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A class of symmetric difference-closed sets related to commuting involutions

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Recent research on the combinatorics of finite sets has explored the structure of symmetric difference-closed sets, and recent research in combinatorial group theory has concerned the enumeration of commuting involutions in S_n and A_n. In this article, we consider an interesting combination of these two subjects, by introducing classes of symmetric difference-closed sets of elements which correspond in a natural way to commuting involutions in S_n and A_n. We consider the natural combinatorial problem of enumerating symmetric difference-closed sets consisting of subsets of sets consisting of pairwise disjoint 2-subsets of $[n]$, and the problem of enumerating symmetric difference-closed sets consisting of elements which correspond to commuting involutions in A_n. We prove explicit combinatorial formulas for symmetric difference-closed sets of these forms, and we prove a number of conjectured properties related to such sets which had previously been discovered experimentally using the On-Line Encyclopedia of Integer Sequences.

Keywords: symmetric difference-closed set, commuting involution, Klein four-group, permutation group, combinatorics of finite sets

1 Introduction

Combinatorial properties concerning symmetric difference-closed (Δ-closed) sets were explored recently in Gamble and Simpson (2015) and Buck and Godbole (2014). In this article, we consider an interesting class of Δ-closed sets related to commuting involutions in the symmetric group S_n and the alternating group A_n.

The study of combinatorial properties associated with pairs of commuting involutions in S_n and A_n is an interesting subject in part because this area is related to the classifications of abstract regular polytopes for fixed automorphism groups, as shown in Kiefer and Leemans (2013). In Kiefer and Leemans (2013) it is proven that up to conjugacy, there are

$$-2n+\sum_{k=1}^{n}\left\lfloor\left(\frac{k}{2}+1\right)^2\right\rfloor\cdot(n-k+1)$$

ordered pairs of commuting involutions in S_{2n} and S_{2n+1}. This formula is used in Kiefer and Leemans (2013) to prove new formulas for the number of unordered pairs of commuting involutions up to isomorphism in a given symmetric or alternating group. These formulas may be used to determine the total number of Klein four-subgroups for S_n and A_n.

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In our present article, we consider a natural variation of the problem of counting $(C_2 \times C_2)$-subgroups of a given permutation group. In particular, we consider the problem of enumerating Klein permutation subgroups which are, in a specific sense, analogous to Δ-closed sets. Inspired in part by Kiefer and Leemans (2013), Buck and Godbole (2014), and Gamble and Simpson (2015), we introduce new classes of Δ-closed sets consisting of elements which correspond in a natural way to commuting involutions in S_n and A_n, and we prove new combinatorial formulas for these classes of Δ-closed sets.

1.1 An enumerative problem concerning symmetric difference-closed sets

Given two sets S_1 and S_2, recall that the symmetric difference of S_1 and S_2 is denoted by $S_1 \Delta S_2$, and may be defined so that $S_1 \Delta S_2 = (S_1 \setminus S_2) \cup (S_2 \setminus S_1)$. An n-subset T consisting of sets is a symmetric difference-closed set or a Δ-closed n-subset if $S_1 \Delta S_2 \in T$ for all $S_1, S_2 \in T$. In other words, restricting the Δ operation to $T \times T$ yields a binary operation on T. Our present article is largely motivated by the enumerative problem described below, which may be formulated in a natural way in terms of Δ-closed sets.

Suppose that n people arrive at a meeting, and suppose that these n people arrange themselves into pairs (except for a loner if n is odd) and that these pairs then form various organizations. What is the total number, taken over all possible pairings, of possible collections C of three distinct organizations such that given two organizations in C, a pair P of people belongs to only one of these two organizations iff P is a member of the remaining (third) organization?

The number of possible collections C as given above is also equal to the number of Δ-closed 4-subsets $S \subseteq 2^{2^{[n]}}$ such that there exists a set T consisting of pairwise disjoint 2-subsets of $[n]$ such that $S \subseteq 2^T$. A set S of this form endowed with the Δ operation forms a group which is isomorphic to the Klein four-group $C_2 \times C_2$, and the elements consisting of pairwise disjoint 2-sets in S may be regarded in an obvious way as commuting involutions in S_n. We thus have that the total number of collections C as given above is also equal to the number of subgroups G of the symmetric group S_n such that G is isomorphic to the Klein four-group, and such that there exists a set $T \subseteq S_n$ of pairwise disjoint transpositions such that each element in G is a product of elements in T. We refer to this latter property as the totally disjoint transposition (TDT) property. It is clear that it is not the case that all Klein four-subgroups of S_n satisfy this property. For example, the permutation subgroup

$$\{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$$

forms a Klein four-subgroup of S_4, but the 2-sets $\{3, 4\}$ and $\{1, 3\}$ are not pairwise disjoint.

The enumerative problem given above may be formulated in a more symmetric way in the following manner. The total number of possible collections C as described in this problem is also equal to the number of 3-sets of the form

$$\{\mathsf{A} \cup \mathsf{B}, \mathsf{A} \cup \mathsf{C}, \mathsf{B} \cup \mathsf{C}\}$$

such that A, B, and C are pairwise disjoint sets contained in a set of pairwise disjoint 2-subsets of $[n]$, and at most one of A, B, and C can be empty.

The sequence labeled A267840 which we contributed to OEIS Foundation Inc. (2011) enumerates Δ-closed sets of the form described above. Accordingly, let $\mathrm{A267840}_n$ denote the number of Klein four-subgroups of S_n satisfying the TDT property. In the OEIS entry for A267840, we provided the following

intriguing, exotic triple sum for A267840_n for $n \in \mathbb{N}$, letting δ denote the Kronecker delta function:

$$n! \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=1}^{i} \sum_{k=\max\left(\lceil \frac{i}{2} \rceil, i+j-\lfloor \frac{n}{2} \rfloor\right)}^{\min\left(j, \lfloor \frac{1}{4}(2i+2j-1) \rfloor\right)} \frac{2^{k-i-j}}{k!(i-k)!(j-k)!(n-2i-2j+2k)!(\delta_{i,j}+\delta_{i,2k}+1)!}. \tag{1}$$

In our present article, we offer an elegant proof of the formula indicated above, by introducing a new class of integer partitions which we refer to as "Klein partitions". After our formula indicated in (1) was added to OEIS Foundation Inc. (2011), Václav Kotěšovec used this formula together with the `plinrec` Mathematica function to determine a conjectural linear recurrence with polynomial coefficients for the integer sequence $(\text{A267840}_n : n \in \mathbb{N})$ (Kotěšovec (2016)).

Kotěšovec also used the formula given in (1) to construct a conjectured exponential generating function (EGF) for A267840 using the `dsolve` Maple function, together with the `rectodiffeq` Maple command. Amazingly, the integer sequence $(\text{A267840}_n)_{n \in \mathbb{N}}$ seems to have a surprisingly simple EGF, in stark contrast to the intricacy of the above triple summation given in (1):

$$\text{EGF}(\text{A267840}; x) \stackrel{?}{=} \frac{e^x}{3} - \frac{e^{\frac{x(x+2)}{2}}}{2} + \frac{e^{\frac{x(3x+2)}{2}}}{6}.$$

We refer to this conjecture as Kotěšovec's conjecture. We present a combinatorial proof of this conjecture in our article, and we use this result to prove a conjectural asymptotic formula for $(\text{A267840}_n : n \in \mathbb{N})$ given by Václav Kotěšovec in OEIS Foundation Inc. (2011).

1.2 A class of symmetric difference-closed sets related to commuting even involutions

Since the number of pairs of commuting involutions in the alternating group A_n up to isomorphism is also considered in Kiefer and Leemans (2013), it is also natural to consider analogues of the results given above for even products of transpositions.

The sequence labeled A266503 which we contributed to OEIS Foundation Inc. (2011) enumerates subgroups G of the alternating group A_n such that G is isomorphic to the Klein four-subgroup $C_2 \times C_2$, and each element in G is the product of the elements in a subset of a fixed set of pairwise disjoint transpositions in A_n. It is important to note that this is not equal to the number of Klein four-subgroups of A_n. This may be proven by the same counterexample as above in the case of the symmetric group.

Letting A266503_n denote the n^{th} entry in the OEIS sequence A266503, A266503_n is also equal to the number of Δ-closed subsets $S \subseteq 2^{2^{[n]}}$ such that there exists a set T consisting of pairwise disjoint 2-subsets of $[n]$ such that $S \subseteq 2^T$, and each element in S is of even order.

In the OEIS sequence labeled A266503, we provided the following beautiful expression for A266503_n for $n \in \mathbb{N}$:

$$n! \sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \sum_{j=1}^{i} \sum_{k=\max\left(i, 2i+2j-\lfloor \frac{n}{2} \rfloor\right)}^{\min\left(2j, \lfloor \frac{1}{4}(4i+4j-1) \rfloor\right)} \frac{2^{k-2i-2j}}{k!(2i-k)!(2j-k)!(n-4i-4j+2k)!(\delta_{i,j}+\delta_{i,k}+1)!}.$$

Václav Kotěšovec used the above formula to conjecture that the EGF for $(\text{A266503}_n)_{n\in\mathbb{N}}$ is equal to the following expression OEIS Foundation Inc. (2011).

$$\frac{e^x}{3}-\frac{e^{\frac{-x(x-2)}{2}}}{8}-\frac{e^{\frac{x(x+2)}{2}}}{4}+\frac{e^{\frac{x(3x+2)}{2}}}{24}.$$

We also offer a combinatorial proof of this conjectured EGF formula. This EGF formula may be used to prove other conjectured results concerning A266503, such as an asymptotic formula for A266503.

2 A proof of Kotěšovec's conjecture

Let A000085_n denote the n^{th} entry in the OEIS sequence labeled A000085 for $n\in\mathbb{N}_0$, so that A000085_n is equal to the number of self-inverse permutations on n and the number of Young tableaux with n cells. Similarly, let A115327_n denote the n^{th} entry in the OEIS sequence labeled A115327, which is defined so that the EGF of this sequence is $e^{\frac{3}{2}x^2+x}$. Since the EGF for the integer sequence $(\text{A000085}_n : n\in\mathbb{N}_0)$ is $e^{\frac{x^2}{2}+x}$, we find that Kotěšovec's conjecture concerning the integer sequence $(\text{A267840}_n : n\in\mathbb{N})$ is equivalent to the following conjecture, as noted by Kotěšovec in OEIS Foundation Inc. (2011).

Conjecture 2.1. (Kotěšovec, 2016) The n^{th} entry in A267840 is equal to $\frac{1}{3}-\frac{\text{A000085}_n}{2}+\frac{\text{A115327}_n}{6}$.

As indicated in OEIS Foundation Inc. (2011), based on results introduced in Leaños et al. (2012), we have that A115327_n is equal to the number of square roots of an arbitrary element $\sigma\in S_{3n}$ such that the disjoint cycle decomposition of σ consists of $n\in\mathbb{N}_0$ three-cycles. We thus find that Kotěšovec's conjecture relates Δ-closed sets as given by A267840 to combinatorial objects such as Young tableaux and permutation roots, in an unexpected and yet simple manner.

To prove Kotěšovec's conjecture, our strategy is to make use of known summation formulas for the OEIS sequences labeled A000085 and A115327. The known formula

$$\text{A000085}_n=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n!}{(n-2k)!2^k k!}$$

is given in OEIS Foundation Inc. (2011). The new summation formula

$$\text{A115327}_n=n!\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{3^k}{2^k k!(n-2k)!}$$

is also given in OEIS Foundation Inc. (2011), based on a result proven in Leaños et al. (2012) concerning mth roots of permutations. So, we find that Kotěšovec's conjecture is equivalent to the following conjectural formula:

$$\text{A267840}_n\stackrel{?}{=}\sum_{k=2}^{\lfloor\frac{n}{2}\rfloor}\left(\frac{3^{k-1}-1}{2}\right)\cdot\frac{n!}{2^k(n-2k)!k!}.$$

We are thus lead to consider the number triangle given by expressions of the form $\frac{n!}{2^k(n-2k)!k!}$, for $k\in\mathbb{N}_0$ such that $k\leq\lfloor\frac{n}{2}\rfloor$. From OEIS Foundation Inc. (2011), we find that this number triangle is precisely

the triangle of Bessel numbers $T(n,k)$, whereby $T(n,k)$ is the number of k-matchings of the complete graph K_n, with $T(n,k) = \frac{n!}{k!(n-2k)!2^k}$. So, we find that Kotěšovec's conjecture may be formulated in the following equivalent manner.

Theorem 2.1. *For $n \in \mathbb{N}$ such that $n \geq 4$, we have that* $\text{A267840}_n = \sum_{k=2}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{3^{k-1}-1}{2} \right) \cdot T(n,k)$.

Proof: Recall that $T(n,k)$ is the number of k-matchings of the complete graph K_n. We claim that

$$\text{A267840}_n = \sum_{k=2}^{\lfloor \frac{n}{2} \rfloor} c_{n,k} \cdot T(n,k),$$

where $c_{n,k}$ is the number of Klein four-subgroups of S_n satisfying the TDT property consisting precisely of the following transpositions: $(1,2),(3,4),\ldots,(2k-1,2k)$. This is easily proven bijectively, in the following way.

Given a k-matching of the complete graph K_n, letting this matching be denoted with the k pairwise disjoint transpositions

$$(x_1 < x_2) <_\ell (x_3 < x_4) <_\ell \ldots <_\ell (x_{2k-1} < x_{2k}),$$

which we order lexicographically, and given a Klein-subgroup of S_n satisfying the TDT property consisting of the transpositions $(1,2),(3,4),\ldots,(2k-1,2k)$, we obtain another Klein-subgroup of S_n satisfying the TDT property consisting of the transpositions

$$(x_1,x_2) <_\ell (x_3,x_4) <_\ell \ldots <_\ell (x_{2k-1},x_{2k})$$

by replacing each occurrence of $(i,i+1)$ with (x_i,x_{i+1}) for $i = 1,3,\ldots,2k-1$. This defines a bijection

$$\phi\colon \mathcal{C}_{n,k} \times \mathcal{T}_{n,k} \to \mathcal{A}_{n,k},$$

where $\mathcal{C}_{n,k}$ is the set of TDT Klein four-subgroups of S_n as given by the coefficient $c_{n,k}$, $\mathcal{T}_{n,k}$ is the set of all k-matchings of K_n, and $\mathcal{A}_{n,k}$ is the set of all TDT Klein-four subgroups of the symmetric group S_n consisting of exactly k transpositions in total.

We claim that

$$c_{n,k} = 1 + 3 + \cdots + 3^{k-2}$$

for all $n \geq 4$ and $k \in \mathbb{N}_{\geq 2}$ whereby $k \leq \lfloor \frac{n}{2} \rfloor$. We proceed by induction on k. In the case whereby $k = 2$, we have that $c_{n,k}$ is the number of Klein-subgroups of S_n satisfying the TDT property consisting of the transpositions $(1,2) \in S_n$ and $(3,4) \in S_n$. But it is clear that there is only one such group, namely:

$$\{(12)(34),(12),(34),\text{id}\} \cong C_2 \times C_2.$$

So, we find that $c_{n,2} = 1$, as desired. We may inductively assume that

$$c_{n,k} = 1 + 3 + \cdots + 3^{k-2}$$

for some $n \geq 4$ and $k \in \mathbb{N}_{\geq 2}$ whereby $k < \lfloor \frac{n}{2} \rfloor$. For each Klein-subgroup $K \leq S_n$ satisfying the TDT property consisting of the transpositions $(1,2),(3,4),\ldots,(2k-1,2k) \in S_n$, we may create three distinct

Klein-subgroups of S_n satisfying the TDT property, by adjoining the transposition $(2k+1, 2k+2)$ to two nonempty products of transpositions in $K \leq S_n$ in three different ways to produce three TDT Klein four-subgroups. For example, we may adjoin the permutation (56) to two non-identity elements within the TDT $(C_2 \times C_2)$-subgroup

$$\{(12)(34), (12), (34), \mathrm{id}\}$$

in three different ways to obtain three additional subgroups which are isomorphic to $C_2 \times C_2$ and which satisfy the TDT property:

$$\{(12)(34)(56), (12)(56), (34), \mathrm{id}\},$$
$$\{(12)(34)(56), (12), (34)(56), \mathrm{id}\},$$
$$\{(12)(34), (12)(56), (34)(56), \mathrm{id}\}.$$

So, we may obtain

$$3 \cdot (1 + 3 + \cdots + 3^{k-2}) = 3 + 3^2 + \cdots + 3^{k-1}$$

new TDT Klein four-subgroups in this manner. But there is a unique remaining Klein four-subgroup satisfying the TDT property consisting of the transpositions $(1,2), (3,4), \ldots, (2k-1, 2k), (2k+1, 2k+2)$ which cannot be obtained in the preceding manner, namely:

$$\{(2k+1, 2k+2), (12)(34)\cdots(2k-1, 2k)(2k+1, 2k+2), (12)(34)\cdots(2k-1, 2k), \mathrm{id}\}.$$

So, this shows that the total number of Klein four-subgroups satisfying the TDT property consisting of the transpositions $(1,2), (3,4), \ldots, (2k-1, 2k), (2k+1, 2k+2)$ is equal to

$$1 + 3 + 3^2 + \cdots + 3^{k-1},$$

thus completing our proof by induction. So, since

$$\mathrm{A267840}_n = \sum_{k=2}^{\left\lfloor \frac{n}{2} \right\rfloor} c_{n,k} \cdot T(n,k),$$

and

$$c_{n,k} = 1 + 3 + \cdots + 3^{k-2} = \frac{3^{k-1} - 1}{2},$$

we thus have that

$$\mathrm{A267840}_n = \sum_{k=2}^{\left\lfloor \frac{n}{2} \right\rfloor} \left(\frac{3^{k-1} - 1}{2} \right) \cdot T(n,k),$$

as desired. □

Corollary. *The EGF for* $(\mathrm{A267840}_n)_{n \in \mathbb{N}}$ *is* $\frac{e^x}{3} - \frac{e^{\frac{x(x+2)}{2}}}{2} + \frac{e^{\frac{x(3x+2)}{2}}}{6}$.

Proof: This follows immediately from Theorem 2.1, since the equality given in Kotěšovec's conjecture is equivalent to the equation given in Theorem 2.1. □

Kotěšovec also provided the following conjectural asymptotic expression for the integer sequence $(\text{A267840}_n : n \in \mathbb{N})$ in OEIS Foundation Inc. (2011):

$$\text{A267840}_n \sim 2^{-\frac{3}{2}} 3^{\frac{n}{2}-1} \exp\left(\sqrt{\frac{n}{3}} - \frac{n}{2} - \frac{1}{12}\right) n^{\frac{n}{2}}. \tag{2}$$

The above conjectured asymptotic formula may be proven using the EGF given in Corollary 2. In particular, the `algolib` Maple package together with the Maple command `equivalent` may be used to derive (2) from Corollary 2 (Kotěšovec (2016)). Kotěšovec discovered the following unexpected recurrence with polynomial coefficients for the sequence $(\text{A267840}_n)_{n\in\mathbb{N}}$ using the Mathematica function `plinrec` (Kotěšovec (2016)):

$$\begin{aligned}(n-4)(n-2)\text{A267840}_n &= 3(n^2-5n+5)\text{A267840}_{n-1} \\ &\quad + (n-1)(4n^2-27n+41)\text{A267840}_{n-2} \\ &\quad - (n-2)(n-1)(8n-29)\text{A267840}_{n-3} \\ &\quad - (n-3)(n-2)(n-1)(3n-16)\text{A267840}_{n-4} \\ &\quad + 3(n-4)(n-3)(n-2)(n-1)\text{A267840}_{n-5}.\end{aligned}$$

We leave it as an easy computational exercise to verify this recurrence using a computer algebra system (CAS) together with Corollary 2, by comparing the EGF for the left-hand side of the above equation with the EGF for the right-hand side of this equality using Corollary 2.

3 Kotěšovec's conjecture for TDT Klein four-subgroups of alternating groups

Let A000085 and A115327 be as given above. Similarly, for $n \in \mathbb{N}_0$, let A001464_n denote the n^{th} term given by the sequence labeled A001464 in OEIS Foundation Inc. (2011). This sequence is defined so that the EGF for this sequence is $\exp(-x - \frac{1}{2}x^2)$. So it is clear that the problem of proving the above conjectural expression for the EGF of A267840 is equivalent to the problem of proving the following identity given in OEIS Foundation Inc. (2011).

Conjecture 3.1. (Kotěšovec, 2016) For $n \in \mathbb{N}$, $\text{A266503}_n = \frac{1}{3} + \frac{1}{8}(-1)^{n+1}\text{A001464}_n - \frac{\text{A000085}_n}{4} + \frac{\text{A115327}_n}{24}$.

The following formula for A001464 is given by Benoit Cloitre in OEIS Foundation Inc. (2011):

$$\text{A001464}_n = (-1)^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k (2k-1)!! \binom{n}{2k}.$$

Also, recall that

$$\text{A000085}_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2k)!2^k k!}$$

and

$$\text{A115327}_n = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{3^k}{2^k k! (n-2k)!}.$$

So, we have that the problem of proving Conjecture 3.1 is equivalent to the problem of proving the following identity.

Theorem 3.1. *For* $n \in \mathbb{N}$, $\text{A266503}_n = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{24} \left(-6 - 3(-1)^k + 3^k \right) T(n,k)$.

Proof: Recall that $T(n,k)$ denotes the number of k-matchings of the complete graph K_n. We claim that

$$\text{A267840}_n = \sum_{k=2}^{\lfloor \frac{n}{2} \rfloor} d_{n,k} \cdot T(n,k),$$

where $d_{n,k}$ is the number of Klein-subgroups of A_n satisfying the TDT property consisting of the following transpositions: $(1,2),(3,4),\ldots,(2k-1,2k)$. Again, this is easily seen bijectively, just as in the proof of Theorem 2.1.

Letting $n \in \mathbb{N}$ be sufficiently large, we proceed to construct an expression for the coefficient $d_{n,k}$ in terms of $d_{n,k-1}$ and $d_{n,k-2}$. For each Klein-subgroup of A_n satisfying the TDT property consisting of the transpositions $(1,2),(3,4),\ldots,(2k-5,2k-4)$, we obtain 3 TDT Klein four-subgroups of A_n by adjoining the product $(2k-3,2k-2)(2k-1,2k)$ twice in three different ways, in essentially the same manner as in the proof Theorem 2.1.

Now consider the remaining Klein four-subgroups of A_n consisting of the transpositions

$$(1,2),(3,4),\ldots,(2k-1,2k)$$

which cannot be obtained from the $d_{n,k-2}$-subgroups in the manner described above. For a subgroup S of this form, exactly two separate products in S contain $(2k-3,2k-2)$ as a factor, and exactly one product in S does not contain $(2k-3,2k-2)$ as a factor. The factor $(2k-1,2k)$ cannot be in a same product as $(2k-3,2k-2)$ twice within S, because otherwise S could be obtained from a $d_{n,k-2}$-subgroup, as above. But since there is exactly one product in S which does not contain $(2k-3,2k-2)$ as a factor, by the pigeonhole principle, it cannot be the case that that factor $(2k-1,2k)$ is never in a same product as $(2k-3,2k-2)$ within S. So, we may conclude that $(2k-1,2k)$ is an a same product as $(2k-3,2k-2)$ within S exactly once.

So, by deleting the unique product of the form $(2k-3,2k-2)(2k-1,2k)$ in each such remaining subgroup, and then replacing the unique remaining transposition of the form $(2k-1,2k)$ with $(2k-3,2k-2)$, we obtain:

(i) Exactly one multiset consisting of an additional "empty product", which corresponds to the unique subgroup

$$\{\text{id},(2k-1,2k),(12)(34)\cdots(2k),(12)(34)\cdots(2k-3,2k-2)\} \cong C_2 \times C_2$$

which cannot be obtained from the $d_{n,k-2}$-subgroups and which contains an element of the form $(2k-1,2k)$; and

(ii) Exactly two copies of each $d_{n,k-1}$-subgroup, given by two "interchangeable" positions for

$$2k \mapsto 2k-1$$

and $2k-1$.

The process outlined above is illustrated in Example 3.1.

So, we have that $d_{n,k} = 2d_{n,k-1} + 3d_{n,k-2} + 1$. The integer sequence given by coefficients of the form $\frac{1}{24}\left(-6 - 3(-1)^k + 3^k\right)$ satisfies the recurrence whereby $a(n) = 2a(n-1) + 3a(n-2) + 1$. By considering the base cases, we have that $d_{n,k} = \frac{1}{24}\left(-6 - 3(-1)^k + 3^k\right)$, as desired. □

Example 3.1. The subgroups corresponding to $d_{n,5} = 10$ are:

$$\{(12)(34)(56)(78), (12)(34)(56)(9,10), (78)(9,10), \mathrm{id}\},$$
$$\{(12)(34)(56)(78), (12)(34)(78)(9,10), (56)(9,10), \mathrm{id}\},$$
$$\{(12)(34)(56)(78), (12)(56)(78)(9,10), (34)(9,10), \mathrm{id}\},$$
$$\{(12)(34)(56)(78), (34)(56)(78)(9,10), (12)(9,10), \mathrm{id}\},$$
$$\{(12)(34)(56)(9,10), (12)(34)(78)(9,10), (56)(78), \mathrm{id}\},$$
$$\{(12)(34)(56)(9,10), (34)(56)(78)(9,10), (12)(78), \mathrm{id}\},$$
$$\{(12)(34)(78)(9,10), (34)(56)(78)(9,10), (12)(56), \mathrm{id}\},$$
$$\{(12)(56)(78)(9,10), (34)(56)(78)(9,10), (12)(34), \mathrm{id}\},$$
$$\{(12)(34)(78)(9,10), (12)(56)(78)(9,10), (34)(56), \mathrm{id}\},$$
$$\{(12)(34)(56)(9,10), (12)(56)(78)(9,10), (34)(78), \mathrm{id}\}.$$

Exactly three of these subgroups may be obtained from $d_{n,3}$ in the manner suggested above, by adjoining the product $(78)(9,10)$ twice in three different ways, leaving us with the following seven subgroups:

$$\{(12)(34)(56)(78), (12)(34)(56)(9,10), (78)(9,10), \mathrm{id}\},$$
$$\{(12)(34)(56)(78), (12)(34)(78)(9,10), (56)(9,10), \mathrm{id}\},$$
$$\{(12)(34)(56)(78), (12)(56)(78)(9,10), (34)(9,10), \mathrm{id}\},$$
$$\{(12)(34)(56)(78), (34)(56)(78)(9,10), (12)(9,10), \mathrm{id}\},$$
$$\{(12)(34)(56)(9,10), (12)(34)(78)(9,10), (56)(78), \mathrm{id}\},$$
$$\{(12)(34)(56)(9,10), (34)(56)(78)(9,10), (12)(78), \mathrm{id}\},$$
$$\{(12)(34)(56)(9,10), (12)(56)(78)(9,10), (34)(78), \mathrm{id}\}.$$

Removing products of the form $(78)(9,10)$ from the above permutations yields one multiset with an additional empty product. The remaining sets are given below:

$$\{(12)(34)(56)(78), (12)(34), (56)(9,10), \mathrm{id}\},$$
$$\{(12)(34)(56)(78), (12)(56), (34)(9,10), \mathrm{id}\},$$
$$\{(12)(34)(56)(78), (34)(56), (12)(9,10), \mathrm{id}\},$$

$$\{(12)(34)(56)(9,10),(12)(34),(56)(78),\mathrm{id}\},$$
$$\{(12)(34)(56)(9,10),(34)(56),(12)(78),\mathrm{id}\},$$
$$\{(12)(34)(56)(9,10),(12)(56),(34)(78),\mathrm{id}\}.$$

By comparing the positions of (78) and $(9,10)$, we find that reducing each expression of the form $(9,10)$ to (78) yields two copies of each subgroup in $d_{n,k-1}$. This shows that $d_{5,k}=2d_{4,k}+3d_{3,k}+1$.

We may use the above EGF evaluation for A266503 together with a CAS such as Maple to prove the following asymptotic result conjectured by Kotěšovec in OEIS Foundation Inc. (2011):

$$\mathrm{A266503}_n \sim 2^{-\frac{7}{2}}3^{\frac{n}{2}-1}\exp\left(\sqrt{\frac{n}{3}}-\frac{n}{2}-\frac{1}{12}\right)n^{\frac{n}{2}}.$$

Kotěšovec also discovered the following interesting recursive formula for the sequence $(\mathrm{A266503}_n)_{n\in\mathbb{N}}$:

$$\begin{aligned}(n-6)(n-4)(n-2)\mathrm{A266503}_n &= (2n-7)(2n^2-14n+15)\mathrm{A266503}_{n-1}\\ &+3(n-7)(n-1)(n^2-7n+11)\mathrm{A266503}_{n-2}\\ &-(n-2)(n-1)(9n^2-85n+189)\mathrm{A266503}_{n-3}\\ &+(n-3)(n-2)(n-1)(n^2-n-22)\mathrm{A266503}_{n-4}\\ &-2(n-4)^2(n-3)(n-2)(n-1)\mathrm{A266503}_{n-5}\\ &-(n-5)(n-4)(n-3)(n-2)(n-1)(3n-19)\mathrm{A266503}_{n-6}\\ &+3(n-6)(n-5)(n-4)(n-3)(n-2)(n-1)\mathrm{A266503}_{n-7}.\end{aligned}$$

This recursion also may be proven using the EGF for A266503 together with a CAS.

4 A triple summation formula for TDT Klein four-subgroups

Let $n\geq 4$. Let S be a Δ-closed 4-set consisting of the empty set together with subsets of a set consisting of pairwise disjoint 2-subsets of $\{1,2,\ldots,n\}$. Then one of the following two situations must occur.

(i) The Δ-closed set S is of the form

$$\{\varnothing,\ \{t_1,t_2,\ldots,t_i\},\ \{t_{i+1},t_{i+2},\ldots,t_j\},\ \{t_1,t_2,\ldots,t_i,t_{i+1},t_{i+2},\ldots,t_j\}\}$$

where $\{t_1,t_2,\ldots,t_j\}$ is a set consisting of j distinct pairwise disjoint 2-sets in $\{1,2,\ldots,n\}$, or

(ii) The 4-set S consists of $\varnothing$ together with non-empty elements of the following forms, letting

$$\{t_1,t_2,\ldots,t_k\}$$

be a set consisting of k distinct pairwise disjoint 2-sets in $\{1,2,\ldots,n\}$:

$$\{t_1,t_2,\ldots,t_i,t_{i+1},t_{i+2},\ldots,t_j\},$$
$$\{t_{i+1},t_{i+2},\ldots,t_j,t_{j+1},t_{j+2},\ldots,t_k\},$$
$$\{t_1,t_2,\ldots,t_i,t_{j+1},t_{j+2},\ldots,t_k\}.$$

Given an arbitrary Δ-closed 4-set S consisting of subsets of a set consisting of pairwise disjoint 2-subsets of $[n]$, we define the *partition type* type(S) of S as the unique partition $\lambda = \text{type}(S)$ of length 3 such that a largest set of 2-sets in S consists of $\frac{\lambda_1}{2}$ pairwise disjoint 2-sets consisting of a total of λ_1 elements, a second-largest set of 2-sets in S consists of $\frac{\lambda_2}{2}$ 2-sets, and a third-largest set of 2-sets in S consists of $\frac{\lambda_3}{3}$ 2-sets. For example,

$$\begin{aligned}
&\text{type}\left(\{\{\{1,3\},\{2,4\}\},\{\{1,3\}\},\{\{2,4\}\},\varnothing\}\right) = (4,2,2) \vdash 8,\\
&\text{type}\left(\{\{\{1,3\},\{2,4\}\},\{\{2,4\},\{5,6\}\},\{\{1,3\},\{5,6\}\},\varnothing\}\right) = (4,4,4) \vdash 12,\\
&\text{type}\left(\{\{\{1,3\},\{2,4\},\{5,6\}\},\{\{1,3\},\{5,6\}\},\{2,4\},\varnothing\}\right) = (6,4,2) \vdash 12.
\end{aligned}$$

We define a *Klein partition* for $n \in \mathbb{N}$ as a partition λ such that $\lambda = \text{type}(S)$ for some set S of the form described above.

Lemma 4.1. *A partition λ is a Klein partition for $n \in \mathbb{N}$ if and only if*

(a) The length $\ell(\lambda)$ of λ is 3;

(b) Each entry of λ is even;

(c) The first entry λ_1 of λ satisfies $\lambda_1 \leq 2\lfloor \frac{n}{2} \rfloor$; and

(d) There exists an index i in $\left[0, \frac{\lambda_2}{2}\right]$ such that $\lambda_1 + \lambda_2 - 4i = \lambda_3$ and $\lambda_1 + \lambda_2 - 2i \leq n$.

Proof: ($\Longrightarrow$) Suppose that λ is a Klein partition for $n \in \mathbb{N}$. We thus have that $\lambda = \text{type}(S)$ for some Δ-closed 4-set S consisting of $\varnothing$ together with subsets of a set consisting of pairwise disjoint 2-subsets of $\{1, 2, \ldots, n\}$. By definition of the partition type of a set of this form, we have that $\lambda = \text{type}(S)$ must be of length 3 and must have even entries. The first entry λ_1 of λ is equal to the total number of elements among all 2-sets in a largest set of 2-sets in S. If n is even, then the maximal total number of elements among all 2-sets in a largest set of 2-sets in S is n, and otherwise, λ_1 is at most $n-1$. We thus have that $\lambda_1 \leq 2\left\lfloor \frac{n}{2} \right\rfloor$. Let p_1, p_2, and p_3 be pairwise distinct nontrivial sets of 2-sets such that p_1 is a largest set of 2-sets in S, p_2 is a second-largest set of 2-sets in S, and p_3 is a smallest set of 2-sets in S. Note that it is possible that p_1, p_2, and p_3 are all sets of equal cardinality. Also observe that $p_1 \Delta p_2 = p_3 \in S$. Suppose that p_1 and p_2 share exactly $j \in \mathbb{N}_0$ 2-sets in common. It is easily seen that $j > 0$ since S forms a group under the binary operation $\Delta\colon S \times S \to S$, and since the number of 2-sets of p_1 is greater than or equal to the number of 2-sets of p_2 and greater than or equal to the number of 2-sets of p_3. Since $\lambda_2 \leq \lambda_1$ it is thus clear that $j \in \left[0, \frac{\lambda_2}{2}\right]$. Since p_1 and p_2 share exactly j 2-sets, we thus have that total number λ_3 of elements among all 2-sets in p_3 is $(\lambda_1 - 2j) + (\lambda_2 - 2j)$. Now consider the total number of elements among the 2-sets in either p_1 or p_2. Since p_1 and p_2 share exactly j 2-sets, by the principle of inclusion-exclusion, we have that the total number of elements among the 2-sets in either p_1 or p_2 is equal to $\lambda_1 + \lambda_2 - 2j$. We thus have that $\lambda_1 + \lambda_2 - 2j \leq n$, and we thus have that there exists an index i in $\left[0, \frac{\lambda_2}{2}\right]$ such that

$$\lambda_1 + \lambda_2 - 4i = \lambda_3, \lambda_1 + \lambda_2 - 2i \leq n$$

as desired.

($\Longleftarrow$) Conversely, suppose that λ is a partition such that (a) The length of λ is 3; (b) Each entry of λ is even; (c) The first entry λ_1 of λ satisfies $\lambda_1 \leq 2\lfloor \frac{n}{2} \rfloor$; and (d) There exists an index i in $\left[0, \frac{\lambda_2}{2}\right]$ such that $\lambda_1 + \lambda_2 - 4i = \lambda_3, \lambda_1 + \lambda_2 - 2i \leq n$. Let p_1 denote the following set of pairwise disjoint 2-sets:

$$p_1 = \{\{1,2\},\{3,4\},\ldots,\{\lambda_1 - 1, \lambda_1\}\}.$$

Since λ_1 is even (since each entry of λ is even), our above definition of p_1 is well-defined. Since $\lambda_1 \leq 2\left\lfloor \frac{n}{2} \right\rfloor$, we thus have that p_1 is a set consisting of pairwise disjoint 2-subsets of $\{1,2,\ldots,n\}$. Since there exists an integer i in the interval $\left[0, \frac{\lambda_2}{2}\right]$ such that

$$\lambda_1 + \lambda_2 - 4i = \lambda_3, \lambda_1 + \lambda_2 - 2i \leq n$$

by assumption, let $j \in \left[0, \frac{\lambda_2}{2}\right]$ denote a fixed integer satisfying $\lambda_1 + \lambda_2 - 4j = \lambda_3$ and $\lambda_1 + \lambda_2 - 2j \leq n$. Now let p_2 denote the following set of pairwise disjoint 2-sets:

$$p_2 = \big\{ \{\lambda_1 - 2j + 1, \lambda_1 - 2j + 2\}, \{\lambda_1 - 2j + 3, \lambda_1 - 2j + 4\}, \ldots,$$

$$\{\lambda_1 - 2j + \lambda_2 - 1, \lambda_1 - 2j + \lambda_2\} \big\}.$$

The total number of elements among the distinct 2-sets in p_2 is thus

$$(\lambda_1 - 2j + \lambda_2) - (\lambda_1 - 2j + 1) + 1 = \lambda_2$$

and since λ_1 and λ_2 are both even, the above definition of p_2 is thus well-defined. Furthermore, since $\lambda_1 + \lambda_2 - 2j \leq n$, we thus have that p_2 is a set consisting of pairwise disjoint 2-subsets of $\{1,2,\ldots,n\}$. Now consider the expression $p_1 \Delta p_2 \in S$. The total number of elements among the 2-sets in the expression $p_1 \Delta p_2$ is equal to

$$(\lambda_1 - 2j) + (-2j + \lambda_2) = \lambda_1 + \lambda_2 - 4j$$

and thus since $\lambda_1 + \lambda_2 - 4j = \lambda_3$ we have that the expression $p_1 \Delta p_2 \in S$ consists of $\frac{\lambda_3}{2}$ pairwise disjoint 2-sets consisting of a total of λ_3 entries. Now consider the expression type(S). The set p_1 consists of $\frac{\lambda_1}{2}$ 2-sets, the set p_2 consists of $\frac{\lambda_2}{2}$ 2-sets, and the set p_3 consists of $\frac{\lambda_3}{2}$ 2-sets, with

$$\frac{\lambda_1}{2} \geq \frac{\lambda_2}{2} \geq \frac{\lambda_3}{2}$$

since $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is a partition. We thus have that type$(S) = \lambda$, thus proving that λ is a Klein partition for $n \in \mathbb{N}$. $\square$

Lemma 4.2. *For $n \in \mathbb{N}$, the Klein partitions for n are precisely tuples of the form*

$$(2a, 2b, 2a + 2b - 4i)$$

such that:

1. $1 \leq a \leq \left\lfloor \frac{n}{2} \right\rfloor$;
2. $1 \leq b \leq a$; *and*

3. $\max\left(a+b-\left\lfloor\frac{n}{2}\right\rfloor,\left\lceil\frac{a}{2}\right\rceil\right)\leq i\leq\min\left(b,\left\lfloor\frac{2a+2b-1}{4}\right\rfloor\right).$

Proof: Let $n\in\mathbb{N}$. Let $\lambda=(2a,2b,2a+2b-4i)$ be a tuple satisfying the conditions (1), (2), and (3) given above. We have that $1\leq\lambda_2\leq\lambda_1$ from condition (2). We have that $2a+2b-4i\leq 2b$ since $\frac{a}{2}\leq i$ since $\left\lceil\frac{a}{2}\right\rceil\leq i$ from condition (3), and we thus have that $\lambda_3\leq\lambda_2\leq\lambda_1$ as desired. We have that $1\leq 2a+2b-4i$ since $i\leq\frac{2a+2b-1}{4}$ since $i\leq\left\lfloor\frac{2a+2b-1}{4}\right\rfloor$ from condition (3), and we thus have that

$$1\leq\lambda_3\leq\lambda_2\leq\lambda_1$$

thus proving that that the tuple

$$\lambda=(2a,2b,2a+2b-4i)$$

is in fact an integer partition. We proceed to make use of Lemma 4.1. Certainly, λ is of length 3, and each entry of λ is even. The first entry $\lambda_1=2a$ of λ satisfies $\lambda_1=2a\leq 2\left\lfloor\frac{n}{2}\right\rfloor$ since $a\leq\left\lfloor\frac{n}{2}\right\rfloor$ from condition (1). Certainly, $\lambda_1+\lambda_2-4i=\lambda_3=2a+2b-4i$. Furthermore, we have that $\lambda_1+\lambda_2-2i\leq n$ since $a+b-\frac{n}{2}\leq i$ since $a+b-\left\lfloor\frac{n}{2}\right\rfloor\leq i$ from condition (3). By Lemma 4.1, we thus have that the partition λ is a Klein partition for $n\in\mathbb{N}$.

Conversely, let λ be a Klein partition for n. By Lemma 4.1, we thus have that:

(a) The length $\ell(\lambda)$ of λ is 3;

(b) Each entry of λ is even;

(c) The first entry λ_1 of λ satisfies $\lambda_1\leq 2\lfloor\frac{n}{2}\rfloor$; and

(d) There exists an integer $i\in\left[0,\frac{\lambda_2}{2}\right]$ such that:

$$\lambda_1+\lambda_2-4i=\lambda_3,\lambda_1+\lambda_2-2i\leq n.$$

Begin by rewriting the entries of $\lambda=(\lambda_1,\lambda_2,\lambda_3)$ as follows. By condition (b) we may thus write $\lambda_1=2a$ and $\lambda_2=2b$, letting $a,b\in\mathbb{N}$. Let $i\in\left[0,\frac{\lambda_2}{2}\right]$ be as given in condition (d) above. We thus have that $\lambda_3=2a+2b-4i$ and we thus have that the integer partition $\lambda=(\lambda_1,\lambda_2,\lambda_3)$ is a tuple of the following form:

$$\lambda=(2a,2b,2a+2b-4i).$$

Since λ is an integer partition, we have that $1\leq a$. Since the first entry λ_1 of λ satisfies $\lambda_1\leq 2\left\lfloor\frac{n}{2}\right\rfloor$ by condition (c) above, we thus have that $a\leq\left\lfloor\frac{n}{2}\right\rfloor$ and we thus have that the first condition given in Lemma 4.2 holds. Since λ is an integer partition, we have that $1\leq b\leq a$, and we thus have that the second condition given in Lemma 4.2 holds.

From condition (d), we have that

$$\lambda_1+\lambda_2-2i\leq n$$

and we thus have that $2a+2b-2i\leq n$ and thus $i\geq a+b-\frac{n}{2}$ and we thus have that $a+b-\left\lfloor\frac{n}{2}\right\rfloor\leq i$. Since λ is a partition, we have that

$$2a+2b-4i\leq 2b$$

and we thus have that $\frac{a}{2} \leq i$ and therefore $\left\lceil \frac{a}{2} \right\rceil \leq i$. From the inequality $a+b-\left\lfloor \frac{n}{2} \right\rfloor \leq i$ together with the inequality $\left\lceil \frac{a}{2} \right\rceil \leq i$ we thus have that

$$\max\left(a+b-\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{a}{2} \right\rceil\right) \leq i.$$

Since $i \in \left[0, \frac{\lambda_2}{2}\right]$, we thus have that $i \leq b$. Since λ is an integer partition, we have that $1 \leq \lambda_3$. Therefore, $1 \leq 2a+2b-4i$. We thus have that $i \leq \frac{2a+2b-1}{4}$, and we thus have that $i \leq \left\lfloor \frac{2a+2b-1}{4} \right\rfloor$. From the inequality $i \leq \left\lfloor \frac{2a+2b-1}{4} \right\rfloor$ together with the inequality $i \leq b$, we thus have that

$$i \leq \min\left(b, \left\lfloor \frac{2a+2b-1}{4} \right\rfloor\right)$$

thus proving that condition (3) given in Lemma 4.2 holds. □

Definition 4.1. Let λ be a partition. We define the maximum repetition length of λ as the maximum natural number m such that

$$\lambda_{i+1} = \lambda_{i+2} = \cdots = \lambda_{i+m}$$

for some $i \in \mathbb{N}_0$. The maximum repetition length of a partition λ is denoted by $\text{repeat}(\lambda)$.

Example 4.1. For a partition λ of length three, $\text{repeat}(\lambda) = 1$ if all three entries of λ are pairwise distinct, $\text{repeat}(\lambda) = 2$ if two entries of λ are equal but different from the remaining (third) entry, and $\text{repeat}(\lambda) = 3$ if $\lambda_1 = \lambda_2 = \lambda_3$.

Lemma 4.3. *Letting λ be a fixed Klein partition, the number of Δ-closed 4-sets consisting of $\varnothing$ together with subsets of a set consisting of pairwise disjoint 2-subsets of $\{1, 2, \ldots, n\}$ of partition type λ is*

$$\frac{1}{(\text{repeat}(\lambda))!} \frac{\prod_{j=0}^{\frac{\lambda_1}{2}-1} \binom{n-2j}{2}}{\left(\frac{\lambda_1}{2}\right)!} \binom{\frac{\lambda_1}{2}}{\frac{\lambda_1+\lambda_2-\lambda_3}{4}} \frac{\prod_{j=0}^{\frac{\lambda_2-\lambda_1+\lambda_3}{4}-1} \binom{n-\lambda_1-2j}{2}}{\left(\frac{\lambda_2-\lambda_1+\lambda_3}{4}\right)!}.$$

Proof: There are

$$\frac{\prod_{j=0}^{\frac{\lambda_1}{2}-1} \binom{n-2j}{2}}{\left(\frac{\lambda_1}{2}\right)!}$$

distinct sets of 2-sets in S_n of length $\frac{\lambda_1}{2}$. Let i denote the unique index in $\left[0, \frac{\lambda_2}{2}\right]$ such that $\lambda_1+\lambda_2-4i = \lambda_3$ and $\lambda_1+\lambda_2-2i \leq n$. We thus have that there are precisely i "overlap" 2-sets shared among the largest set of sets in a 4-set S of partition type λ and the second-largest set of sets in S. There are

$$\binom{\frac{\lambda_1}{2}}{\frac{\lambda_1+\lambda_2-\lambda_3}{4}}$$

choices for i "overlap" 2-sets, and for each such choice there are

$$\frac{\prod_{j=0}^{\frac{\lambda_2-\lambda_1+\lambda_3}{4}-1} \binom{n-\lambda_1-2j}{2}}{\left(\frac{\lambda_2-\lambda_1+\lambda_3}{4}\right)!}$$

remaining choices for the remaining 2-sets for the second-largest set in S. □

Theorem 4.4. *The number of Δ-closed 4-sets S such that there exists a set T consisting of pairwise disjoint 2-subsets of $[n]$ such that each element in S is contained in T is*

$$n!\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor}\sum_{j=1}^{i}\sum_{k=\max\left(\lceil \frac{i}{2} \rceil, i+j-\lfloor \frac{n}{2} \rfloor\right)}^{\min\left(j,\lfloor \frac{1}{4}(2i+2j-1) \rfloor\right)} \frac{2^{k-i-j}}{k!(i-k)!(j-k)!(n-2i-2j+2k)!(\delta_{i,j}+\delta_{i,2k}+1)!}$$

for arbitrary $n \in \mathbb{N}$.

Proof: From the above lemma, we have that the number of Δ-closed 4-sets consisting of the empty set together with subsets of a set consisting of pairwise disjoint 2-subsets of $\{1, 2, \ldots, n\}$ is

$$\sum_{\lambda} \frac{1}{(\text{repeat}(\lambda))!} \frac{\prod_{j=0}^{\frac{\lambda_1}{2}-1} \binom{n-2j}{2}}{\left(\frac{\lambda_1}{2}\right)!} \binom{\frac{\lambda_1}{2}}{\frac{\lambda_1+\lambda_2-\lambda_3}{4}} \frac{\prod_{j=0}^{\frac{\lambda_2-\lambda_1+\lambda_3}{4}-1} \binom{n-\lambda_1-2j}{2}}{\left(\frac{\lambda_2-\lambda_1+\lambda_3}{4}\right)!}$$

where the above sum is over all Klein partitions λ for n. By Lemma 4.2, we thus have that the above summation may be rewritten as:

$$\sum_{a=1}^{\lfloor \frac{n}{2} \rfloor}\sum_{b=1}^{a}\sum_{i=\max\left(a+b-\lfloor \frac{n}{2} \rfloor, \lceil \frac{a}{2} \rceil\right)}^{\min\left(b,\lfloor \frac{2a+2b-1}{4} \rfloor\right)} \frac{1}{(\delta_{0,a-b}+\delta_{0,2i-a}+1)!} \frac{\prod_{j=0}^{a-1} \binom{n-2j}{2}}{a!} \binom{a}{i} \frac{\prod_{j=0}^{b-i-1} \binom{n-2a-2j}{2}}{(b-i)!}$$

Rewriting the above expression by evaluating the products in the summand yields the desired result. □

The integer sequence

$$(0, 0, 0, 3, 15, 105, 525, 3255, 17703, 112455, 669735, 4485195, 29023995, 205768563, \ldots)$$

given by the number of Δ-closed 4-sets consisting of the empty set together with subsets of a set consisting of pairwise disjoint 2-subsets of $\{1, 2, \ldots, n\}$ is given in the On-Line Encyclopedia of Integer Sequences sequence A267840 which we contributed. For example, there are $\text{A267840}_n = 15$ symmetric difference-closed 4-sets of this form in the case whereby $n = 5$:

$$\{\varnothing, \{\{1,2\}\}, \{\{3,4\}\}, \{\{1,2\}, \{3,4\}\}\}, \{\varnothing, \{\{1,2\}\}, \{\{3,5\}\}, \{\{1,2\}, \{3,5\}\}\},$$
$$\{\varnothing, \{\{1,2\}\}, \{\{4,5\}\}, \{\{1,2\}, \{4,5\}\}\}, \{\varnothing, \{\{1,3\}\}, \{\{2,4\}\}, \{\{1,3\}, \{2,4\}\}\},$$
$$\{\varnothing, \{\{1,3\}\}, \{\{2,5\}\}, \{\{1,3\}, \{2,5\}\}\}, \{\varnothing, \{\{1,3\}\}, \{\{4,5\}\}, \{\{1,3\}, \{4,5\}\}\},$$
$$\{\varnothing, \{\{1,4\}\}, \{\{2,3\}\}, \{\{1,4\}, \{2,3\}\}\}, \{\varnothing, \{\{1,4\}\}, \{\{2,5\}\}, \{\{1,4\}, \{2,5\}\}\},$$
$$\{\varnothing, \{\{1,4\}\}, \{\{3,5\}\}, \{\{1,4\}, \{3,5\}\}\}, \{\varnothing, \{\{1,5\}\}, \{\{2,3\}\}, \{\{1,5\}, \{2,3\}\}\},$$
$$\{\varnothing, \{\{1,5\}\}, \{\{2,4\}\}, \{\{1,5\}, \{2,4\}\}\}, \{\varnothing, \{\{1,5\}\}, \{\{3,4\}\}, \{\{1,5\}, \{3,4\}\}\},$$
$$\{\varnothing, \{\{2,3\}\}, \{\{4,5\}\}, \{\{2,3\}, \{4,5\}\}\}, \{\varnothing, \{\{2,4\}\}, \{\{3,5\}\}, \{\{2,4\}, \{3,5\}\}\},$$
$$\{\varnothing, \{\{2,5\}\}, \{\{3,4\}\}, \{\{2,5\}, \{3,4\}\}\}.$$

It it natural to use Lemma 4.2 to determine "even" analogues of the above results.

Lemma 4.5. *For $n \in \mathbb{N}$, the Klein partitions for n corresponding to 4-sets consisting of $\varnothing$ together with subsets of a set consisting of pairwise disjoint 2-subsets of $[n]$ are precisely tuples of the form $(4d, 4e, 4d+4e-4i)$ such that:*

1. $1 \leq d \leq \left\lfloor \frac{n}{4} \right\rfloor$;
2. $1 \leq e \leq d$; *and*
3. $\max\left\{2d+2e-\left\lfloor \frac{n}{2} \right\rfloor, d\right\} \leq i \leq \min\left\{2e, \left\lfloor \frac{4d+4e-1}{4} \right\rfloor\right\}$.

Proof: The above lemma follows immediately from Lemma 4.2 by letting $a = 2d$ and $b = 2e$. □

Theorem 4.6. *The number of Δ-closed 4-sets consisting of even-order subsets of a set consisting of pairwise disjoint 2-subsets of $\{1, 2, \ldots, n\}$ is*

$$n! \sum_{i=1}^{\left\lfloor \frac{n}{4} \right\rfloor} \sum_{j=1}^{i} \sum_{k=\max\left(i, 2i+2j-\left\lfloor \frac{n}{2} \right\rfloor\right)}^{\min\left(2j, \left\lfloor \frac{1}{4}(4i+4j-1) \right\rfloor\right)} \frac{2^{k-2i-2j}}{k!(2i-k)!(2j-k)!(n-4i-4j+2k)!(\delta_{i,j}+\delta_{i,k}+1)!}$$

for arbitrary $n \in \mathbb{N}$.

Proof: The above theorem follows from Lemma 4.3 by analogy with Theorem 4.4. □

The corresponding integer sequence is given below, and is given in the sequence A266503 which we contributed to OEIS Foundation Inc. (2011).

$$(0, 0, 0, 0, 0, 15, 105, 735, 4095, 26775, 162855, 1105335, 7187895, 51126075, 356831475, \ldots).$$

5 Conclusion

The number of Klein partitions for $n = 1, 2, \ldots$ is given by the following integer sequence:

$$(0, 0, 0, 1, 1, 3, 3, 6, 6, 10, 10, 16, 16, 23, 23, 32, 32, 43, 43, 56, 56, 71, 71, 89, \ldots).$$

We have previously noted that the corresponding integer sequence

$$(0, 1, 3, 6, 10, 16, 23, 32, 43, 56, 71, 89, 109, 132, 158, \ldots)$$

coincides with the sequence A034198 given by the number of binary codes of a given length with 3 words, as indicated in OEIS Foundation Inc. (2011). We currently leave it as an open problem to use Lemma 4.2 to prove this. Proving this problem is nontrivial in the following sense. It may be difficult to construct a closed-form formula for the number of Klein partitions of $n \in \mathbb{N}$, since the definition of a Klein partition is somewhat complicated. Moreover, it may not be obvious as to how to relate such a formula to a known formula for the sequence A034198.

Interestingly, there are known connections between the OEIS sequence A034198 and Klein four-subgroups. In particular, A034198_n is the number of orbits of Klein subgroups of C_2^n under automorphisms of C_2^n, and A034198_n is the number of faithful representations of $K_4 = C_2^2$ of dimension n up to equivalence by automorphisms of C_2^2 (OEIS Foundation Inc. (2011)).

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