),\ldots,(g_i,h_{j_k})\}$ in $H(g_i)$. For each i $(1\leq i\leq 3)$, let $Q^1(g_i)$ be the path in $H(g_i)$ corresponding to Q^1 in H, and let $Q^2(g_i)$ be the path in $H(g_i)$ corresponding to Q^2 in H, and let $Q^3(g_i)$ be the path in $H(g_i)$ corresponding to Q^3 in H. For each i $(1\leq i\leq k)$, let (g_i,h^*) be the path in $H(g_i)$ corresponding to h^* in H.

On the other hand, since there is an S_G-Steiner tree of size $d_G(S_G)$ in G, it follows that there exists an Steiner tree of size $d_G(S_G)$ connecting $\{(g_1,h^*),(g_2,h^*),\ldots,(g_k,h^*)\}$ in $G(h^*)$, say $T(h^*)$. Furthermore, the subgraph induced by the edges in

$$\left(\bigcup_{i=4}^{a}E(T(g_i))\right)\cup E(Q^1(g_1))\cup E(Q^2(g_2))\cup E(Q^3(g_3))\cup E(T(h^*))$$

is an S-Steiner tree in $G\Box H$ (see Fig.3 (b)), and hence $d_{G\Box H}(S)\leq d_G(S_G)+(a-2)d_H(S_H)$. Since $r=a-3$, it follows that $d_{G\Box H}(S)\leq d_G(S_G)+(r+1)d_H(S_H)$.

Subcase 2.2. $x=1$ or $x=2$.

Without loss of generality, let $|V(H(g_1))\cap S|=1$ and $(g_{i_1},h_{j_1})=(g_1,h_{j_1})$. Since there is an S_G-Steiner tree of size $d_G(S_G)$ in G, it follows that there exists an Steiner tree of size $d_G(S_G)$ connecting $\{(g_1,h_{j_1}),(g_2,h_{j_1}),\ldots,(g_a,h_{j_1})\}$ in $G(h_{j_1})$, say $T(h_{j_1})$. Since there is an S_H-Steiner tree of size $d_H(S_H)$ in H, it follows that there exists an Steiner tree of size $d_H(S_H)$ connecting $\{(g_2,h_{j_1}),(g_2,h_{j_2}),$

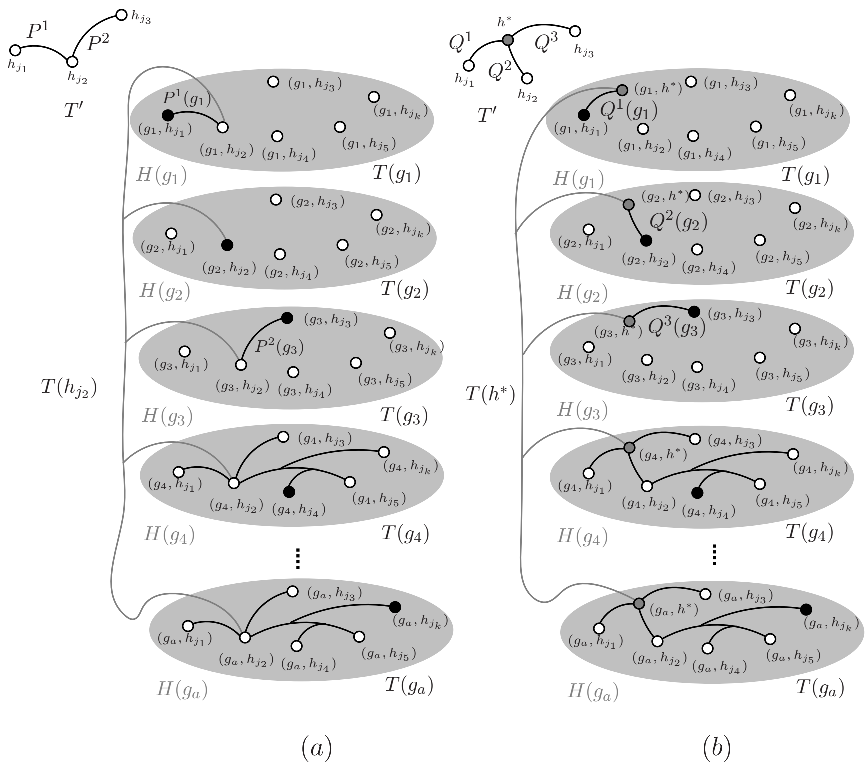

Fig. 3: Graphs for Subcase 2.1 in the proof of Theorem 2.4.

$\ldots, (g_2, h_{j_k})\}$ in $H(g_2)$, say $T(g_2)$. For each i $(3 \leq i \leq a)$, let $T(g_i)$ be the Steiner tree in $H(g_i)$ corresponding to $T(g_2)$ in $H(g_2)$. Note that $T(g_i)$ $(2 \leq i \leq a)$ is the Steiner tree of size $d_H(S_H)$ connecting $\{(g_i, h_{j_1}), (g_i, h_{j_2}), \ldots, (g_i, h_{j_k})\}$ in $H(g_i)$. Furthermore, the subgraph induced by the edges in $(\bigcup_{i=2}^{a} E(T(g_i))) \cup E(T(h_{j_1}))$ is an S-Steiner tree, and hence $d_{G\Box H}(S) \leq d_G(S_G) + (a-1)d_H(S_H)$. From the definition of r, we have $r = a-2$ or $r = a-1$, and hence $d_{G\Box H}(S) \leq d_G(S_G) + (r+1)d_H(S_H)$, as desired.

From the above argument, we conclude that $d_{G\Box H}(S) \leq \min\{d_G(S_G) + (r+1)d_H(S_H), d_H(S_H) + (t+1)d_G(S_G)\}$, as desired. $\Box$

The following corollaries are immediate from Theorem 2.4.

Corollary 2.5 *Let G, H be two connected graphs of order n, m, respectively. Let k be an integer with $3 \leq k \leq mn$. Let $S = \{(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_2}), \ldots, (g_{i_k}, h_{j_k})\}$ be a set of distinct vertices of $G\Box H$. Let $S_G = \{g_{i_1}, g_{i_2}, \ldots, g_{i_k}\}$ and $S_H = \{h_{j_1}, h_{j_2}, \ldots, h_{j_k}\}$. Then*

$$\begin{aligned} d_G(S_G) + d_H(S_H) &\leq d_{G\Box H}(S) \\ &\leq \min\{d_G(S_G) + (k-2)d_H(S_H), d_H(S_H) + (k-2)d_G(S_G)\} \\ &= d_G(S_G) + d_H(S_H) + (k-3)\min\{d_H(S_H), d_G(S_G)\}. \end{aligned}$$

Corollary 2.6 *Let G, H be two connected graphs, and let (g,h), (g',h') and (g'',h'') be three vertices of $G\Box H$. Let $S_G = \{g, g', g''\}$, $S_H = \{h, h', h''\}$, and $S = \{(g,h), (g',h'), (g'',h'')\}$. Then*

$$d_{G\Box H}(S) = d_G(S_G) + d_H(S_H)$$

To show the sharpness of the above upper and lower bound, we consider the following example.

Example 1: (1) For $k = 3$, from Corollary 2.6, we have $d_{G\Box H}(S) = d_G(S_G) + d_H(S_H)$, which implies that the upper and lower bounds in Corollary 2.5 and Theorem 2.4 are sharp.

(2) Let $G = P_n$ and $H = K_{1,m-1}$, where $P_n = g_1 g_2 \cdots g_n$, $h_1, h_2, \cdots, h_{m-1}$ are the leaves of H, and h_m is the center of H. Choose $S = \{(g_1,h_1), (g_1,h_2), (g_1,h_m)\} \cup \{(g_n,h_1), (g_n,h_2), (g_n,h_m)\} \cup \{(g_i,h_1), (g_i,h_2), (g_i,h_m) \,|\, 2 \leq i \leq x-2\}$, where $4 \leq x \leq n$. Then $d_G(S_G) = n-1$, $d_H(S_H) = 2$, $r = x-1$, $t = 3$ and $d_{G\Box H}(S) = n - 1 + 2x = n - 1 + 2 + \min\{2(x-1), 3(n-1)\} = d_G(S_G) + d_H(S_H) + \min\{rd_H(S_H), td_G(S_G)\}$, which implies that the upper bound in Corollary 2.5 are sharp.

2.2 Steiner diameter of Cartesian product graphs

For Steiner k-diameter, we have the following.

Theorem 2.7 *Let k, m, n be an integer with $3 \leq k \leq mn$ and $n \leq m$. Let G, H be two connected graphs of order n, m, respectively.*

(1) *If $k \leq n$, then*

$$\begin{aligned} & sdiam_k(G) + sdiam_k(H) \\ \leq\ & sdiam_k(G\Box H) \\ \leq\ & sdiam_k(G) + sdiam_k(H) + (k-3)\min\{sdiam_k(G), sdiam_k(H)\}. \end{aligned}$$

(2) *If* $n < k \leq m$, *then*

$$\begin{aligned} n-1+sdiam_k(H) &\leq sdiam_k(G\Box H) \\ &\leq n-1+sdiam_k(H)+(k-3)\min\{n-1, sdiam_k(H)\}. \end{aligned}$$

(3) *If* $m < k \leq mn$, *then*

$$n+m-2 \leq sdiam_k(G\Box H) \leq m-1+(k-2)(n-1).$$

(4) *If* $mn-\kappa(G\Box H)+1 \leq k \leq mn$, *then* $sdiam_k(G\Box H) = k-1$.

Proof. We first consider all the upper bounds in this theorem. From the definition of $sdiam_k(G\Box H)$, there exists a vertex subset $S \subseteq V(G\Box H)$ with $|S| = k$ such that $d_{G\Box H}(S) = sdiam_k(G\Box H)$. Let $S = \{(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_2}), \ldots, (g_{i_k}, h_{j_k})\}$, and let $S_G = \{g_{i_1}, g_{i_2}, \ldots, g_{i_k}\}$ and $S_H = \{h_{j_1}, h_{j_2}, \ldots, h_{j_k}\}$. From Corollary 2.5, we have

$$sdiam_k(G\Box H) = d_{G\Box H}(S) \leq \min\{d_G(S_G)+(k-2)d_H(S_H), (k-2)d_G(S_G)+d_H(S_H)\}.$$

For (1), since $k \leq n$, it follows that $d_G(S_G) \leq sdiam_k(G)$ and $d_H(S_H) \leq sdiam_k(H)$, and hence

$$\begin{aligned} & sdiam_k(G\Box H) = d_{G\Box H}(S) \\ \leq\ & \min\{d_G(S_G)+(k-2)d_H(S_H), (k-2)d_G(S_G)+d_H(S_H)\} \\ \leq\ & \min\{sdiam_k(G)+(k-2)sdiam_k(H), (k-2)sdiam_k(G)+sdiam_k(H)\} \\ =\ & sdiam_k(G)+sdiam_k(H)+(k-3)\min\{sdiam_k(G), sdiam_k(H)\}. \end{aligned}$$

For (2), since $n < k \leq m$, it follows that $d_G(S_G) \leq n-1$ and $d_H(S_H) \leq sdiam_k(H)$, and hence

$$\begin{aligned} sdiam_k(G\Box H) &= d_{G\Box H}(S) \\ &\leq \min\{d_G(S_G)+(k-2)d_H(S_H), (k-2)d_G(S_G)+d_H(S_H)\} \\ &\leq \min\{n-1+(k-2)sdiam_k(H), (k-2)(n-1)+sdiam_k(H)\} \\ &= n-1+sdiam_k(H)+(k-3)\min\{n-1, sdiam_k(H)\}. \end{aligned}$$

For (3), since $m < k \leq mn$, it follows that $d_G(S_G) \leq n-1$ and $d_H(S_H) \leq m-1$, and hence

$$\begin{aligned} sdiam_k(G\Box H) &= d_{G\Box H}(S) \\ &\leq \min\{d_G(S_G)+(k-2)d_H(S_H), (k-2)d_G(S_G)+d_H(S_H)\} \\ &\leq \min\{n-1+(k-2)(m-1), (k-2)(n-1)+(m-1)\} \\ &= m-1+(k-2)(n-1). \end{aligned}$$

Next, we consider the lower bounds in this theorem. For (1), we suppose $k \leq n \leq m$. From the definition of $sdiam_k(G)$, it follows that there exists a vertex subset $S_G \subseteq V(G)$ with $|S_G| = k$ such that $d_G(S_G) = sdiam_k(G)$. Similarly, there exists a vertex subset $S_H \subseteq V(H)$ with $|S_H| = k$ such that $d_H(S_H) = sdiam_k(H)$. Without loss of generality, let $S_G = \{g_1, g_2, \ldots, g_k\}$ and $S_H =$

$\{h_1, h_2, \ldots, h_k\}$. Then $S = \{(g_1, h_1), (g_2, h_2), \ldots, (g_k, h_k)\} \subseteq V(G\square H)$ and $|S| = k$. From Lemma 2.3 and the definition of Steiner k-diameter, we have

$$sdiam_k(G) + sdiam_k(H) = d_G(S_G) + d_H(S_H) \leq d_{G\square H}(S) \leq sdiam_k(G\square H).$$

For (2), we suppose $n < k \leq m$. Let $S = \{(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_2}), \ldots, (g_{i_k}, h_{j_k})\}$ be a set of distinct vertices of $G\square H$ such that $V(G) \subseteq \{g_{i_1}, g_{i_2}, \ldots, g_{i_k}\} = S_G$ and $d_H(S_H) = sdiam_k(H)$, where $S_H = \{h_{j_1}, h_{j_2}, \ldots, h_{j_k}\}$. From Lemma 2.3, we have

$$n - 1 + sdiam_k(H) = d_G(S_G) + d_H(S_H) \leq d_{G\square H}(S) \leq sdiam_k(G\square H).$$

For (3), we suppose $m < k \leq mn$. Let $S = \{(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_2}), \ldots, (g_{i_k}, h_{j_k})\}$ be a set of distinct vertices of $G\square H$ such that $V(G) \subseteq S_G$ and $V(H) \subseteq S_H$, where $S_G = \{g_{i_1}, g_{i_2}, \ldots, g_{i_k}\}$ and $S_H = \{h_{j_1}, h_{j_2}, \ldots, h_{j_k}\}$. From Lemma 2.3, we have

$$n + m - 2 = (n-1) + (m-1) = d_G(S_G) + d_H(S_H) \leq d_{G\square H}(S) \leq sdiam_k(G\square H),$$

as desired.

For (4), we suppose $mn - \kappa(G\square H) + 1 \leq k \leq mn$. For any $S \subseteq V(G\square H)$ with $|S| = k$, we have $|V(G)| - |S| \leq \kappa(G\square H) - 1$, and hence $G[S]$ is connected. Therefore, we have $d_{G\square H}(S) \leq k - 1$, and hence $sdiam_k(G\square H) \leq k - 1$ by the arbitrariness of S. So, we have $sdiam_k(G\square H) = k - 1$. □

The following corollary is immediate from Theorem 2.7.

Corollary 2.8 *Let G, H be two connected graphs of order at least 3. Then*

$$sdiam_3(G\square H) = sdiam_3(G) + sdiam_3(H).$$

To show the sharpness of the above upper and lower bound, we consider the following example.

Example 2: (1) For $k = 3$, from Corollary 2.8, we have $sdiam_k(G\square H) = sdiam_k(G) + sdiam_k(H)$, which implies that the upper and lower bounds in Theorem 2.7 are sharp.

(2) Let $G = P_n$ and $H = P_m$ with $5 \leq n \leq m$. Then $sdiam_4(G) = n - 1$, $sdiam_4(H) = m - 1$ and $sdiam_4(G\square H) = 2(n-1) + (m-1)$, which implies that all the upper bounds in Theorem 2.7 are sharp.

3 Results for lexicographic product

From the definition, the lexicographic product graph $G \circ H$ is the graph obtained by replacing each vertex of G by a copy of H and replacing each edge of G by a complete bipartite graph $K_{m,m}$, where $m = |V(H)|$.

Lemma 3.1 Hammack et al. (2011) *Let G and H be two graphs, and let (g, h) and (g', h') be two vertices of $G \circ H$. Then*

$$d_{G\circ H}((g,h),(g',h')) = \begin{cases} d_G(g,g'), & \text{if } g \neq g'; \\ d_H(h,h'), & \text{if } g = g' \text{ and } deg_G(g) = 0; \\ \min\{d_H(h,h'), 2\}, & \text{if } g = g' \text{ and } deg_G(g) \neq 0. \end{cases}$$

A *weak homomorphism* $\varphi : G \to H$ is a map $\varphi : V(G) \to V(H)$ for which $uv \in E(G)$ implies $\varphi(u)\varphi(v) \in E(H)$ or $\varphi(u) = \varphi(v)$. Observe that the projection $p : G \circ H \to G$ is a weak homomorphism. For more details, we refer to Hammack et al. (2011) (p.32,p.57).

Lemma 3.2 Hammack et al. (2011) *Let G and H be two graphs, and let (g, h) and (g', h') be two vertices of $G \circ H$. Then*

$$d_{G\circ H}((g, h), (g', h')) \geq d_G(g, g').$$

3.1 Steiner distance of lexicographic product graphs

The following lemma is a generalization of Lemma 3.2, which is a natural lower bound of $d_{G\circ H}(S)$ for $S \subseteq V(G \circ H)$ and $|S| = k$.

Lemma 3.3 *Let $k \geq 2$ be an integer, G be a connected graph, and H be a graph. Let $S = \{(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_2}), \ldots, (g_{i_k}, h_{j_k})\}$ be a set of distinct vertices of $G \circ H$. Let $S_G = \{g_{i_1}, g_{i_2}, \ldots, g_{i_k}\}$. Then*

$$d_{G\circ H}(S) \geq d_G(S_G).$$

Proof. We note that $g_{i_1}, g_{i_2}, \ldots, g_{i_k}$ are not necessarily distinct. Let T be a minimum S-Steiner tree in $G \circ H$. So T has $d_{G\circ H}(S)$ edges. Let Z be the minor obtained from $G \circ H$ by contracting edge in $H(g)$ for every g of G. (Equivalently, identifying all the vertices in $H(g)$ into a single vertex g and delete multiple edges in the resulting graph.) Then Z is isomorphic to G. Now T becomes Y, a connected subgraph of Z containing the vertices corresponding to $g_{i_1}, g_{i_2}, \ldots, g_{i_k}$ in G. Thus $E(Y) \geq d_G(S_G)$. Since $E(T) \geq E(Y)$, the result follows. □

Anand et al. (2012) obtained the following formula.

Lemma 3.4 Anand et al. (2012) *Let $k \geq 2$. Let G, H be two graphs such that G is connected. Let $S = \{(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_2}), \ldots, (g_{i_k}, h_{j_k})\}$ be a set of distinct vertices of $G \circ H$ such that $g_{i_p} \neq g_{i_q}$ $(1 \leq p, q \leq k)$. Let $S_G = \{g_{i_1}, g_{i_2}, \ldots, g_{i_k}\}$. Then*

$$d_{G\circ H}(S) = d_G(S_G).$$

For general case, we have the following formula for Steiner distance of lexicographic product graphs.

Theorem 3.5 *Let k, n, m be three integers with $2 \leq k \leq mn$. Let G be a connected graph of order n, and H be a graph of order m. Let $S = \{(g_{i_1}, h_{i_1}), (g_{i_2}, h_{i_2}), \ldots, (g_{i_k}, h_{i_k})\}$ be a set of distinct vertices of $G \circ H$. Let $S_G = \{g_{i_1}, g_{i_2}, \ldots, g_{i_k}\}$ and $S_H = \{h_{j_1}, h_{j_2}, \ldots, h_{j_k}\}$ (note that S_G, S_H are both multi-sets). Let r be the number of distinct vertices in S_G, where $1 \leq r \leq k$.*

(1) If $r = 1$ and $H[S_H]$ is connected in H, then $d_{G\circ H}(S) = k - 1$.

(2) If $r = 1$ and $H[S_H]$ is not connected in H, then $d_{G\circ H}(S) = k$.

(3) If $r \geq 2$, then $d_{G\circ H}(S) = d_G(S_G) + k - r$.

Proof. (1) Since $r = 1$, it follows that $g_{i_1} = g_{i_2} = \ldots = g_{i_k}$, and hence $S = \{(g_{i_1}, h_{i_1}), (g_{i_2}, h_{i_2}), \ldots, (g_{i_k}, h_{i_k})\} = \{(g_{i_1}, h_{i_1}), (g_{i_1}, h_{i_2}), \ldots, (g_{i_1}, h_{i_k})\} \subseteq V(H(g_{i_1}))$. Since $H[S_H]$ is connected in H, it follows that the subgraph induced by the vertices in $\{(g_{i_1}, h_{i_1}), (g_{i_2}, h_{i_2}), \ldots, (g_{i_k}, h_{i_k})\}$ is connected in $H(g_{i_1})$, and hence $d_{G\circ H}(S) = k - 1$.

(2) Since $H[S_H]$ is not connected in H, it follows that the subgraph induced by the vertices in $\{(g_{i_1}, h_{i_1}), (g_{i_2}, h_{i_2}), \ldots, (g_{i_k}, h_{i_k})\}$ is not connected in $H(g_{i_1})$, and hence $d_{G\circ H}(S) \geq k$. Since G is a connected graph of order at least 2, it follows that there exists a vertex $g^* \in V(G)$ such that $g_{i_1}g^* \in E(G)$.

From the structure of $G \circ H$, the tree induced by the edges in $\{(g_{i_p}, h_{i_p})(g^*, h_1) \,|\, 1 \leq p \leq k\} = \{(g_{i_1}, h_{i_p})(g^*, h_1) \,|\, 1 \leq p \leq k\}$ is an S-Steiner tree in $G \circ H$, and hence $d_{G\circ H}(S) \leq k$. So, we have $d_{G\circ H}(S) = k$.

(3) Since $r \geq 2$, it follows that the vertices in S belong to at least two copies of H in $G \circ H$. From the definition of r, we can assume that $H(g_1), H(g_2), \ldots, H(g_r)$ satisfy $S \cap V(H(g_i)) \neq \emptyset$ for each g_i $(1 \leq i \leq r)$, and $S \cap V(H(g_i)) = \emptyset$ for each g_i $(r+1 \leq i \leq n)$. Let $S'_G = \{g_1, g_2, \ldots, g_r\}$. Then $S'_G = S_G$ when we regard S_G as a normal set, not a multi-set. Clearly, $d_G(S_G) = d_G(S'_G)$, and $S = \{(g_{i_1}, h_{i_1}), (g_{i_2}, h_{i_2}), \ldots, (g_{i_k}, h_{i_k})\} \subseteq \bigcup_{i=1}^{r} V(H(g_i))$. Without loss of generality, we can assume $(g_{i_a}, h_{i_a}) \in V(H(g_a))$ for each a $(1 \leq a \leq r)$. Then $(g_{i_a}, h_{i_a}) = (g_a, h_{i_a})$ for each a $(1 \leq a \leq r)$. Let $S' = \{(g_a, h_{i_a}) \,|\, 1 \leq a \leq r\}$. Then $(g_{i_{r+1}}, h_{i_{r+1}}), (g_{i_{r+2}}, h_{i_{r+2}}), \ldots, (g_{i_k}, h_{i_k}) \in (\bigcup_{i=1}^{r} V(H(g_i)) \setminus S'$. Note that there exists an S'_G-Steiner tree T_G of size $d_G(S'_G) = d_G(S_G)$ in G. Without loss of generality, let $V(T_G) = \{g_1, g_2, \ldots, g_t\}$, where $r \leq t \leq n$. In order to select $d_G(S'_G)$ edges in $G \circ H$ to form an S'-Steiner tree T' in $G \circ H$ isomorphic to T_G in G such that $V(T') \subseteq \bigcup_{i=1}^{t} V(H(g_i))$, we define a function $f : E(T_G) \longrightarrow E(T')$:

$$f(g_a g_b) = \begin{cases} (g_a, h_{i_a})(g_b, h_{i_b}), & \text{if } g_a \in S_G \text{ and } g_b \in S_G; \\ (g_a, h_{i_a})(g_b, h_1), & \text{if } g_a \in S_G \text{ and } g_b \notin S_G; \\ (g_a, h_1)(g_b, h_1), & \text{if } g_a \notin S_G \text{ and } g_b \notin S_G, \end{cases}$$

for each $g_a g_b \in E(T_G)$ $(1 \leq a \neq b \leq t)$. Note that T' is an S'-Steiner tree in $G \circ H$.

We now extend the tree T' to an S-Steiner tree T by adding $|S| - |S'| = k - r$ edges in $G \circ H$. For each vertex $(g_{i_a}, h_{j_a}) \in S \setminus S'$ $(r+1 \leq a \leq k)$, since there exists a vertex $g_{i_b} \in V(T_G)$ $(1 \leq b \neq a \leq t)$ in G such that $g_{i_a} g_{i_b} \in E(T_G)$, we select an edge

$$e_a = \begin{cases} (g_a, h_{i_a})(g_b, h_{i_b}), & \text{if } g_b \in S_G; \\ (g_a, h_{i_a})(g_b, h_1), & \text{if } g_b \notin S_G \end{cases}$$

in $G \circ H$, and then add it into T'. Observe that the tree induced by the edges in $E(T') \cup \{e_a \,|\, r+1 \leq i \leq k\}$ is an S-Steiner tree T in $G \circ H$. Since $|E(T)| = d_G(S_G) + k - r$, it follows that $d_{G\circ H}(S) \leq d_G(S_G) + k - r$.

It remains us to show that $d_{G\circ H}(S) \geq d_G(S_G) + k - r$. Recall that $V(G) = \{g_1, g_2, \ldots, g_n\}$. Without loss of generality, we assume that $H(g_1), H(g_2), \ldots, H(g_r)$ be the H copies such that $V(H(g_i)) \cap S \neq \emptyset$, $1 \leq i \leq r$. Clearly, $S = \{(g_{i_1}, h_{i_1}), (g_{i_2}, h_{i_2}), \ldots, (g_{i_k}, h_{i_k})\} \subseteq \bigcup_{i=1}^{r} V(H(g_i))$. Set $|S \cap V(H(g_i))| = x_i$. Then $\sum_{i=1}^{r} x_i = k$. Without loss of generality, let $S_i = S \cap V(H(g_i)) = \{(g_i, h_j) \,|\, 1 \leq j \leq x_i\}$ for each g_i $(1 \leq i \leq r)$. In order to find an S-Steiner tree T in $G \circ H$, we need the edges between some $H(g_i)$ and $H(g_j)$, $1 \leq i \neq j \leq r$. Note that $S_i \subseteq V(H(g_i))$ for each i $(1 \leq i \leq r)$. Clearly, there exists a subtree T' connecting S' in T such that $E(T') \cap (\bigcup_{i=1}^{r} E(H(g_i))) = \emptyset$, where $|S' \cap S_i| = 1$ $(1 \leq i \leq r)$. Since $|E(T')| \geq d_G(S_G)$ and $|S| - |S'| = k - r$, it follows that T is an S-Steiner tree of size $d_G(S_G) + k - r$ in $G \circ H$, and hence $d_{G\circ H}(S) \geq d_G(S_G) + k - r$.

From the above argument, we conclude that $d_{G\circ H}(S) = d_G(S_G) + k - r$. □

In Theorem 3.5, we assume that G is a connected graph. For $k = 3$, we have the following by assuming that G is not connected.

Proposition 3.6 *Let G and H be two graphs such that G is connected, and let (g,h), (g',h') and (g'',h'') be three vertices of $G\circ H$. Let $S=\{(g,h),(g',h'),(g'',h'')\}$, $S_G=\{g,g',g''\}$ and $S_H=\{h,h',h''\}$. Then*

$$d_{G\circ H}(S)=\begin{cases} d_H(S_H), & \text{if } g=g'=g'' \text{ and } deg_G(g)=0;\\ \min\{d_H(S_H),3\}, & \text{if } g=g'=g'' \text{ and } deg_G(g)\neq 0;\\ \infty, & \text{if } g\neq g',\ g'=g'' \text{ and } d_G(g,g')=\infty;\\ d_G(g,g')+1, & \text{if } g\neq g',\ g'=g'' \text{ and } d_G(g,g')\neq\infty;\\ d_G(S_G), & \text{if } g\neq g',\ g\neq g'' \text{ and } g'\neq g''. \end{cases}$$

Proof. Suppose that $g=g'=g''$ and $deg_G(g)=0$. Since g is isolated, it follows that $H(g)$ is a component of $G\circ H$, and hence $d_{G\circ H}(S)=d_H(S_H)$.

Suppose that $g=g'=g''$ and $deg_G(g)\geq 1$. Since $deg_G(g)\geq 1$, there exists a vertex g^* in G such that $gg^*\in E(G)$, and hence the tree induced by the edges in

$$\{(g,h)(g^*,h),(g,h')(g^*,h),(g,h'')(g^*,h)\}$$

is an S-Steiner tree. Therefore, $d_{G\circ H}(S)\leq 3$. On the other hand, from Observation 1.1, $d_{G\circ H}(S)\geq 2$. So $d_{G\circ H}(S)=2$ or $d_{G\circ H}(S)=3$. Since $d_H(S_H)\geq 2$ by Observation 1.1, it follows that $d_{G\circ H}(S)=\min\{d_H(S_H),3\}$.

Suppose that $g\neq g'$, $g'=g''$ and $d_G(g,g')=\infty$. Then there is no path connecting g and g' in G. Note that $(g,h)\in V(H(g))$ and $(g',h'),(g'',h'')\in V(H(g'))$. Clearly, there is no S-Steiner tree in $G\circ H$. Therefore, $d_{G\circ H}(S)=\infty$.

Suppose that $g\neq g'$, $g'=g''$ and $d_G(g,g')\neq\infty$. Set $d_G(g,g')=\ell$. Let $P=gg_1g_2\cdots g_{\ell-1}g'$ be a path connecting g and g' in G. Then the tree induced by the edges in

$$\{(g,h)(g_1,h),(g_1,h)(g_2,h),\cdots,(g_{\ell-2},h)(g_{\ell-1},h),(g_{\ell-1},h)(g',h'),(g_{\ell-1},h)(g'',h'')\}$$

is an S-Steiner tree. Therefore, $d_{G\circ H}(S)\leq\ell+1$. It suffices to show $d_{G\circ H}(S)\geq\ell+1$. From Observation 2.1, any minimal S-Steiner tree T is a path or there exists a vertex $(g^*,h^*)\in V(G\circ H)\setminus S$ such that the tree T consists of three paths connecting (g^*,h^*) and $(g,h),(g',h'),(g'',h'')$, respectively. If T is a path, then we can assume that (g',h') be the internal vertex of the path T. Since $g'=g''$, it follows that $(g',h'),(g'',h'')\in V(H(g'))$. One can see that the length of the path from (g',h') to (g'',h'') is at least 1. By Lemma 3.2, $d_{G\circ H}(S)=d_{G\circ H}((g,h)(g',h'))+1\geq d_G(g,g')+1=\ell+1$, as desired. Suppose that T is a tree and there exists a vertex $(g^*,h^*)\in V(G\circ H)\setminus S$ such that T consists of three paths connecting (g^*,h^*) and $(g,h),(g',h'),(g'',h'')$, respectively. Then

$$\begin{aligned} d_{G\circ H}(S) &= d_{G\circ H}((g,h)(g^*,h^*))+d_{G\circ H}((g',h')(g^*,h^*))+d_{G\circ H}((g'',h'')(g^*,h^*))\\ &\geq d_G(g,g^*)+d_G(g',g^*)+1 \text{ (by Lemma 3.2)}\\ &\geq \ell+1. \end{aligned}$$

Suppose that $g\neq g'$, $g\neq g''$ and $g'\neq g''$. From Lemma 3.4, we have $d_{G\circ H}(S)=d_G(S_G)$, as desired. The proof is now complete. □

3.2 Steiner diameter of lexicographic product graphs

By Theorem 3.5, we can derive the following results for Steiner diameter of lexicographic product graphs.

Theorem 3.7 *Let k, n, m be three integers with $2 \leq k \leq mn$. Let G be a connected graph of order n, and H be a graph of order m. Then*

(1)

$$sdiam_k(G \circ H) \leq \begin{cases} sdiam_k(G) + k - 2, & \text{if } 2 \leq k \leq n; \\ \max\{n + k - 3, k\}, & \text{if } n < k \leq mn; \end{cases}$$

Furthermore, if $n \geq 3$, then $sdiam_k(G \circ H) \leq n + k - 3$.

(2)

$$sdiam_k(G \circ H) \geq \begin{cases} sdiam_k(G), & \text{if } 2 \leq k \leq n; \\ k - 1, & \text{if } n < k \leq nm. \end{cases}$$

Moreover, if

$$r = \min_{2 \leq x \leq n}\{x \mid sdiam_x(G) = n - 1\},$$

then

$$sdiam_k(G \circ H) \geq$$

$$\begin{cases} sdiam_k(G), & \text{if } 2 \leq k \leq r; \\ n - 1 + k - r, & \text{if } r < k \leq rm; \\ n - 1 + r(m-1) + \lfloor \frac{k - rm}{m} \rfloor (m-1) + \max\{k - (r + \lfloor \frac{k-rm}{m} \rfloor)m - 1, 0\}, & \text{if } rm < k \leq nm. \end{cases}$$

Proof. (1) From the definition of $sdiam_k(G \circ H)$, there exists a vertex subset $S \subseteq V(G \circ H)$ with $|S| = k$ such that $d_{G \circ H}(S) = sdiam_k(G \circ H)$. Let $S = \{(g_{i_1}, h_{i_1}), (g_{i_2}, h_{i_2}), \ldots, (g_{i_k}, h_{i_k})\}$, and $S_G = \{g_{i_1}, g_{i_2}, \ldots, g_{i_k}\}$. Let s be the number of distinct vertices in S_G. We apply Theorem 3.5. (Here s plays the role of r in Theorem 3.5.) If $s \geq 2$, then $sdiam_k(G \circ H) = d_{G \circ H}(S) = d_G(S_G) + k - s \leq d_G(S_G) + k - 2$. Furthermore, if $k \leq n$, then $d_G(S_G) \leq sdiam_k(G)$, and hence $sdiam_k(G \circ H) \leq d_G(S_G) + k - 2 \leq sdiam_k(G) + k - 2$. If $n < k \leq mn$, then $d_G(S_G) \leq n - 1$, and hence $sdiam_k(G \circ H) \leq d_G(S_G) + k - 2 \leq (n-1) + k - 2 = n + k - 3$. Note that if $s = 1$, then $k \leq m$, and hence $sdiam_k(G \circ H) = d_{G \circ H}(S) \leq k$. From the above argument, we conclude that $sdiam_k(G \circ H) \leq sdiam_k(G) + k - 2$ if $k \leq n$, and $sdiam_k(G \circ H) \leq \max\{n + k - 3, k\}$ if $n < k \leq mn$, as desired.

(2) If $2 \leq k \leq n$, then we let $S = \{(g_{i_1}, h_{i_1}), (g_{i_2}, h_{i_2}), \ldots, (g_{i_k}, h_{i_k})\}$ be a set of distinct vertices of $G \circ H$ such that $d_G(S_G) = sdiam_k(G)$, where $S_G = \{g_{i_1}, g_{i_2}, \ldots, g_{i_k}\}$. From Lemma 3.3, we have $sdiam_k(G) = d_G(S_G) \leq d_{G \circ H}(S) \leq sdiam_k(G \circ H)$. If $n \leq k \leq nm$, then it follows from Observation 1.1 that $k - 1 \leq d_{G \circ H}(S) \leq sdiam_k(G \circ H)$ for any $S \subseteq V(G \circ H)$ and $|S| = k$.

Now for the "moreover" part of the result. Let $r = \min_{2 \leq x \leq n}\{x \mid sdiam_x(G) = n - 1\}$. Suppose $sdiam_r(G) = n - 1$ $(2 \leq r \leq n)$. If $2 \leq k \leq r$, then $2 \leq k \leq r \leq n$, and hence $sdiam_k(G \circ H) \geq sdiam_k(G)$. Suppose $r < k \leq rm$. Since $sdiam_r(G) = n - 1$, it follows that there exists a vertex set $S' = \{g_1, g_2, \ldots, g_r\} \subseteq V(G)$ such that $d_G(S') = n - 1 = sdiam_k(G)$. Let $S = S_1 \cup S_2 \subseteq \bigcup_{i=1}^{r} V(H(g_i))$ such that $S_1 = \{(g_i, h_1) \mid 1 \leq i \leq r\}$ and $S_2 \subseteq \bigcup_{i=1}^{r} V(H(g_i)) - S_1$ and $|S_2| = k - r$.

Since $r \geq 2$ and $sdiam_r(G) = n-1$, it follows that $sdiam_k(G \circ H) \geq d_{G\circ H}(S) = d_G(S_G) + k - r = d_G(S') + k - r = n - 1 + k - r$, as desired. Suppose $rm < k \leq nm$. Since $sdiam_r(G) = n-1$, it follows that there exists a vertex set $S' = \{g_1, g_2, \ldots, g_r\} \subseteq V(G)$ such that $d_G(S') = n - 1 = sdiam_k(G)$. Let $S = S_1 \cup S_2 \subseteq \bigcup_{i=1}^{r} V(H(g_i))$ such that $S_1 = \bigcup_{i=1}^{r} V(H(g_i))$ and $S_2 \subseteq \bigcup_{i=r+1}^{x} V(H(g_i))$ and $|S_2| = k - rm$, where $x = \lceil \frac{k-rm}{m} \rceil$. Then $sdiam_k(G\circ H) \geq d_{G\circ H}(S) = d_G(S') + r(m-1) + \lfloor \frac{k-rm}{m} \rfloor (m-1) + \max\{k - (r + \lfloor \frac{k-rm}{m} \rfloor)m - 1, 0\} = n - 1 + r(m-1) + \lfloor \frac{k-rm}{m} \rfloor (m-1) + \max\{k - (r + \lfloor \frac{k-rm}{m} \rfloor)m - 1, 0\}$.
□

To show the sharpness of the upper and lower bounds in Theorem 3.7, we consider the following example.

Example 3: Let $G = P_n$, and H be a graph of order m. If $k \leq \min\{2m, n\}$, then $sdiam_k(G \circ H) = n + k - 3 = sdiam_k(G) + k - 2$. If $\max\{n, m+1\} \leq k \leq 2m$, then $sdiam_k(G \circ H) = n + k - 3 = \max\{n + k - 3, k\}$. These implies that the upper bounds in Theorem 3.7 are sharp.

Example 4: Let $G = K_n$ and $H = K_m$. Then $G \circ H$ is a complete graph of order mn. If $2 \leq k \leq n$, then $sdiam_k(G) = k - 1 = sdiam_k(G \circ H)$. If $n \leq k \leq nm$, then $sdiam_k(G \circ H) = k - 1$. These implies that the lower bounds in Theorem 3.7 are sharp.

Example 5: Let $G = P_n$ $(n \geq 3)$, and H be a graph of order m. From the definition of r, we have $r = 2$. For $2 \leq k \leq r$, we have $k = r = 2$, and hence $sdiam_2(G) = n - 1 = sdiam_2(G \circ H)$. For $r < k \leq rm$, we have $n - 1 + k - 2 \leq sdiam_k(G \circ H) \leq n + k - 3$, and hence $sdiam_k(G \circ H) = n + k - 3$. Let $G' = P_n$ $(n \geq 3)$, and $H' = P_2$. For $rm < k \leq nm$, we let $k = 2t$. From Theorem 3.7, we have $sdiam_k(G \circ H) \geq n - 1 + t$. One can easily check that $sdiam_k(G \circ H) = n - 1 + t$. These implies that the lower bounds for parameter r in Theorem 3.7 are sharp.

The following result is immediate from Proposition 3.6.

Proposition 3.8 *Let G, H be two connected graphs. Then*

$$sdiam_3(G \circ H) = \begin{cases} diam(G) + 1 & \text{if } G = P_n,\ diam(G) \geq 2, \\ sdiam_3(G) & \text{if } G \neq P_n,\ diam(G) \geq 2, \\ \min\{sdiam_3(H), 3\} & \text{if } G = K_n. \end{cases}$$

4 Applications

In this section, we demonstrate the usefulness of the proposed constructions by applying them to some instances of Cartesian and lexicographical product networks.

The following results are immediate.

Proposition 4.1 *Let k, n be two integers with $2 \leq k \leq n$.*

(1) *For a complete graph K_n, $sdiam_k(K_n) = k - 1$;*

(2) *For a path P_n, $sdiam_k(P_n) = n - 1$;*

(3) *For a cycle C_n, $sdiam_k(C_n) = \lfloor \frac{n(k-1)}{k} \rfloor$.*

4.1 Two-dimensional grid graph

A *two-dimensional grid graph* $G_{n,m}$ is the Cartesian product graph $P_n \Box P_m$ of path graphs on m and n vertices. For more details on grid graph, we refer to Calkin and Wilf (1998); Itai and Rodeh (1988). The network $P_n \circ P_m$ is the lexicographical product of P_n and P_m; see Mao (2016).

Proposition 4.2 *Let k, n, m be three integers with $3 \leq k \leq mn$, $n \geq 3$, and $m \geq 3$.*
(1) *For network $P_n \Box P_m$,*

$$m + n - 2 \leq sdiam_k(P_n \Box P_m) \leq m + n - 2 + (k-3)\min\{m-1, n-1\}.$$

(2) *For network $P_n \circ P_m$,*

$$n + k - 3 \geq sdiam_k(P_n \circ P_m) \geq \begin{cases} k-1, & \text{if } n+1 \leq k \leq mn; \\ n-1, & \text{if } 2 \leq k \leq n. \end{cases}$$

Proof. (1) From (2) of Proposition 4.1, we have $sdiam_k(P_n) = n - 1$ and $sdiam_k(P_m) = m - 1$. By Theorem 2.7, $sdiam_k(P_n \Box P_m) \geq sdiam_k(P_n) + sdiam_k(P_m) = m + n - 2$ and $sdiam_k(P_n \Box P_m) \leq m + n - 2 + (k-3)\min\{m-1, n-1\}$.
(2) Set $G = P_n$ and $H = P_m$. From Theorem 3.7, the result holds. □

4.2 r-dimensional mesh

An *r-dimensional mesh* is the Cartesian product of r paths. By this definition, two-dimensional grid graph is a 2-dimensional mesh. An r-dimensional hypercube is a special case of an r-dimensional mesh, in which the r linear arrays are all of size 2; see Johnsson and Ho (1989).

Proposition 4.3 *Let $k, m_1, m_2, \cdots, m_r$ be the integers with $m_1 \geq m_2 \geq \cdots \geq m_r$ and $3 \leq k \leq \prod_{i=1}^{r} m_i$.*
(1) *For an r-dimensional mesh $P_{m_1} \Box P_{m_2} \Box \cdots \Box P_{m_r}$,*

$$\sum_{i=1}^{r} m_i - r \leq sdiam_k(P_{m_1} \Box P_{m_2} \Box \cdots \Box P_{m_r}) \leq (k-2)\left(\sum_{i=2}^{r} m_i - r + 1\right) + m_1 - 1.$$

(2) *For an r-dimensional network $P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_r}$,*

$$m_1 + k - 2 \geq sdiam_k(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_r}) \geq \begin{cases} k-1, & \text{if } m_1 + 1 \leq k \leq \prod_{i=1}^{r} m_i; \\ m_1 - 1, & \text{if } 2 \leq k \leq m_1. \end{cases}$$

Proof. (1) From (2) of Proposition 4.1, $sdiam_k(P_{m_i}) = m_i - 1$ for each i $(1 \leq i \leq r)$. From Theorem 2.7, we have $sdiam_k(P_{m_1} \Box P_{m_2} \Box \cdots \Box P_{m_r}) \geq \sum_{i=1}^{r} sdiam_k(P_{m_i}) = \sum_{i=1}^{r} m_i - r$, and $sdiam_k(G \Box H) \leq sdiam_k(G) + sdiam_k(H) + (k-3)\min\{sdiam_k(G), sdiam_k(H)\}$ for two connected graphs G and H, and hence

$$\begin{aligned} & sdiam_k(P_{m_1} \Box P_{m_2} \Box \cdots \Box P_{m_r}) \\ = \; & sdiam_k((P_{m_1} \Box P_{m_2} \Box \cdots \Box P_{m_{r-1}}) \Box P_{m_r}) \\ \leq \; & sdiam_k(P_{m_1} \Box P_{m_2} \Box \cdots \Box P_{m_{r-1}}) + sdiam_k(P_{m_r}) \end{aligned}$$

$$
\begin{aligned}
& +(k-3)\min\{sdiam_k(P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_{r-1}}), sdiam_k(P_{m_r})\} \\
= \ & sdiam_k(P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_{r-1}}) + (k-2)(m_r-1) \\
\leq \ & sdiam_k(P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_{r-2}}) + sdiam_k(P_{m_{r-1}}) + (k-2)(m_r-1) \\
& +(k-3)\min\{sdiam_k(P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_{r-2}}), sdiam_k(P_{m_{r-1}})\}
\end{aligned}
$$

$$
\begin{aligned}
= \ & sdiam_k(P_{m_1}\Box P_{m_2}\Box\cdots\Box P_{m_{r-2}}) + (k-2)(m_{r-1}-1) + (k-2)(m_r-1) \\
\leq \ & \cdots \\
\leq \ & sdiam_k(P_{m_1}) + (k-2)\left(\sum_{i=2}^{r} m_i - r + 1\right) \\
= \ & (k-2)\left(\sum_{i=2}^{r} m_i - r + 1\right) + m_1 - 1.
\end{aligned}
$$

(2) From Theorem 2.7, the result holds. □

4.3 r-dimensional torus

An r-*dimensional torus* is the Cartesian product of r cycles $C_{m_1}, C_{m_2}, \cdots, C_{m_r}$ of size at least three. The cycles C_{m_i} are not necessary to have the same size. Ku et al. (2003) showed that there are r edge-disjoint spanning trees in an r-dimensional torus. The network $C_{m_1} \circ C_{m_2} \circ \cdots \circ C_{m_r}$ is investigated in Mao (2016). Here, we consider the networks constructed by $C_{m_1}\Box C_{m_2}\Box\cdots\Box C_{m_r}$ and $C_{m_1}\circ C_{m_2}\circ\cdots\circ C_{m_r}$.

Proposition 4.4 *Let $k, m_1, m_2, \cdots, m_r$ be the integers with $m_1 \geq m_2 \geq \cdots \geq m_r \geq 3$ and $3 \leq k \leq \prod_{i=1}^{r} m_i$.*

(1) *For network $C_{m_1}\Box C_{m_2}\Box\cdots\Box C_{m_r}$,*

$$
\begin{aligned}
\sum_{i=1}^{r}\left\lfloor\frac{(k-1)m_i}{k}\right\rfloor & \leq sdiam_k(C_{m_1}\Box C_{m_2}\Box\cdots\Box C_{m_r}) \\
& \leq \left\lfloor\frac{m_1(k-1)}{k}\right\rfloor + (k-2)\sum_{i=2}^{n}\left\lfloor\frac{m_i(k-1)}{k}\right\rfloor,
\end{aligned}
$$

where m_i is the order of C_{m_i} and $1 \leq i \leq n$.

(2) *For network $C_{m_1}\circ C_{m_2}\circ\cdots\circ C_{m_r}$,*

$$
sdiam_k(C_{m_1}\circ C_{m_2}\circ\cdots\circ C_{m_r}) \leq \begin{cases} \left\lfloor\frac{(k-1)m_1}{k}\right\rfloor + k - 2, & \text{if } k \leq m_1; \\ m_1 + k - 3, & \text{if } m_1 < k \leq \prod_{i=1}^{r} m_i. \end{cases}
$$

and

$$
sdiam_k(C_{m_1}\circ C_{m_2}\circ\cdots\circ C_{m_r}) \geq \begin{cases} \left\lfloor\frac{(k-1)m_1}{k}\right\rfloor, & \text{if } 2 \leq k \leq m_1; \\ k - 1, & \text{if } m_1 + 1 \leq k \leq \prod_{i=1}^{r} m_i. \end{cases}
$$

Proof. (1) From (3) of Proposition 4.1, $sdiam_k(C_{m_i}) = \left\lfloor \frac{(k-1)m_i}{k} \right\rfloor$ for each i $(1 \leq i \leq r)$. By Theorem 2.7, we have

$$sdiam_k(C_{m_1} \square C_{m_2} \square \cdots \square C_{m_r}) \geq \sum_{i=1}^{r} sdiam_k(C_{m_i}) = \sum_{i=1}^{r} \left\lfloor \frac{(k-1)m_i}{k} \right\rfloor.$$

and

$$sdiam_k(C_{m_1} \square C_{m_2} \square \cdots \square C_{m_r}) \leq \left\lfloor \frac{(k-1)m_1}{k} \right\rfloor + (k-2) \sum_{i=2}^{r} \left\lfloor \frac{(k-1)m_i}{k} \right\rfloor.$$

(2) The result follows from Theorem 3.7. □

4.4 r-dimensional generalized hypercube

Let K_m be a clique of m vertices, $m \geq 2$. An *r-dimensional generalized hypercube* or *Hamming graph* Day and Al-Ayyoub (1997); Fragopoulou et al. (1996) is the product of r cliques. We have the following:

Proposition 4.5 *Let $k, m_1, m_2, \cdots, m_r$ be the integers with $m_1 \geq m_2 \geq \cdots \geq m_r \geq k \geq 2$.*
(1) *For network $K_{m_1} \square K_{m_2} \square \cdots \square K_{m_r}$,*

$$r(k-1) \leq sdiam_k(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_r}) \leq (k-1)(kr-2r-k+3).$$

(2) *For network $K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_r}$,*

$$sdiam_k(K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_r}) = k-1.$$

Proof. (1) From (1) of Proposition 4.1, $sdiam_k(K_{m_i}) = k-1$ for each i $(1 \leq i \leq r)$. From Theorem 2.7, we have

$$sdiam_k(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_r}) \geq \sum_{i=1}^{r} sdiam_k(K_{m_i}) = r(k-1)$$

and

$$sdiam_k(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_r}) \leq (k-2)(k-1)(r-1) + (k-1) = (k-1)(kr-2r-k+3).$$

(2) From the definition of lexicographical product, $K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_n}$ is a complete graph, and hence $sdiam_k(K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_n}) = k-1$. □

4.5 n-dimensional hyper Petersen network

An *n-dimensional hyper Petersen network* HP_n $(n \geq 3)$ is defined as follows (see Das et al. (1995)).

- HP_3 is the Petersen graph (see Fig.4 (a));
- HP_n is the Cartesian product of the Petersen graph PG and an $(n-3)$-dimensional hypercube Q_{n-3}, that is, $HP_n = PG \square Q_{n-3}$, where $n \geq 4$.

The hyper Petersen network HP_4 are depicted in Fig.4 (b).

The network HL_n $(n \geq 3)$ is defined as follows (see Mao (2016)).

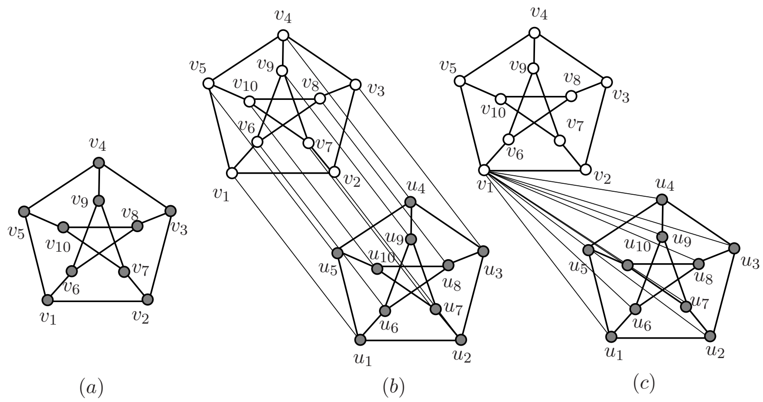

Fig. 4: (a) Petersen graph; (b) The network HP_4; (c) The structure of HL_4.

- HL_3 is the Petersen graph;
- HL_n is the lexicographic product of the Petersen graph PG and an $(n-3)$-dimensional hypercube Q_{n-3}, that is, $HP_n = PG \circ Q_{n-3}$, where $n \geq 4$.

Note that HL_4 is a graph obtained from two copies of the Petersen graph by add one edge between one vertex in a copy of the Petersen graph and one vertex in another copy. See Figure 4 (c) for an example (We only show the edges v_1u_i $(1 \leq i \leq 10)$).

Similarly to the proof of (4) of Theorem 2.7, we can get the following observation.

Observation 4.6 *Let G be a connected graph of order n. If $n - \kappa(G) + 1 \leq k \leq n$, then $sdiam_k(G) = k - 1$.*

Proposition 4.7 (1) *For network HP_3 and HL_3,*

$$sdiam_k(HP_3) = sdiam_k(HL_3) = \begin{cases} k+1, & \text{if } k = 3,4; \\ k, & \text{if } k = 5,6,7; \\ k-1, & \text{if } 8 \leq k \leq 10. \end{cases}$$

(2) *For network HL_4,*

$$sdiam_k(HL_4) = \begin{cases} k, & \text{if } 3 \leq k \leq 7; \\ k-1, & \text{if } 8 \leq k \leq 20. \end{cases}$$

(3) *For network HP_4,*

$$\begin{cases} sdiam_k(HP_4) = 5, & \text{if } k = 3; \\ k-1 \leq sdiam_k(HP_4) \leq 9 + \lfloor k/2 \rfloor, & \text{if } 4 \leq k \leq 16; \\ sdiam_k(HP_4) = k-1, & \text{if } 17 \leq k \leq 20. \end{cases}$$

Proof. (1) Observe that HL_3 is just the Petersen graph. Set $G = HL_3$. Choose $S = \{v_1, v_3, v_9\}$. One can see that any S-Steiner tree must use at least 4 edges of G, and hence $sdiam_3(G) \geq d_G(S) \geq 4$. One can check that $d_G(S) \leq 4$ for any $S \subseteq V(G)$ and $|S| = 3$. Therefore, $sdiam_3(G) \leq 4$, and hence $sdiam_3(G) = sdiam_3(HL_3) = 4$. Since $HL_3 = HP_3$, we have $sdiam_3(HP_3) = sdiam_3(HL_3) = 4$. Since $\kappa(G) = 3$, it follows from Observation 4.6 that $sdiam_k(G) = k - 1$ if $8 \leq k \leq 10$. If $k = 4$, then we choose $S = \{v_1, v_4, v_7, v_8\}$. One can see that any S-Steiner tree must use at least 5 edges of G, and hence $sdiam_4(G) \geq d_G(S) \geq 5$. One can check that $d_G(S) \leq 5$ for any $S \subseteq V(G)$ and $|S| = 3$. So, we have $sdiam_4(G) = 5$. Similarly, we can prove that $sdiam_k(G) = k$ if $5 \leq k \leq 7$.

(2) For network HL_4, there are two copies of Petersen graphs, say HL_3 and HL'_3. Set $G = HL_4$, $V(HL_3) = \{v_i \,|\, 1 \leq i \leq 10\}$ and $V(HL'_3) = \{u_i \,|\, 1 \leq i \leq 10\}$. Choose $S = \{v_1, v_2, v_9\}$. One can see that any S-Steiner tree must use at least 3 edges of G, and hence $sdiam_3(G) \geq d_G(S) \geq 3$. It suffices to show that $d_G(S) \leq 3$ for any $S \subseteq V(G)$ and $|S| = 3$. Suppose $S \subseteq V(HL_3)$ or $S \subseteq V(HL'_3)$. Without loss of generality, let $S = \{v_1, v_2, v_3\} \subseteq V(HL_3)$. If $d_{HL_3}(S) = 4$, then the tree induced by the edges in $\{u_1v_1, u_1v_2, u_1v_3\}$ is an S-Steiner tree, and hence $d_G(S) \leq 3$. Otherwise, $d_G(S) \leq d_{HL_3}(S) \leq 3$, as desired. Suppose $|S \cap V(HL_3)| = 2$ or $|S \cap V(HL'_3)| = 2$. Without loss of generality, let $|S \cap V(HL_3)| = 2$ and $S = \{v_1, v_2, u_1\}$. Then the tree induced by the edges in $\{u_1v_1, u_1v_2\}$ is an S-Steiner tree, and hence $d_G(S) \leq 2$, as desired. So $sdiam_3(HL_4) = 3$. Since $\kappa(G) = 13$, it follows from Observation 4.6 that $sdiam_k(G) = k - 1$ if $8 \leq k \leq 20$. One can also prove that $sdiam_k(G) = k$ if $3 \leq k \leq 7$.

(3) For network HP_4, there are two copies of Petersen graphs, say HP_3 and HP'_3. Set $G = HP_4$, $V(HP_3) = \{v_i \,|\, 1 \leq i \leq 10\}$ and $V(HP'_3) = \{u_i \,|\, 1 \leq i \leq 10\}$. Choose $S = \{u_1, u_3, v_{10}\}$. One can see that any S-Steiner tree must use at least 5 edges of G, and hence $sdiam_3(G) \geq d_G(S) \geq 5$. One can check that $d_G(S) \leq 5$ for any $S \subseteq V(G)$ and $|S| = 3$. Then $sdiam_3(HP_4) \leq 5$, and hence $sdiam_3(HP_4) = 5$. Since $\kappa(G) = 4$, it follows from Observation 4.6 that $sdiam_k(G) = k - 1$ if $17 \leq k \leq 20$. For $4 \leq k \leq 16$, we have $sdiam_k(HP_4) \geq k - 1$, and for any $S \subseteq V(G)$ with $|S| = k$, we let $S \cap V(HP_3) = S_1$ and $S \cap V(HP'_3) = S_2$. Without loss of generality, let $|S_1| \geq \lceil \frac{k}{2} \rceil$. Let $S_2 = S \cap V(HP'_3) = \{u_1, u_2, \ldots, u_x\}$, where $x \leq \lfloor k/2 \rfloor$. Since HP_3 is connected, it follows that it contains a spanning tree T of size 9. Then the tree induced by the edges in $E(T) \cup \{u_iv_i \,|\, 1 \leq i \leq x\}$ is an S-Steiner tree in G, and hence $d_G(S) \leq x + 9 \leq \lfloor k/2 \rfloor + 9$. □

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References

P. Ali. The steiner diameter of a graph with prescribed girth. *Discrete Math.*, 313(12):1322–1326, 2013.

P. Ali, P. Dankelmann, and S. Mukwembi. Upper bounds on the steiner diameter of a graph. *Discrete Appl. Math.*, 160(12):1845–1850, 2012.

B. Anand, M. Changat, S. Klavar, and I. Peterin. Convex sets in lexicographic product of graphs. *Graphs Combin.*, 28:77–84, 2012.

F. Bao, Y. Igarashi, and S. R.Öhring. Reliable broadcasting in product network. *Discrete Appl. Math.*, 83: 3–20, 1998.

J. Bondy and U. Murty. *Graph Theory*. Springer, 2008.

F. Buckley and F. Harary. *Distance in Graphs*. Addision-Wesley, 1990.

J. Cáceresa, A. Márquezb, and M. Puertasa. Steiner distance and convexity in graphs. *European J. Combin.*, 29:726–736, 2008.

N. Calkin and H. Wilf. The number of independent sets in a grid graph. *SIAM J. Discrete Math.*, 11(1): 54–60, 1998.

G. Chartrand, O. Oellermann, S. Tian, and H. Zou. Steiner distance in graphs. *'Casopis pro pěstování matematiky*, 114:399–410, 1989.

G. Chartrand, F. Okamoto, and P. Zhang. Rainbow trees in graphs and generalized connectivity. *Networks*, 55:360–367, 2010.

F. Chung. Diameter of graphs: Old problems and new results. In *18th Southeastern Conf. on Combinatorics, Graph Theory and Computing*, 1987.

P. Dankelmann and R. Entringer. Average distance, minimum degree, and spanning trees. *J. Graph Theory*, 33:1–13, 2000.

P. Dankelmann, H. Swart, and O. Oellermann. Bounds on the steiner diameter of a graph. In *Combinatorics and Graph Theory and Algorithms I and II*, pages 269–279, Kalamazoo, MI, 1999. New Issues Press.

S. Das, S. Öhring, and A. Banerjee. into hyper petersen network: Yet another hypercube-like interconnection topology. *VLSI Design*, 2(4):335–351, 1995.

A. D'Atri and M. Moscarini. Distance-hereditary graphs and, steiner trees and connected domination. *SIAM J. Comput.*, 17(3):521–538, 1988.

D. Day, O. Oellermann, and H. Swart. Steiner distance-hereditary graphs. *SIAM J. Discrete Math.*, 7(3): 437–442, 1994.

K. Day and A.-E. Al-Ayyoub. The cross product of interconnection networks. *IEEE Trans. Parall. Distr. Sys.*, 8(2):109–118, 1997.

D. Du, Y. Lyuu, and D. Hsu. Line digraph iteration and connectivity analysis of de Bruijn and Kautz graphs. *IEEE Trans. Comput.*, 42:612–616, 1993.

P. Fragopoulou, S. Akl, and H. Meijer. Optimal communication primitives on the generalized hypercube network. *IEEE Trans. Parall. Distr. Comput.*, 32(2):173–187, 1996.

B. Furtula, I. Gutman, and V. Katanić. Three-center harary index and its applications. *Iranian J. Math. Chem.*, 7(1):61–68, 2016.

M. Garey and D. Johnson. *Computers and Intractibility: A Guide to the Theory of NP-Completeness*. Freeman & Company New York, 1979.

W. Goddard and O. Oellermann. Distance in graphs. In M. Dehmer, editor, *Structural Analysis of Complex Networks*, pages 49–72. Birkhäuser, Dordrecht, 2011.

W. Goddard and O. Oellrmann. Steiner distance stable graphs. *Discrete Math.*, 132:65–73, 1994.

T. Gologranc. Steiner convex sets and cartesian product. *Bull. Malays. Math. Sci. Soc.*, 41(2):627–636, 2018.

I. Gutman. On steiner degree distance of trees. *Appl. Math. Comput.*, 283:163–167, 2016.

I. Gutman, B. Furtula, and X. Li. Multicenter wiener indices and their applications. *J. Serb. Chem. Soc.*, 80:1009–1017, 2015.

S. Hakimi. Steiner's problem in graph and its implications. *Networks*, 1:113–133, 1971.

R. Hammack, W. Imrich, and S. Klavžar. *Handbook of product graphs*. CRC Press, 2011.

D. Hsu. On container width and length in graphs, groups, and networks. *IEICE Transaction on Fundamentals of Electronics, Communications and Computer Science*, E77-A:668–680, 1994.

D. Hsu and T. Łuczak. Note on the k-diameter of k-regular k-connected graphs. *Discrete Math.*, 133: 291–296, 1994.

F. Hwang, D. Richards, and P. Winter. *The Steiner Tree Problem*. North-Holland, Amsterdam, 1992.

A. Itai and M. Rodeh. The multi-tree approach to reliability in distributed networks. *Inform. Comput.*, 79: 43–59, 1988.

S. Johnsson and C. Ho. Optimum broadcasting and personaized communication in hypercubes. *IEEE Trans. Comput.*, 38(9):1249–1268, 1989.

S. Ku, B. Wang, and T. Hung. Constructing edge-disjoint spanning trees in product networks. *IEEE Trans. Parall. Distr. Sys.*, 14(3):213–221, 2003.

A. Levi. Algorithm for shortest connection of a group of graph vertices. *Sov. Math. Dokl.*, 12:1477–1481, 1971.

X. Li, Y. Mao, and I. Gutman. The steiner wiener index of a graph. *Discuss. Math. Graph Theory*, 36: 455–465, 2016.

X. Li, Y. Mao, and I. Gutman. Inverse problem on the steiner wiener index. *Discuss. Math. Graph Theory*, 38(1):83–95, 2017.

Y. Mao. *Steiner Distance in Graphs–A Survey*. arXiv:1708.05779.

Y. Mao. Path-connectivity of lexicographical product graphs. *Int. J. Comput. Math.*, 93(1):27–39, 2016.

Y. Mao. The steiner diameter of a graph. *Bull. Iran. Math. Soc.*, 43(2):439–454, 2017.

Y. Mao and K. Das. Steiner gutman index. *MATCH Commun. Math. Comput. Chem.*, 79(3):779–794, 2018.

Y. Mao, Z. Wang, and I. Gutman. Steiner wiener index of graph products. *Trans. Combin.*, 5(3):39–50, 2016.

Y. Mao, Z. Wang, I. Gutman, and A. Klobučar. Steiner degree distance. *MATCH Commun. Math. Comput. Chem.*, 78(1):221–230, 2017a.

Y. Mao, Z. Wang, I. Gutman, and H. Li. Nordhaus-gaddum-type results for the steiner wiener index of graphs. *Discrete Appl. Math.*, 219:167–175, 2017b.

Y. Mao, C. Melekian, and E. Cheng. A note on the steiner $(n-k)$-diameter of a graph. *Int. J. Comput. Math.:CST*, 3(1):41–46, 2018.

F. Meyer and D. Pradhan. Flip trees. *IEEE Trans. Comput.*, 37(3):472–478, 1987.

O. Oellermann. From steiner centers to steiner medians. *J. Graph Theory*, 20(2):113–122, 1995.

O. Oellermann. On steiner centers and steiner medians of graphs. *Networks*, 34:258–263, 1999.

O. Oellermann and S. Tian. Steiner centers in graphs. *J. Graph Theory*, 14(5):585–597, 1990.