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This is the second of two papers in which we determine all 242 Wilf classes of triples of 4-letter permutation patterns by showing that there are 32 non-singleton Wilf classes. There are 317 symmetry classes of triples of 4-letter patterns and after computer calculation of initial terms, the problem reduces to showing that counting sequences that appear to be the same (i.e., agree in the first 16 terms) are in fact identical. This amounts to counting avoiders for 107 representative triples. The insertion encoding algorithm (INSENC) applies to many of them and some others have been previously counted. There remain 36 triples and the first paper dealt with the first 18. In this paper, we find the generating function for the last 18 triples which turns out to be algebraic in each case. Our methods are both combinatorial and analytic, including decompositions by left-right maxima and by initial letters. Sometimes this leads to an algebraic equation for the generating function, sometimes to a functional equation or a multi-index recurrence that succumbs to the kernel method. A particularly nice so-called cell decomposition is used in one of the cases (Case 238).

Keywords: pattern avoidance, Wilf equivalence, kernel method, insertion encoding algorithm

1 Introduction

In recent decades pattern avoidance has received a lot of attention. It has a prehistory in the work of MacMahon [16] and Knuth [12], but the paper that really sparked the current interest is by Simion and Schmidt [25]. They thoroughly analyzed 3-letter patterns in permutations, including a bijection between 123- and 132-avoiding permutations, thereby explaining the first (nontrivial) instance of what is, in modern terminology, a Wilf class. Since then the problem has been addressed on several other discrete structures, such as compositions, k-ary words, and set partitions; see, e.g., the texts [10, 17] and references contained therein.

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Permutations avoiding a single 4-letter pattern have been well studied (see, e.g., [27, 28, 31, 33]). There are 56 symmetry classes of pairs of 4-letter patterns, for all but 8 of which the avoiders have been enumerated. Le [15] established that these 56 symmetry classes form 38 distinct Wilf classes. Vatter [30] showed that of these 38, 12 can be enumerated with so-called regular insertion encodings (the INSENC algorithm, see also [1]). These generating functions were computed in part by hand by Kremer and Shiu [13]. See [34] for results concerning the enumeration of permutations avoiding a 3-letter and/or a 4-letter pattern, and also the references [2, 21–23, 29, 32] for further related results.

Much less is known about larger sets/longer patterns. Here, we consider the 317 symmetry classes of triples of 4-letter patterns and determine their Wilf classes. First, we used the software of Kuszmaul [14] to compute the initial terms $\{|S_n(T)|\}_{n=1}^{16}$ for a choice of T in each of the 317 symmetry classes. These results are available for reference in Table 2 in the Appendix of the arXiv full-length version [8] of the paper. There are 242 distinct 16-term sequences among the 317. This gives a lower bound of 242 on the number of Wilf classes, but we will show that whenever two sequences in Table 2 [8] agree in the first 16 terms, they are in fact identical, and so there are exactly 242 Wilf classes. To do so, we find the generating function for every triple whose 16-term counting sequence is repeated in Table 2 [8]. Thirty-eight of them can be found by INSENC; some others have already been counted and are referenced in Table 1 of the arXiv version [8]. We remark that in fact the first 9 terms of $\{|S_n(T)|\}_{n\geq 1}$ suffice to distinguish the Wilf classes for triples of 4-letter patterns; see concluding section for a further discussion.

There remain 36 triples to enumerate with 15 distinct counting sequences (cases). The first 9 cases (50, 55, 166, 171, 174, 177, 191, 196, 201 in the full table of 242 counting sequences [8]), treated in the first paper, cover 18 of these 36 triples. In this paper, we treat the remaining 6 cases (203, 218, 229, 234, 235, 238) which cover the last 18 triples, see Table 1 below.

To summarize, we say a Wilf class (of triples of 4-letter patterns) is *small* if it contains just one symmetry class and *large* if it contains more than one symmetry class. There are then 242 Wilf classes of triples of 4-letter patterns, of which 210 are small and 32 are large. All 32 generating functions for large Wilf classes turn out to be algebraic but the generating function for at least one small Wilf class – $S_n(4123, 4231, 4312)$ – is conjectured not to be differentiably finite (D-finite) and hence not algebraic [9]. Enumeration of some other small Wilf classes will be treated in forthcoming work; see [4, 5] for partial results.

Theorem 1 (Main Theorem). *There are exactly* 242 *Wilf classes of triples of* 4*-letter permutation patterns. Of these* 242 *Wilf classes,* 210 *consist of a single symmetry class and* 32 *consist of two or more symmetry classes.*

Tab. 1: Large Wilf classes of three 4-letter patterns from Case 203 to Case 239, numbering taken from Table 2 [8].

Start of Table			
No.	T	$\sum_{n\geq 0} \lvert S_n(T)\rvert x^n$	Reference
203	{3142,1432,1324}, {3124,1423,1234}	$\frac{1-x}{2-2x-(1-x-x^2)C(x)}$	Thm. 2, 5
215	{1243,2134,2143}, {1234,1243,2143} {1423,2314,2413}, {1423,1432,4123}	$\frac{1-4x+2x^2}{(1-x)(1-4x+x^2)}$	INSENC
218	{1342,2314,2413}, {3142,1324,1423} {3124,1423,1243}	$\frac{(1-2x)(1+\sqrt{1-4x})}{x^2+(2-4x+x^2)\sqrt{1-4x}}$	Thm. 6, 7, 10

Continuation of Table 1

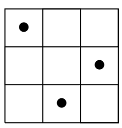
No.	T	$\sum_{n\geq0}\lvert S_n(T)\rvert x^n$	Reference
221	{2413,3142,1324}, {2143,3142,1324} {2143,1324,1423}, {3142,4132,1243} {3142,4123,1423}, {4132,1432,1243} {4132,1342,1324}	$1+\frac{1-2x}{2(1-x)}\left(\frac{1}{\sqrt{1-4x}}-1\right)$	[3]
229	{2413,3142, 2341}, {2143,1342,1423} {2134,1342,1423}	$\frac{1-2x+2x^2-\sqrt{1-8x+20x^2-24x^3+16x^4-4x^5}}{2x(1-x+x^2)}$	Thm. 11, 14, 17
233	{2143,1324,1243}, {2134,1324,1243} {2134,1243,1234}, {3142,4132,1432} {3142,4132,1342}, {3142,4132,1423} {3142,1342,1324}, {3124,1342,1324} {3124,1324,1423}, {4132,1432,1324} {4132,4123,1423}, {1342,4123,1423}	$\frac{2(1-4x)}{2-9x+4x^2-x\sqrt{1-4x}}$	[7]
234	{2143,2413,2314}, {3142,1342,1243}	$\frac{(1-x)^2-\sqrt{(1-x)^4-4x(1-2x)(1-x)}}{2x(1-x)}$	Thm. 18, 19
235	{1423,1432,2143}, {3142,1432,1423} {1234,1243,2314}	$F_T(x)=1-x+xF_T(x)$ $+x(1-2x)F_T^2(x)+x^2F_T^3(x)$	Thm. 23, 24, 27
236	{1423,3124,4123}, {1342,1432,4132} {1324,1423,1432}, {1243,1324,1423} {1234,1243,1423}	$\frac{1-5x+(1+x)\sqrt{1-4x}}{1-5x+(1-x)\sqrt{1-4x}}$	[6]
238	{1423,2413,3142}, {2134,2143,2413} {1342,1423,1234}, {1342,1423,1324} {1342,1423,1243}	$\frac{3-2x-\sqrt{1-4x}-\sqrt{2-16x+4x^2+(2+4x)\sqrt{1-4x}}}{2(1-\sqrt{1-4x})}$	Thm. 28, 29, 32, 33, 34
239	{2413,3412,3142}, {4312,3412,4132} {3412,3142,1342}, {3142,1432,1342} {3142,1342,1423}, {3124,1324,1243} {1432,1423,1243}, {1324,1423,1234} {4123,1423,1243}	$\frac{2}{1+x+\sqrt{1-6x+5x^2}}$	[20]

End of Table

2 Preliminaries and Notation

We say a permutation is *standard* if its support set is an initial segment of the positive integers, and for a permutation π whose support is any set of positive integers, St(π) denotes the standard permutation obtained by replacing the smallest entry of π by 1, the next smallest by 2, and so on. As usual, a standard permutation π *avoids* a standard permutation τ if there is no subsequence ρ of π for which St(ρ) $=\tau$. In this context, τ is called a *pattern*, and for a list T of patterns, $S_n(T)$ denotes the set of permutations of $[n]=\{1,2,\ldots,n\}$ that avoid all the patterns in T.

A permutation has an obvious representation as a matrix diagram,



matrix diagram of the permutation 312

and it will often be convenient to use such diagrams where shaded areas always indicate regions that contain no entries (blank regions may generally contain entries but in a few cases, as noted and clear from the context, they don't).

The eight symmetries of a square, generated by rotation and reflection, partition patterns and sets of patterns into symmetry classes on each of which the counting sequence for avoiders is obviously constant. Thus if π avoids τ then, for example, π^{-1} avoids τ^{-1} since inversion corresponds to flipping the matrix diagram across a diagonal. It sometimes happens (and remarkably often) that different symmetry classes have the same counting sequence, and all symmetry classes with a given counting sequence form a *Wilf class*. Thus Wilf classes correspond to counting sequences.

Throughout, $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ denotes the generating function for the Catalan numbers $C_n := \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$. As is well known [34], $C(x)$ is the generating function for $(|S_n(\pi)|)_{n\geq 0}$ where π is any one of the six 3-letter patterns.

A permutation π expressed as $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)}$ where $i_1 < i_2 < \cdots < i_m$ and $i_j > \max(\pi^{(j)})$ for $1 \leq j \leq m$ is said to have m *left-right maxima* (at $i_1, i_2, \ldots, i_m$). Given nonempty sets of numbers S and T, we will write $S < T$ to mean $\max(S) < \min(T)$ (with the inequality vacuously holding if S or T is empty). In this context, we will often denote singleton sets simply by the element in question. Also, for a number k, $S - k$ means the set $\{s - k : s \in S\}$. An *ascent* in π is a pair of adjacent increasing entries, thus 413625 has 3 ascents, 13, 36 and 25.

Our approach is ultimately recursive. In each case, we examine the structure of an avoider, usually by splitting the class of avoiders under consideration into subclasses according to a judicious choice of parameters which may involve, for example, left-right maxima, initial letters, ascents, and whether resulting subpermutations are empty or not. The choice is made so that each member of a subclass can be decomposed into independent parts. The generating function for the subclass (a summand of the full generating function) is then the product of the generating functions for the parts, and we speak of the "contribution" of the various parts to the generating function for that subclass. For Case 238, we use a cell decomposition, described in that subsection. From the structure, we are able to find an equation for the generating function $F_T(x) := \sum_{n\geq 0} |S_n(T)|x^n$, where T is the triple under consideration. This equation is often algebraic and, if linear or quadratic, as it is here in all but one case, easy to solve explicitly once found (the exception being the cubic equation for the triples in Case 235). It also frequently comes in the form of a functional equation requiring the *kernel method* (see, e.g., [11] for an exposition). In every case, the generating function turns out to be algebraic.

Furthermore, in several cases, especially those where recurrences are made use of, we have in fact counted members of the avoidance class in question according to the distribution of one or more statistics, specific to the class, and have assumed particular values of the parameters to obtain the avoidance result. In some of these cases, to aid in solving the recurrence, certain auxiliary arrays related to the statistic are introduced. This leads to systems of linear functional equations to which we apply the kernel method, adapted for a system. See, for example, the proof below of the first triple in Case 235. Also, in instances where the kernel method is used, it is usually possible (if desired) to solve the functional equation in its full generality yielding a polynomial generalization of the avoidance result.

We now proceed to the proofs for the 6 cases listed in the Introduction.

3 Proofs

3.1 Case 203

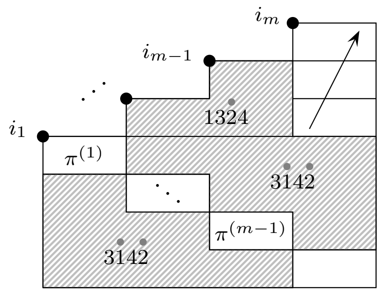
The two representative triples T are:
$\{1324,1432,3142\}$ (Theorem 2)
$\{1234,1342,2314\}$ (Theorem 5)

3.1.1 $\mathbf{T = \{1324, 1432, 3142\}}$

Theorem 2. *Let* $T = \{1324, 1432, 3142\}$. *Then*

$$F_T(x) = \frac{1-x}{2-2x-(1-x-x^2)C(x)}.$$

Proof: Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now let us write an equation for $G_m(x)$ with $m \geq 2$. Suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)}$ is a permutation that avoids T with $m \geq 2$ left-right maxima. Then $\pi^{(j)}$ avoids 132 for all $j = 1, 2, \ldots, m-1$ or else i_m is the 4 of a 1324. All the letters greater than i_1 in $\pi^{(m)}$ are increasing (to avoid 1432) and all the letters less than i_1 in $\pi^{(m)}$ are $<$ all letters in other π's (to avoid 3142), and $i_1 > \pi^{(1)} > \pi^{(2)} > \cdots > \pi^{(m-1)}$ (see figure, where the shaded regions are empty to avoid the indicated pattern with the gray bullets).



Also, at most one of the $m-1$ rectangles covered by the arrow can be occupied: ab in $\pi^{(m)}$ with b in a higher such rectangle than a makes ab the 24 of a 1324, and b in a lower rectangle than a makes ab the 32 of a 1432. So we distinguish two cases:

- all of these rectangles except possibly the top one are empty, i.e., there is no letter in $\pi^{(m)}$ between i_1 and i_{m-1}. In this case $\pi^{(m)}$ can be decomposed as

$$\beta^{(1)}(i_{m-1}+1)\beta^{(2)}(i_{m-1}+2)\cdots\beta^{(i_m-i_{m-1}-1)}(i_m-1)\beta^{(i_m-i_{m-1})}$$

  such that $\pi^{(m-1)} > \beta^{(1)} > \cdots > \beta^{(i_m-i_{m-1})}$, $\beta^{(j)}$ avoids 132 for $j = 1, 2, \ldots, i_m - i_{m-1} - 1$ and $\beta^{(i_m-i_{m-1})}$ avoids T. Since $\beta^{(j)}$ avoids 132, each $\beta^{(j)}(i_{m-1}+j)$ contributes $xC(x)$ and since there are zero or more of them, their contribution is $\frac{1}{1-xC(x)}$. Hence, this case contributes $\frac{x^mC(x)^{m-1}F_T(x)}{1-xC(x)}$.

- There is a letter in $\pi^{(m)}$ between i_p and i_{p+1} for some $p \in [m-2]$. Then $\pi^{(p+1)} = \cdots = \pi^{(m-1)} = \emptyset$ (3142) and $\pi^{(m)}$ can be decomposed as

$$\beta^{(1)}(i_p+1)\beta^{(2)}(i_p+2)\cdots\beta^{(i_{p+1}-i_p-1)}(i_{p+1}-1)\beta^{(i_{p+1}-i_p)}$$

such that $\pi^{(p)} > \beta^{(1)} > \cdots > \beta^{(i_{p+1}-i_p)}$ where all except the last $\beta^{(j)}$ avoid 132 and $\beta^{(i_{p+1}-i_p)}$ avoids T. This time there is at least one $\beta^{(j)}(i_p+j)$ and so we have an overall contribution of $\frac{x^{m+1}C(x)^{p+1}F_T(x)}{1-xC(x)}$.

Since $C(x) = \frac{1}{1-xC(x)}$, we find that

$$G_m(x) = x^m C(x)^m F_T(x) + \sum_{p=1}^{m-2} x^{m+1}C(x)^{p+2}F_T(x), \qquad m \geq 2,$$

with $G_1(x) = xF_T(x)$ and $G_0(x) = 1$.

From $F_T(x) = \sum_{m\geq 0} G_m(x)$, we deduce

$$F_T(x) = 1 + xF_T(x) + x^2C(x)^3F_T(x) - \frac{x^2C(x)F_T(x)}{1-x} + x^2C(x)^2F_T(x),$$

with solution

$$F_T(x) = \frac{1-x}{2-2x-(1-x-x^2)C(x)}.$$

□

3.1.2 $\mathbf{T = \{1234, 1342, 2314\}}$

A permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is said to have an *ascent* at index i if $\pi_i < \pi_{i+1}$, where $1 \leq i \leq n-1$. The letter π_{i+1} is called an *ascent top*. In order to count the members of $S_n(T)$, we categorize them by the nature of their leftmost ascent (i.e., smallest i such that $\pi_i < \pi_{i+1}$). If $n \geq 2$ and $1 \leq i \leq n-1$, let $a(n;i)$ denote the number of T-avoiding permutations of length n whose leftmost ascent occurs at index i, with $a(n;n) = 1$ for $n \geq 1$ (this accounts for the permutation $n(n-1)\cdots 1$, which is understood to have an ascent at index n). Let $a(n) = \sum_{i=1}^{n} a(n;i)$ for $n \geq 1$, with $a(0) = 1$.

We now consider various restrictions on the ascent top corresponding to the leftmost ascent which will prove helpful in determining a recurrence for $a(n;i)$. Let $A_{n,i}$ denote the subset of permutations of $S_n(T)$ enumerated by $a(n;i)$. If $1 \leq i \leq n-1$, let $b(n;i)$ be the number of members of $A_{n,i}$ in which the leftmost ascent top equals n. If $1 \leq i \leq n-2$, let $c(n;i)$ be the number of members of $A_{n,i}$ not starting with n in which the leftmost ascent top equals $n-1$. Finally, for $1 \leq i \leq n-2$, let $d(n;i)$ be the number of members of $A_{n,i}$ not starting with n in which the leftmost ascent top is less than $n-1$. For example, we have $b(4;2) = 3$, the enumerated permutations being 2143, 3142 and 3241, $c(4;1) = 2$, the permutations being 1324 and 2341 (note that 1342 and 2314 are excluded), and $d(5;3) = 2$, the permutations being 42135 and 43125. Note that by the definitions, we have

$$a(n;i) = a(n-1;i-1) + b(n;i) + c(n;i) + d(n;i), \qquad 1 \leq i \leq n-1, \tag{1}$$

upon considering whether or not a member of $A_{n,i}$ starts with n. The arrays $b(n;i)$, $c(n;i)$ and $d(n;i)$ are determined recursively as follows.

Lemma 3. *We have*

$$b(n;i) = \sum_{j=i}^{n-1} a(n-1;j), \qquad 1 \le i \le n-1, \tag{2}$$

$$c(n;i) = \sum_{j=1}^{n-i-1} a(j-1), \qquad 1 \le i \le n-2, \tag{3}$$

and

$$d(n;i) = c(n-1;i) + c(n-1;i-1) + d(n-1;i) + d(n-1;i-1), \qquad 1 \le i \le n-2. \tag{4}$$

Proof: Let $B_{n,i}$, $C_{n,i}$ and $D_{n,i}$ denote the subsets of $S_n(T)$ enumerated by $b(n;i)$, $c(n;i)$ and $d(n;i)$, respectively. For (2), observe that members of $B_{n,i}$ can be obtained by inserting n directly after the i-th letter of a member of $\cup_{j=i}^{n-1} A_{n-1,j}$, with such an insertion seen not to introduce an occurrence of any of the patterns in T (since the "4" does not correspond to the first ascent within these patterns). This insertion operation is seen to be a bijection and hence (2) follows. To show (3), note that members $\pi \in C_{n,i}$ must be of the form

$$\pi = \alpha j(n-1)\beta n\gamma,$$

where $\alpha = j+i-1, j+i-2, \ldots, j+1$ for some $j \in [n-i-1]$, $\beta = n-2, n-3, \ldots, j+i$, and γ is a T-avoider (on the letters in $[j-1]$). The section α if nonempty consists of a decreasing string of consecutive numbers ending in $j+1$ in order to avoid 2314, with all letters in $[j+i, n-2]$ required to be to the left of n and all letters in $[j-1]$ required to be to the right, in order to avoid 1342 or 2314, respectively. That β is decreasing is required in order to avoid 1234. Furthermore, one may verify that all permutations π of the stated form above avoid the patterns in T. Considering all possible j, we get $\sum_{j=1}^{n-i-1} a(j-1)$ possibilities for π, which gives (3).

Finally, to show (4), first note that one can express $\sigma \in D_{n,i}$ as

$$\sigma = \sigma^{(1)} jk \sigma^{(2)} \sigma^{(3)} \sigma^{(4)},$$

where $\sigma^{(1)}$ is a decreasing sequence of length $i-1$ in $[j+1, n-1]$, $1 \le j < k < n-1$, $\sigma^{(2)}$ is contained within $[j+1, k-1]$, $\sigma^{(3)}$ is a sequence in $[k+1, n]$ that contains n, and $\sigma^{(4)}$ is a permutation of $[j-1]$. Observe that $\sigma^{(3)}$ must decrease in order to avoid 1234 and hence starts with n. If $n-1$ belongs to $\sigma^{(3)}$, then removing n is seen to define a bijection with $C_{n-1,i} \cup D_{n-1,i}$. If $n-1$ belongs to $\sigma^{(1)}$, then removing $n-1$, and replacing n with $n-1$, defines a bijection with $C_{n-1,i-1} \cup D_{n-1,i-1}$. Combining the two previous cases implies (4) and completes the proof. $\square$

Let $a_n(u) = \sum_{i=1}^{n} a(n;i)u^i$ for $n \ge 1$, $b_n(u) = \sum_{i=1}^{n-1} b(n;i)u^i$ for $n \ge 2$, $c_n(u) = \sum_{i=1}^{n-2} c(n;i)u^i$ for $n \ge 3$, and $d_n(u) = \sum_{i=1}^{n-2} d(n;i)u^i$ for $n \ge 3$. For convenience, we take $a_0(u) = 1$.

Then recurrences (1) and (2) imply

$$a_n(u) = ua_{n-1}(u) + b_n(u) + c_n(u) + d_n(u), \qquad n \ge 1, \tag{5}$$

and

$$\begin{aligned} b_n(u) &= \sum_{i=1}^{n-1} u^i \sum_{j=i}^{n-1} a(n-1;j) = \sum_{j=1}^{n-1} a(n-1;j) \sum_{i=1}^{j} u^i \\ &= \frac{u}{1-u}(a_{n-1}(1) - a_{n-1}(u)), \qquad n \geq 2. \end{aligned} \tag{6}$$

Multiplying both sides of (3) by u^i, and summing over $1 \leq i \leq n-2$, yields

$$\begin{aligned} c_n(u) &= \sum_{j=1}^{n-2} a(j-1) \sum_{i=1}^{n-j-1} u^i \\ &= \frac{u}{1-u} \sum_{j=1}^{n-2} a(j-1) - \frac{1}{1-u} \sum_{j=1}^{n-2} a(j-1) u^{n-j}, \qquad n \geq 3. \end{aligned} \tag{7}$$

Finally, recurrence (4) gives

$$d_n(u) = (1+u)(c_{n-1}(u) + d_{n-1}(u)), \qquad n \geq 3. \tag{8}$$

Let $a(x;u) = \sum_{n\geq 0} a_n(u) x^n$. It is determined by the following functional equation.

Lemma 4. *We have*

$$\left(1 + \frac{xu^2}{1-u}\right) a(x;u) = 1 + xu \left(\frac{1}{1-u} + \frac{x^2}{(1-x)(1-xu)(1-x-xu)}\right) a(x;1). \tag{9}$$

Proof: Let $b(x;u) = \sum_{n\geq 2} b_n(u) x^n$, $c(x;u) = \sum_{n\geq 3} c_n(u) x^n$, and $d(x;u) = \sum_{n\geq 3} d_n(u) x^n$. Rewriting recurrences (5)–(8) in terms of generating functions yields the following:

$$\begin{aligned} a(x;u) &= 1 + xua(x;u) + b(x;u) + c(x;u) + d(x;u), \\ b(x;u) &= \frac{xu}{1-u}(a(x;1) - a(x;u)), \\ c(x;u) &= \frac{x^3 u}{(1-x)(1-xu)} a(x;1), \\ d(x;u) &= x(1+u)(c(x;u) + d(x;u)). \end{aligned}$$

Noting

$$c(x;u) + d(x;u) = c(x;u) + \frac{x(1+u)}{1-x(1+u)} c(x;u) = \frac{c(x;u)}{1-x(1+u)},$$

and using the expressions for $b(x;u)$ and $c(x;u)$ in the equation for $a(x;u)$, gives (9). □

We can now determine the generating function for the sequence $a(n)$.

Theorem 5. *Let* $T = \{1234, 1342, 2314\}$. *Then*

$$F_T(x) = \frac{1-x}{2 - 2x - (1-x-x^2)C(x)}.$$

Proof: In the present notation, we must find $a(x;1)$. Applying the kernel method to (9), and setting $u = C(x)$, gives

$$\begin{aligned} a(x;1) &= -\frac{(1-x)(1-u)(1-xu)(1-x-xu)}{xu(1-x)(1-xu)(1-x-xu)+x^3u(1-u)} \\ &= \frac{xu(1-x-xu)(1-x)}{x(1-x)^2 - x^2(1-x)u + x^3u(1-u)} \\ &= \frac{(1-x)(1-xu)}{x+(1-x)^2-2x(1-x)u}, \end{aligned}$$

where we have used the fact $xu^2 = u - 1$ several times. Multiplying the numerator and denominator of the last expression by u gives

$$a(x;1) = \frac{(1-x)(u-xu^2)}{(1-x+x^2)u - 2(1-x)(u-1)} = \frac{1-x}{2-2x-(1-x-x^2)u},$$

as desired. □

3.2 Case 218

The three representative triples T are:
$\{$1342,2314,2413$\}$ (Theorem 6)
$\{$1324,1423,3142$\}$ (Theorem 7)
$\{$1243,1342,2314$\}$ (Theorem 10)

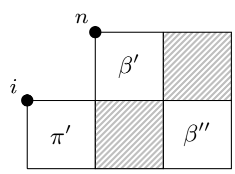
3.2.1 $\mathbf{T = \{1342, 2314, 2413\}}$

Theorem 6. *Let* $T = \{1342, 2314, 2413\}$. *Then*

$$F_T(x) = \frac{(1-2x)(1+\sqrt{1-4x})}{x^2 + (2-4x+x^2)\sqrt{1-4x}}.$$

Proof: Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now let us write an equation for $G_m(x)$ with $m \geq 2$.

For $m = 2$, suppose $\pi = i\pi' n\pi'' \in S_n(T)$ has two left-right maxima. In π'' all letters $> i$ occur before all letters $< i$, for otherwise π'' contains letters a, b with $a < i < b$ and $inab$ is a 2413. Thus, $\pi = i\pi' n\beta'\beta''$ with $\beta' > i > \beta''$:

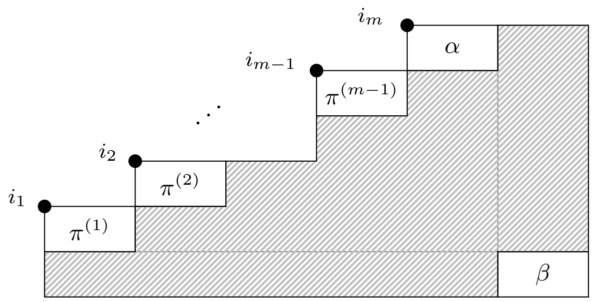


If β' is decreasing, then $\pi = i\pi' n(n-1)\cdots(i+1)\beta''$ and $\pi' i\beta'' \in S_i(T)$, giving a contribution of $\frac{x}{1-x}(F_T(x) - 1)$.

If β' is not decreasing, then $\pi' > \beta''$ (or an ascent ab in β' would be the 34 of a 1342); π' avoids 231 (or n is the 4 of a 2314); β' avoids 231 (or i is the 1 of a 1342), and β'' avoids T. Since β' is not decreasing, its contribution is $C(x) - \frac{1}{1-x}$, and the overall contribution of this case is $x^2C(x)\big(C(x) - \frac{1}{1-x}\big)F_T(x)$. Thus,

$$G_2(x) = \frac{x}{1-x}(F_T(x) - 1) + x^2C(x)\left(C(x) - \frac{1}{1-x}\right)F_T(x)\,.$$

Now, let $m \geq 3$ and suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)}$ is a permutation that avoids T with m left-right maxima. Let α (resp. β) denote the list of letters in $\pi^{(m)}$ that are greater than (resp. less than) i_1. All letters of α occur before all letters of β in $\pi^{(m)}$ (or i_1i_{m-1} are the 23 of a 2314) and so $\pi^{(m)} = \alpha\beta$; $\pi^{(1)} > \beta$ (or $a \in \pi^{(1)}$, $b \in \beta$ with $a < b$ makes ai_2i_mb a 1342); $\pi^{(j)} > i_{j-1}$ for $j = 2, \ldots, m-1$ (or $i_{j-1}i_ji_m$ are the 234 of a 2314); $\alpha > i_{m-1}$ (or $i_1i_{m-1}i_m$ are the 134 of a 1342). Thus, π has the form pictured.



Also, π_j avoids 231, $j = 1, 2, \ldots, m-1$ (or i_m is the 4 of a 2314); α avoids 231 (or i_{m-1} is the 1 of a 1342); β avoids T. Hence,

$$G_m(x) = x^mC^m(x)F_T(x)\,.$$

From $F_T(x) = \sum_{m\geq 0} G_m(x)$, we obtain

$$F_T(x) = 1 + xF_T(x) + \frac{x}{1-x}(F_T(x) - 1) + x^2C(x)\left(C(x) - \frac{1}{1-x}\right)F_T(x) + \sum_{m\geq 3} x^mC^m(x)F_T(x)\,.$$

Solving for $F_T(x)$ yields

$$F_T(x) = \frac{(1-2x)(1-xC(x))}{(1-2x)(1-x) - x(1-2x)(1-x)C(x) - x^2(1-2x)C^2(x) - x^4C^3(x)}\,,$$

which is equivalent to the desired expression. $\square$

3.2.2 $\mathbf{T = \{1324, 1423, 3142\}}$

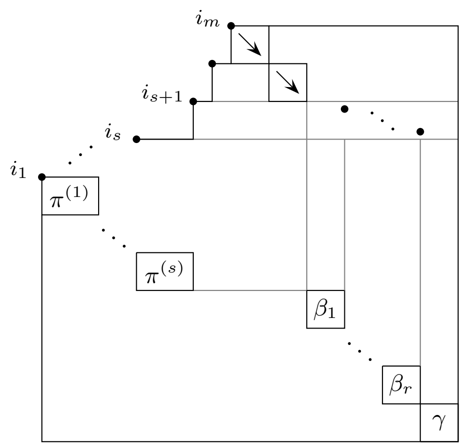
Theorem 7. *Let* $T = \{1324, 1423, 3142\}$. *Then*

$$F_T(x) = \frac{(1-2x)(1+\sqrt{1-4x})}{x^2 + (2-4x+x^2)\sqrt{1-4x}}\,.$$

Proof: Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ has $m \geq 2$ left-right maxima. Then $i_1 > \pi^{(j)}$ for all $j = 1, 2, \ldots, m-1$ to avoid 1324, and the letters $> i_1$ in $\pi^{(m)}$ are decreasing to avoid 1423. We consider two cases for $\pi^{(m)}$:

- Each letter of $\pi^{(m)}$ is either greater than i_{m-1} or smaller than i_1. In this case,
$$\pi^{(m)} = \beta^{(1)}(n-1)\cdots\beta^{(n-1-i_{m-1})}(i_{m-1}+1)\beta^{(n-i_{m-1})},$$
where $\pi^{(1)} > \cdots > \pi^{(m-1)} > \beta^{(1)} > \cdots > \beta^{(n-i_{m-1})}$ and $\pi^{(j)}$ avoids 132 for $j = 1, 2, \ldots, m-1$, $\beta^{(j)}$ avoids 132 for $j = 1, 2, \ldots, n-1-i_{m-1}$ and $\beta^{(n-i_{m-1})}$ avoids T. There are zero or more factors of the form $\beta_j(n-j)$, each contributing $xC(x)$. Hence, the contribution is
$$\frac{x^m C(x)^{m-1}F_T(x)}{1-xC(x)} = x^m C(x)^m F_T(x)\,.$$

- $\pi^{(m)}$ has a letter between i_1 and i_{m-1} (this case only arises for $m \geq 3$). Let $s \in [m-2\,]$ be the smallest index such that $\pi^{(m)}$ has a letter between i_s and i_{s+1}. Then $\pi^{(s+1)} = \cdots = \pi^{(m-1)} = \emptyset$ to avoid 3142, and π has the form



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where blank regions are empty and there is one β for each of the $r := i_{s+1} - i_s - 1$ letters in $[i_s+1, i_{s+1}-1]$, the π's and β's all avoid 132 (due to 1324), γ avoids T, and the arrows indicate decreasing entries. The π's contribute $C(x)^s$; each β and its associated letter between i_s and i_{s+1} contributes $xC(x)$ and there are one or more β's, so they contribute $\frac{xC(x)}{1-xC(x)}$; each of the $m-1-s$ arrows contributes $\frac{1}{1-x}$; γ contributes $F_T(x)$. Thus, for given $s \in [m-2]$, the contribution is
$$\frac{x^m C(x)^s F_T(x)}{(1-x)^{m-1-s}}\frac{xC(x)}{1-xC(x)} = \frac{x^{m+1}C(x)^{s+2}F_T(x)}{(1-x)^{m-1-s}}.$$

Hence, from $F_T(x) = \sum_{m\geq 0} G_m(x)$, we have

$$F_T(x) = 1 + xF_T(x) + \sum_{m\geq 2}\left(x^m C(x)^m F_T(x) + x^{m+1}C(x)^2 F_T(x)\sum_{s=1}^{m-2}\frac{C(x)^s}{(1-x)^{m-1-s}}\right),$$

with solution

$$F_T(x) = \frac{(1-2x)(1-xC(x))}{(1-2x)(1-x) - x(1-2x)(1-x)C(x) - x^2(1-2x)C(x)^2 - x^4C(x)^3},$$

which simplifies to the desired expression. □

3.2.3 $\mathbf{T = \{1243, 1342, 2314\}}$

We will employ an approach similar to that used for the second triple in case 203 above and make use of the same notation. As before, we have

$$a(n;i) = a(n-1;i-1) + b(n;i) + c(n;i) + d(n;i), \qquad 1 \leq i \leq n-1, \tag{10}$$

with $a(n;n) = 1$ for $n \geq 1$. The arrays $b(n;i)$, $c(n;i)$ and $d(n;i)$ are determined recursively as follows and a similar proof applies.

Lemma 8. *We have*

$$b(n;i) = \sum_{j=i}^{n-1} a(n-1;j), \qquad 1 \leq i \leq n-1, \tag{11}$$

$$c(n;i) = a(n-i-2) + \sum_{j=1}^{n-i-2} 2^{n-i-j-2} a(j-1), \qquad 1 \leq i \leq n-2, \tag{12}$$

$$d(n;i) = c(n-1;i) + c(n-1;i-1) + d(n-1;i) + d(n-1;i-1), \qquad 1 \leq i \leq n-2. \tag{13}$$

Note that the recurrences in Lemma 8 are the same as those in Lemma 3 except for a factor of $2^{n-i-j-2}$ appearing in the formula for $c(n;i)$. This accounts for the fact that within the decomposition of a T-avoiding permutation $\pi = \alpha j(n-1)\beta n\gamma$ enumerated by $c(n;i)$, where $\alpha = j+i-1, j+i-2, \ldots, j+1$ for some i, the section β is now a permutation of $[j+i, n-2]$ that avoids the patterns 132 and 231 (instead of just being a decreasing sequence as it was previously). Thus, there are $2^{n-i-j-2}$ possibilities for β whenever it is nonempty. Note that a comparison of the recurrences shows that there are strictly more permutations of length n that avoid $\{1243, 1342, 2314\}$ than there are that avoid $\{1234, 1342, 2314\}$ for $n \geq 5$.

If $a(x;u) = \sum_{n\geq 0} a_n(u)x^n$ as before, then one gets the following functional equation whose proof we omit.

Lemma 9. *We have*

$$\left(1 + \frac{xu^2}{1-u}\right)a(x;u) = 1 + xu\left(\frac{1}{1-u} + \frac{x^2(1-x)}{(1-2x)(1-xu)(1-x-xu)}\right)a(x;1). \tag{14}$$

We can now determine the generating function $F_T(x)$.

Theorem 10. *Let* $T = \{1243, 1342, 2314\}$. *Then*

$$F_T(x) = \frac{(1-2x)(1+\sqrt{1-4x})}{x^2 + (2-4x+x^2)\sqrt{1-4x}}.$$

Proof: Setting $u = C(x)$ in (14), and using the fact $xu^2 = u - 1$, gives

$$\begin{aligned}
a(x;1) &= -\frac{(1-2x)(1-u)(1-xu)(1-x-xu)}{xu(1-2x)(1-xu)(1-x-xu) + x^3u(1-x)(1-u)} \\
&= \frac{x(1-2x)(1-xu)}{x(1-2x)(1-x-xu) + x^2(1-x)(1-u+xu)} \\
&= \frac{(1-2x)(1+\sqrt{1-4x})}{(1-2x)(1-2x+\sqrt{1-4x}) + (1-x)(3x-1+(1-x)\sqrt{1-4x})} \\
&= \frac{(1-2x)(1+\sqrt{1-4x})}{x^2 + (2-4x+x^2)\sqrt{1-4x}},
\end{aligned}$$

as desired. □

3.3 Case 229

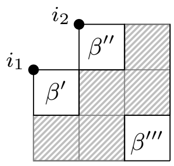
The three representative triples T are:
{2341,2413,3142} (Theorem 11)
{1342,1423,2143} (Theorem 14)
{1342,1423,2134} (Theorem 17)

3.3.1 $\mathbf{T = \{2341, 2413, 3142\}}$

Theorem 11. *Let* $T = \{2341, 2413, 3142\}$. *Then*

$$F_T(x) = \frac{1 - 2x + 2x^2 - \sqrt{1 - 8x + 20x^2 - 24x^3 + 16x^4 - 4x^5}}{2x(1-x+x^2)}.$$

Proof: Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now suppose $m \geq 2$ and $\pi = i_1\pi^{(1)} \cdots i_m\pi^{(m)}$ avoids T. Clearly, there is no letter smaller than i_1 in $\pi^{(3)} \cdots \pi^{(m)}$ (such a letter would be the "1" of a 2341). Moreover, to avoid 2413 and 3142, $\pi^{(1)}i_2\pi^{(2)}$ has the form $\beta' i_2 \beta'' \beta'''$ with $\beta'' > i_1 > \beta' > \beta'''$:



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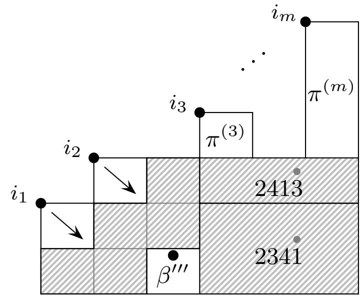
If $\beta''' = \emptyset$, then we have a contribution of $xF_T(x)G_{m-1}(x)$. Otherwise, π has the form



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where dark bullets indicate mandatory entries, shaded regions are empty (gray bullets would form part of a forbidden pattern as indicated), β' is decreasing ($b < c$ in β' implies bci_2a is a 2341 for a in β'''), and β'' is decreasing ($b < c$ in β'' implies i_1bca is a 2341).

Thus, we have a contribution of $\frac{x^2}{(1-x)^2}(F_T(x) - 1)G_{m-2}(x)$. Hence, for $m \geq 2$,

$$G_m(x) = xF_T(x)G_{m-1}(x) + \frac{x^2}{(1-x)^2}(F_T(x) - 1)G_{m-2}(x)\,.$$

By summing over $m \geq 2$, we obtain

$$F_T(x) - 1 - xF_T(x) = xF_T(x)(F_T(x) - 1) + \frac{x^2}{(1-x)^2}(F_T(x) - 1)F_T(x)\,.$$

Solving this quadratic for $F_T(x)$ completes the proof. □

3.3.2 $\mathbf{T = \{1342, 1423, 2143\}}$

Here, and in the subsequent subsection, let $a(n; i_1, i_2, \ldots, i_k)$ denote the number of T-avoiding permutations of length n starting with $i_1, i_2, \ldots, i_k$. Let $a(n) = \sum_{i=1}^{n} a(n; i)$ for $n \geq 1$ be the total number of T-avoiders, with $a(0) = 1$, and $\mathcal{T}_{i,j}$ be the set of permutations enumerated by $a(n; i, j)$. Clearly, $a(n; n) = a(n; n-1) = a(n-1)$ for all $n \geq 2$. We have the following recurrence for the array $a(n; i, j)$.

Lemma 12. *If $n \geq 3$, then*

$$a(n; i, j) = a(n - j + i + 1; i + 1, i) + \sum_{\ell=1}^{i-1} a(n - j + i + 1; i, \ell), \qquad i + 2 \leq j \leq n, \tag{15}$$

$$a(n; i, i - 1) = a(n - 1; i; i - 1) + \sum_{\ell=1}^{i-2} a(n - 1; i - 1, \ell), \qquad 2 \leq i \leq n - 1, \tag{16}$$

and

$$a(n; i, j) = a(n - 1; i - 1, j) + \sum_{r=2}^{i-j} a(n - r; j + 1, j) + \sum_{r=1}^{i-j}\sum_{\ell=1}^{j-1} a(n - r; j, \ell) \tag{17}$$

for $3 \leq i \leq n - 1$ and $1 \leq j \leq i - 2$, with $a(n; i, i + 1) = a(n - 1; i)$ for $1 \leq i \leq n - 1$.

Proof: Clearly, we have $|\mathcal{T}_{i,i+1}| = a(n-1;i)$, as the letter $i+1$ may be deleted. Let x denote the third letter of a member of $\mathcal{T}_{i,j}$. To show (16), first note that members of $\mathcal{T}_{i,i-1}$ must have $x = i+1$ or $x < i-1$. In the first case, the letter $i+1$ may be deleted, implying $a(n-1;i,i-1)$ possibilities, while in the latter, the letter i may be, which gives $\sum_{\ell=1}^{i-2} a(n-1;i-1,\ell)$ possibilities. We now show (15). Note first that one cannot have $x > j$ or $x < i$ within members of $\mathcal{T}_{i,j}$ if $j \geq i+3$, lest there be an occurrence of 1342 or 1423 (as witnessed by $ijx(j-1)$ or $ij(j-2)(j-1)$, respectively). So we must have $x \in [i+1, j-1]$ and thus $x = j-1$ in order to avoid 1423. By similar reasoning, the fourth letter must be $x-1$ if $x \geq i+3$. Repeating this argument shows that the block of letters $j, j-1, \ldots, i+2$ must occur. The next letter z must be $i+1$ or less than i (so as to avoid 1342). If $z = i+1$, then all members of $[i+3, j]$, along with i, are seen to be irrelevant concerning avoidance of T and hence may be deleted, while if $z < i$, then all members of $[i+2, j]$ may be deleted (note that i, z imposes the same requirement on subsequent letters as does $i, i+2$ and $i+2, z$, together). It follows that there are $a(n-j+i+1; i+1, i) + \sum_{\ell=1}^{i-1} a(n-j+i+1; i, \ell)$ members of $\mathcal{T}_{i,j}$ when $j \geq i+2$.

For (17), we consider the following cases for x: (i) $x = j+1$, (ii) $x < j$, (iii) $j+1 < x < i$, and (iv) $x = i+1$. There are clearly $a(n-1; i-1, j)$ possibilities in (i) and $\sum_{\ell=1}^{j-1} a(n-1; j, \ell)$ possibilities in (ii). Reasoning as in the previous paragraph shows in case (iii) that the block of letters $x, x-1, \ldots, j+2$ must occur directly following j. The next letter z may either equal $j+1$ or be less than j. Thus, all members of $[j+3, x]$, along with i, may be deleted in either case. Furthermore, the letter j may also be deleted if $z = j+1$ (since $j+2, j+1$ is more restrictive than i, j), while the letter $j+2$ may be deleted if $z < j$ (since $j+2$ is redundant in light of j, z). Considering all possible x, and letting $r = x - j$, one gets $\sum_{r=2}^{i-j-1} a(n-r; j+1, j)$ possibilities if $z = j+1$, and $\sum_{r=2}^{i-j-1} \sum_{\ell=1}^{j-1} a(n-r; j, \ell)$ possibilities if $z < j$. If $x = i+1$, then the block $x, x-2, x-3, \ldots, j+2$ must occur with the next letter z as in case (iii) above. This implies that there are $a(n-i+j; j+1, j) + \sum_{\ell=1}^{j-1} a(n-i+j; j, \ell)$ possibilities in (iv). Combining all of the previous cases gives (17) and completes the proof. $\square$

In order to solve the recurrence in Lemma 12, we introduce the following auxiliary functions: $b_{n,i}(v) = \sum_{j=1}^{i-1} a(n;i,j)v^j$ for $2 \leq i \leq n-1$, $c_{n,i}(v) = \sum_{j=i+1}^{n} a(n;i,j)v^j$ for $1 \leq i \leq n-1$, $b_n(u,v) = \sum_{i=2}^{n-2} b_{n,i}(v)u^i$ for $n \geq 4$, $c_n(u,v) = \sum_{i=1}^{n-2} c_{n,i}(v)u^i$ for $n \geq 3$, and $d_n(u) = \sum_{i=2}^{n-1} a(n;i,i-1)u^i$ for $n \geq 3$. Let $a_n(u,v) = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} a(n;i,j)u^i v^j$ for $n \geq 2$, with $a_1(u,v) = u$. Note that by the definitions, we have

$$a_n(u,v) = u^{n-1}(1+u)a_{n-1}(v,1) - (uv)^{n-1}(1-v)a_{n-2}(1,1) + b_n(u,v) + c_n(u,v), \qquad n \geq 2. \quad (18)$$

By (16) and (17), we have for $2 \leq i \leq n-2$,

$$b_{n,i}(v) = b_{n-1,i-1}(v) + a(n-1;i,i-1)v^{i-1} + \sum_{j=1}^{i-1} b_{n-1,j}(1)v^j + \sum_{j=1}^{i-2} v^j \sum_{r=2}^{i-j} a(n-r;j+1,j)$$
$$+ \sum_{j=1}^{i-2} v^j \sum_{r=2}^{i-j} b_{n-r,j}(1).$$

Multiplying both sides of the last recurrence by u^i, and summing over $2 \le i \le n-2$, yields

$$\begin{aligned}
b_n(u,v) &= ub_{n-1}(u,v) + \frac{1}{v}d_{n-1}(uv) + \sum_{j=1}^{n-3} b_{n-1,j}(1)\left(\frac{u^{j+1}-u^{n-1}}{1-u}\right)v^j \\
&\quad + \sum_{j=1}^{n-4} v^j \sum_{r=2}^{n-j-1} a(n-r;j+1,j)\left(\frac{u^{j+r}-u^{n-1}}{1-u}\right) + \sum_{j=1}^{n-4} v^j \sum_{r=2}^{n-j-2} b_{n-r,j}(1)\left(\frac{u^{j+r}-u^{n-1}}{1-u}\right) \\
&= ub_{n-1}(u,v) + \frac{1}{v}d_{n-1}(uv) + \frac{u}{1-u}b_{n-1}(uv,1) - \frac{u^{n-1}}{1-u}b_{n-1}(v,1) \\
&\quad + \frac{1}{uv(1-u)}\sum_{r=2}^{n-2}(d_{n-r}(uv)u^r + a(n-r-2)u^n v^{n-r}) \\
&\quad - \frac{u^{n-1}}{v(1-u)}\sum_{r=2}^{n-2}(d_{n-r}(v) + a(n-r-2)v^{n-r}) + \frac{1}{1-u}\sum_{r=2}^{n-3} b_{n-r}(uv,1)u^r \\
&\quad - \frac{u^{n-1}}{1-u}\sum_{r=2}^{n-3} b_{n-r}(v,1) \\
&= ub_{n-1}(u,v) + \frac{1}{v}d_{n-1}(uv) + \frac{1}{1-u}\sum_{r=3}^{n-1} b_r(u,v)u^{n-r} - \frac{u^{n-1}}{1-u}\sum_{r=3}^{n-1} b_r(v,1) \\
&\quad + \frac{1}{uv(1-u)}\sum_{r=2}^{n-2} d_r(uv)u^{n-r} - \frac{u^{n-1}}{v(1-u)}\sum_{r=2}^{n-2} d_r(v), \qquad n \ge 4. \qquad (19)
\end{aligned}$$

By (15), we have

$$c_{n,i}(v) = a(n-1;i)v^{i+1} + \sum_{j=2}^{n-i} a(n-j+1;i+1,i)v^{i+j} + \sum_{j=2}^{n-i} b_{n-j+1,i}(1)v^{i+j}, \qquad 1 \le i \le n-2,$$

and thus

$$\begin{aligned}
&c_n(u,v) \\
&= v\sum_{i=1}^{n-2} a(n-1;i)(uv)^i + \sum_{j=2}^{n-1} v^j \sum_{i=1}^{n-j} a(n-j+1;i+1,i)(uv)^i \\
&\quad + \sum_{j=2}^{n-1} v^j \sum_{i=1}^{n-j} b_{n-j+1,i}(1)(uv)^i \\
&= v(a_{n-1}(uv,1) - a(n-2)(uv)^{n-1}) + \frac{1}{u}\sum_{j=2}^{n-1} v^{j-1}(d_{n-j+1}(uv) + a(n-j-1)(uv)^{n-j+1}) \\
&\quad + \sum_{j=2}^{n-1} v^j(b_{n-j+1}(uv,1) + (a(n-j) - a(n-j-1))(uv)^{n-j}),
\end{aligned}$$

which leads to

$$\begin{aligned}
&c_n(u,v)\\
&= v(a_{n-1}(uv,1) - a(n-2)(uv)^{n-1}) + \frac{1}{u}\sum_{j=2}^{n-1} d_j(uv)v^{n-j} + \sum_{j=2}^{n-1} b_j(uv,1)v^{n-j+1}\\
&\quad + v^n \sum_{j=1}^{n-2} a(j)u^j, \qquad n \geq 3. \qquad (20)
\end{aligned}$$

Multiplying both sides of (16) by u^i, and summing over $2 \leq i \leq n-1$, gives

$$d_n(u) = u^{n-1}a(n-2) + ub_{n-1}(u) + d_{n-1}(u), \qquad n \geq 3. \tag{21}$$

Define generating functions $a(x;u,v) = \sum_{n\geq 1} a_n(u,v)x^n$, $b(x;u,v) = \sum_{n\geq 4} b_n(u,v)x^n$, $c(x;u,v) = \sum_{n\geq 3} c_n(u,v)x^n$, and $d(x;u) = \sum_{n\geq 3} d_n(u)x^n$. Rewriting recurrences (18)–(21) in terms of generating functions yields the following system of functional equations.

Lemma 13. *We have*

$$\begin{aligned}
a(x;u,v) &= xu(1-xv+xv^2) + b(x;u,v) + c(x;u,v) + x(1+u)a(xu;v,1)\\
&\quad - x^2uv(1-v)a(xuv;1,1), \qquad (22)
\end{aligned}$$

$$\begin{aligned}
(1-xu)b(x;u,v) &= \frac{x}{(1-u)(1-xu)}(ub(x;uv,1) - b(xu;v,1)) + \frac{x(1-u+xu^2)}{v(1-u)(1-xu)}d(x;uv)\\
&\quad - \frac{x^2u}{v(1-u)(1-xu)}d(xu;v), \qquad (23)
\end{aligned}$$

$$\begin{aligned}
c(x;u,v) &= -x^2uv^2 + xva(x;uv,1) + \left(\frac{x^2v^2}{1-xv} - x^2uv^2\right)a(xuv;1,1) + \frac{xv^2}{1-xv}b(x;uv,1)\\
&\quad + \frac{xv}{u(1-xv)}d(x;uv), \qquad (24)
\end{aligned}$$

$$(1-x)d(x;u) = x^2ua(xu;1,1) + xub(x;u,1). \tag{25}$$

We can now determine the generating function $F_T(x)$.

Theorem 14. *Let* $T = \{1342, 1423, 2143\}$. *Then*

$$F_T(x) = \frac{1-2x+2x^2-\sqrt{1-8x+20x^2-24x^3+16x^4-4x^5}}{2x(1-x+x^2)}.$$

Proof: In the notation above, we seek to determine $1 + a(x; 1, 1)$. Setting $u = v = 1$ in (22), (24) and (25), and solving the resulting system for $b(x; 1, 1)$, $c(x; 1, 1)$ and $d(x; 1)$, yields

$$\begin{aligned} b(x;1,1) &= \frac{(1-5x+7x^2-5x^3+x^4)a(x;1,1)-x(1-x)^3}{1-x+x^2}, \\ c(x;1,1) &= \frac{x(2-2x+x^2)((1-x)a(x;1,1)-x)}{1-x+x^2}, \\ d(x;1) &= \frac{x(1-x)^2((1-x)a(x;1,1)-x)}{1-x+x^2}. \end{aligned}$$

Substituting the expression for $d(x; u)$ from (25) into (23) at $v = 1$, we find

$$\begin{aligned} &\left(1-x-\frac{x}{(1-u)(1-x)}-\frac{x^2(1-u+xu)}{(1-u)(1-x)(u-x)}\right)b(x/u;u,1) \\ &= \frac{x^3(1-u+xu)}{u(1-u)(1-x)(u-x)}a(x;1,1)-\frac{x}{u(1-u)(1-x)}b(x;1,1)-\frac{x^2}{u(1-u)(1-x)}d(x;1). \end{aligned}$$

Applying the kernel method to the preceding equation, and setting

$$u = u_0 = \frac{1-2x+\sqrt{1-8x+20x^2-24x^3+16x^4-4x^5}}{2(1-x)^2},$$

we obtain

$$b(x;1,1) = \frac{x^2(1-u_0+xu_0)}{u_0-x}a(x;1,1)-xd(x;1).$$

Substituting out the expressions above for $b(x; 1, 1)$ and $d(x; 1)$, and then solving the equation that results for $a(x; 1, 1)$, yields

$$a(x;1,1) = \frac{x(1-x)^2(u_0-x)}{(1-4x+4x^2-2x^3)u_0-x(1-x)^3}.$$

Substituting the expression for u_0 into the last equation gives the desired formula for $1 + a(x; 1, 1)$ and completes the proof. ☐

Remark: Once $a(x; 1, 1)$ is known, it is possible to find $b(x; u, 1)$, and thus $d(x; u)$, $a(x; u, 1)$ and $c(x; u, 1)$. This in turn allows one to solve the system (22)–(25) for all u and v, and thus obtain a generating function formula for the joint distribution of the statistics recording the first two letters.

3.3.3 $\mathbf{T = \{1342, 1423, 2134\}}$

Clearly, $a(n; n) = a(n-1)$ for all $n \geq 1$. We have the following recurrence for the array $a(n; i, j)$ where $i < n$.

Lemma 15. *If $n \geq 3$, then*

$$a(n;i,n) = a(n-1;i,n-1)+\sum_{j=1}^{i-1}a(n-1;i,j), \qquad 1 \leq i \leq n-2, \tag{26}$$

$$a(n;i,i-1) = a(n-1;i-1,n-1) + \sum_{j=1}^{i-2} a(n-1;i-1,j), \qquad 2 \le i \le n-1, \tag{27}$$

and

$$a(n;i,j) = a(n-1,i-1,j) + a(n-1;j,n-1) + \sum_{\ell=1}^{j-1} a(n-1;j,\ell)$$
$$+ \sum_{\ell=2}^{i-j-1} a(n-\ell;j+1,j), \qquad 1 \le j \le i-2 \quad \text{and} \quad 3 \le i \le n-1. \tag{28}$$

Furthermore, we have $a(n;i,i+1) = a(n-1;i)$ *for* $1 \le i \le n-1$ *and* $a(n;i,j) = a(n-j+i+1;i+1,i)$ *for* $i+2 \le j \le n-1$.

Proof: Throughout, let x denote the third letter of a member of $\mathcal{T}_{i,j}$. To show (26), first note that for members of $\mathcal{T}_{i,n}$, we must have $x = n-1$ or $x < i$. There are $a(n-1;i,n-1)$ possibilities in the first case as the letter n is extraneous concerning avoidance of T, whence it may be deleted, and $\sum_{j=1}^{i-1} a(n-1;i,j)$ possibilities in the latter case as again n may be deleted (note that the presence of i,j imposes a stronger restriction on the order of subsequent letters than does i,n). To show (27), first note that members of $\mathcal{T}_{i,i-1}$ for $2 \le i \le n-1$ must have $x = n$ or $x < i-1$. There are $a(n-1;i-1,n-1)$ possibilities in the former case and $\sum_{j=1}^{i-2} a(n-1;i-1,j)$ possibilities in the latter since the letter i may be deleted in either case as the restriction it imposes is redundant.

Next, we show (28). For this, we consider the following cases: (i) $x = j+1$, (ii) $x = n$, (iii) $x < j$, and (iv) $j+1 < x < i$. The first three cases are readily seen to be enumerated by the first three terms, respectively, on the right-hand side of (28). For case (iv), let y denote the fourth letter of $\pi \in \mathcal{T}_{i,j}$. First note that one cannot have $y > x$, for otherwise π would contain 1342 as witnessed by the subsequence $jxy(x-1)$. It is also not possible to have $y < j$, for otherwise π would again contain 1342, this time with the subsequence $jxn(j+1)$, since all letters to the right of y and larger than j would have to occur in decreasing order (due to the presence of j,y). So we must have $j < y < x$ and thus $y = x-1$ in order to avoid 1423. By similar reasoning, the next letter must be $x-2$ if $x > j+2$. Repeating this argument shows that the block $x, x-1, \ldots, j+1$ must occur directly following j, with each of these letters, except the last two, seen to be extraneous concerning the avoidance or containment of patterns in T. Note further that the presence of $j+2, j+1$ imposes a stricter requirement on subsequent letters than does i,j when $i \ge j+3$, whence the i and j are also extraneous. Deleting all members of $[j+3,x]$ from π, along with i and j, implies that there are $a(n-\ell;j+1,j)$ possibilities where $\ell = x-j$. Summing over all possible values of ℓ gives the last term on the right-hand side of (28).

There are clearly $a(n-1;i)$ members of $\mathcal{T}_{i,j}$ if $j = i+1$, as the letter $i+1$ may be deleted. If $j \ge i+2$, then similar reasoning as before shows that the block $j, j-1, \ldots, i+1$ must occur when $j < n$, and thus all members of $[i+3,j]$, along with i, may be deleted. This implies that there are $a(n-j+i+1;i+1,i)$ members of $\mathcal{T}_{i,j}$ in this case, which completes the proof. □

In order to solve the recurrence in Lemma 15, we introduce the following functions: $b_{n,i}(v) = \sum_{j=1}^{i-1} a(n;i,j)v^j$ for $2 \le i \le n-1$, $c_{n,i}(v) = \sum_{j=i+1}^{n-1} a(n;i,j)v^j$ for $1 \le i \le n-2$, $b_n(u,v) =$

$\sum_{i=2}^{n-1} b_{n,i}(v)u^i$ for $n \geq 3$, $c_n(u,v) = \sum_{i=1}^{n-2} c_{n,i}(v)u^i$ for $n \geq 3$, and $d_n(u) = \sum_{i=1}^{n-1} a(n;i,n)u^i$ for $n \geq 2$. Let $a_n(u,v) = \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} a(n;i,j)u^iv^j$ for $n \geq 2$, with $a_1(u,v) = u$. Note that by the definitions, we have

$$a_n(u,v) = u^n a_{n-1}(v,1) + b_n(u,v) + c_n(u,v) + v^n d_n(u), \qquad n \geq 2. \tag{29}$$

In order to determine a formula for $b_n(u,v)$, first note that (27) and (28) imply

$$\begin{aligned} b_{n,i}(v) &= b_{n-1,i-1}(v) + \sum_{j=1}^{i-1} a(n-1;j,n-1)v^j + \sum_{j=1}^{i-1} b_{n-1,j}(1)v^j \\ &\quad + \sum_{j=1}^{i-3} v^j \sum_{\ell=2}^{i-j-1} a(n-\ell;j,n-\ell), \qquad 2 \leq i \leq n-1, \end{aligned}$$

where we have used the fact $a(m;j+1,j) = a(m;j,m)$ in the last sum. Multiplying both sides of the last recurrence by u^i, and summing over $2 \leq i \leq n-1$, gives

$$\begin{aligned} b_n(u,v) &= ub_{n-1}(u,v) + \sum_{j=1}^{n-2} a(n-1;j,n-1)\left(\frac{u^{j+1}-u^n}{1-u}\right)v^j + \sum_{j=1}^{n-2} b_{n-1,j}(1)\left(\frac{u^{j+1}-u^n}{1-u}\right)v^j \\ &\quad + \sum_{j=1}^{n-4} v^j \sum_{\ell=2}^{n-j-1} a(n-\ell;j,n-\ell)\left(\frac{u^{j+\ell+1}-u^n}{1-u}\right) \\ &= ub_{n-1}(u,v) + \frac{u}{1-u}(d_{n-1}(uv) - u^{n-1}d_{n-1}(v)) + \frac{u}{1-u}(b_{n-1}(uv,1) - u^{n-1}b_{n-1}(v,1)) \\ &\quad + \frac{u}{1-u}\sum_{\ell=2}^{n-2} d_\ell(uv)u^{n-\ell} - \frac{u^n}{1-u}\sum_{\ell=2}^{n-2} d_\ell(v), \qquad n \geq 3, \end{aligned} \tag{30}$$

where we have replaced the index ℓ by $n-\ell$ in the last sum.

By Lemma 15, we have

$$c_{n,i}(v) = a(n-1;i)v^{i+1} + \sum_{j=i+2}^{n-1} a(n-j+i+1;i+1,i)v^j, \qquad 1 \leq i \leq n-2,$$

and thus

$$\begin{aligned} c_n(u,v) &= \sum_{i=1}^{n-2} a(n-1;i)u^iv^{i+1} + \sum_{i=1}^{n-3} u^i \sum_{j=2}^{n-i-1} a(n-j+1;i+1,i)v^{i+j} \\ &= v(a_{n-1}(uv,1) - (uv)^{n-1}a(n-2)) + \sum_{j=2}^{n-2} v^j \sum_{i=1}^{n-j-1} a(n-j+1;i,n-j+1)(uv)^i, \end{aligned}$$

which implies

$$\begin{aligned}c_n(u,v) &= v(a_{n-1}(uv,1)-(uv)^{n-1}a(n-2))+\sum_{j=2}^{n-2}v^j(d_{n-j+1}(uv)-(uv)^{n-j}a(n-j-1))\\&= v(a_{n-1}(uv,1)-(uv)^{n-1}a(n-2))+\sum_{j=3}^{n-1}v^{n-j+1}(d_j(uv)-(uv)^{j-1}a(j-2)),\qquad n\geq 3.\end{aligned}\tag{31}$$

Multiplying both sides of (26) by u^i, and summing over $1 \leq i \leq n-2$ implies

$$d_n(u) = u^{n-1}a(n-2)+b_{n-1}(u,1)+d_{n-1}(u),\qquad n\geq 2.\tag{32}$$

Define generating functions

$$a(x;u,v)=\sum_{n\geq 1}a_n(u,v)x^n,\qquad b(x;u,v)=\sum_{n\geq 3}b_n(u,v)x^n,$$
$$c(x;u,v)=\sum_{n\geq 3}c_n(u,v)x^n,\qquad d(x;u)=\sum_{n\geq 2}d_n(u)x^n.$$

Rewriting recurrences (29)–(32) in terms of generating functions yields the following system of functional equations.

Lemma 16. *We have*

$$a(x;u,v)=xu+xua(xu;v,1)+b(x;u,v)+c(x;u,v)+d(xv;u),\tag{33}$$

$$\begin{aligned}(1-xu)b(x;u,v) &= \frac{xu}{1-u}(b(x;uv,1)-b(xu;v,1))+\frac{xu(1-xu+xu^2)}{(1-u)(1-xu)}d(x;uv)\\&\quad-\frac{xu}{(1-u)(1-xu)}d(xu;v),\end{aligned}\tag{34}$$

$$c(x;u,v)=xva(x;uv,1)-\frac{x^2uv^2}{1-xv}(a(xuv;1,1)+1)+\frac{xv^2}{1-xv}d(x;uv),\tag{35}$$

$$(1-x)d(x;u)=x^2u(a(xu;1,1)+1)+xb(x;u,1).\tag{36}$$

We can now determine the generating function $F_T(x)$.

Theorem 17. *Let* $T=\{1342,1423,2134\}$. *Then*

$$F_T(x)=\frac{1-2x+2x^2-\sqrt{1-8x+20x^2-24x^3+16x^4-4x^5}}{2x(1-x+x^2)}.$$

Proof: In the notation above, we seek to determine $1 + a(x; 1, 1)$. By (36), we have

$$d(x; u) = \frac{x^2 u}{1-x}(a(xu; 1, 1) + 1) + \frac{x}{1-x} b(x; u, 1).$$

Thus, equation (34) with $v = 1$ gives

$$\begin{aligned}&\left(1 - x - \frac{x}{1-u} - \frac{x^2(1-x+xu)}{(1-u)(1-x)(u-x)}\right) b(x/u; u, 1)\\ &\quad = -\left(\frac{x}{1-u} + \frac{x^2}{(1-u)(1-x)^2}\right) b(x; 1, 1)\\ &\qquad + \left(\frac{x^3(1-x+xu)}{(1-u)(1-x)(u-x)} - \frac{x^3}{(1-u)(1-x)^2}\right)(a(x; 1, 1) + 1).\end{aligned}$$

Applying the kernel method to this last equation, and setting

$$u = u_0 = \frac{1 - 2x + \sqrt{1 - 8x + 20x^2 - 24x^3 + 16x^4 - 4x^5}}{2(1-x)^2},$$

we obtain

$$b(x; 1, 1) = \frac{x^2(1-u_0)(a(x; 1, 1) + 1)}{u_0 - x}.$$

Note that $c(x; 1, 1) = \frac{x((1-2x)a(x;1,1) - x + d(x;1))}{1-x}$ by (35), and

$$a(x; 1, 1) = x + xa(x; 1, 1) + b(x; 1, 1) + c(x; 1, 1) + d(x; 1)$$

by (33). Substituting out $c(x; 1, 1)$, and then $d(x; 1)$ and $b(x; 1, 1)$, in the preceding equation and solving the equation that results for $a(x; 1, 1)$ yields

$$a(x; 1, 1) = \frac{x^3 + x(1-x)^2 u_0}{x(2x^2 - 2x + 1) - (1-x)^3 u_0}.$$

Substituting the expression for u_0 into the last equation gives the desired formula for $1 + a(x; 1, 1)$ and completes the proof. □

3.4 Case 234

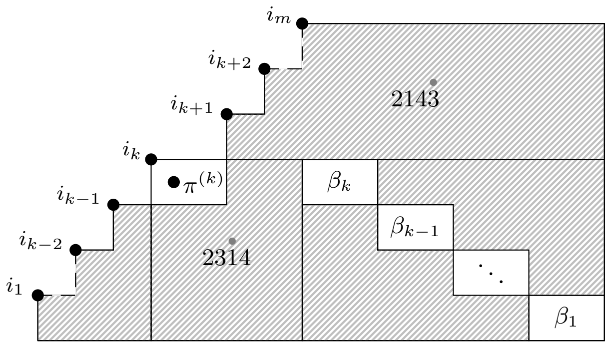
The two representative triples T are:
{2143,2314,2413} (Theorem 18)
{1243,1342,3142} (Theorem 19)

Theorem 18. *Let* $T = \{2143, 2314, 2413\}$. *Then*

$$F_T(x) = \frac{(1-x)^2 - \sqrt{(1-x)^4 - 4x(1-2x)(1-x)}}{2x(1-x)}.$$

Proof: Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now suppose $\pi = i_1\pi^{(1)}\cdots i_m\pi^{(m)}$ is a permutation that avoids T with $m \geq 2$ left-right maxima. Then $\pi^{(m)}$ has the form $\beta_m\beta_{m-1}\cdots\beta_1$ with $\beta_1 < i_1 < \beta_2 < i_2 < \cdots < \beta_m < i_m$ because $c < d$ in $\pi^{(m)}$ with $c < i_j < d$ implies $i_j i_m cd$ is a 2413.

If $\pi^{(1)} = \cdots = \pi^{(m-1)} = \emptyset$, the contribution is $(xF_T(x))^m$. Otherwise, let k be minimal such that $\pi^{(k)} \neq \emptyset$. Then π has the form



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where dark bullets indicate mandatory entries and some shaded regions are empty because the gray bullet would form part of the indicated pattern, $\pi^{(k)}i_m\beta_k$ avoids T and does not start with its largest entry, and $\beta_{k-1}, \ldots, \beta_1$ all avoid T. Thus, the contribution for fixed $k \in [m]$ is given by $x^{m-1}(F_T(x) - 1 - xF_T(x))F_T(x)^{k-1}$.

Hence, for $m \geq 2$,

$$G_m(x) = (xF_T(x))^m + x^{m-1}(F_T(x) - 1 - xF_T(x)) \sum_{k=0}^{m-1} F_T(x)^k.$$

Summing over $m \geq 0$, we obtain

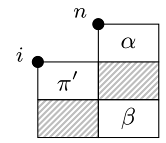
$$F_T(x) = 1 + \frac{xF_T(x)}{1 - xF_T(x)} + \frac{\big(F_T(x) - 1 - xF_T(x)\big)\left(\frac{x}{1-x} - \frac{xF_T(x)}{1-xF_T(x)}\right)}{1 - F_T(x)},$$

which has the desired solution. □

Theorem 19. *Let* $T = \{1243, 1342, 3142\}$. *Then*

$$F_T(x) = \frac{(1-x)^2 - \sqrt{(1-x)^4 - 4x(1-2x)(1-x)}}{2x(1-x)}.$$

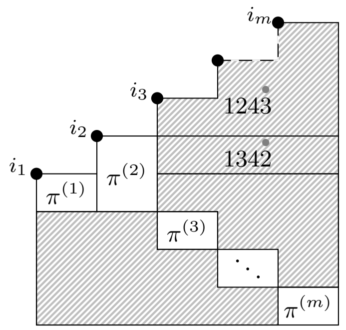
Proof: Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. For $m = 2$, suppose $\pi = i\pi' n\pi''$ is a permutation in $S_n(T)$ with two left-right maxima. Let β denote the subsequence of letters less than i in π''. Then $\beta < \pi'$ ($a \in \pi'$ and $b \in \beta$ with $a < b$ implies $ianb$ is a 3142) and so π is as in the figure.



If $\alpha = \emptyset$, then π' and β avoid T and the contribution is $x^2F_T(x)^2$. If $\alpha \neq \emptyset$ so that $i+1 \in \alpha$, then π' is decreasing (or $n(i+1)$ would be the 43 of a 1243), and St$(i\pi'')$ is a T-avoider that does not start with its maximal element. Hence, the contribution is $\frac{x}{1-x}\big(F_T(x) - 1 - xF_T(x)\big)$. Thus,

$$G_2(x) = x^2F_T(x)^2 + \frac{x}{1-x}\big(F_T(x) - 1 - xF_T(x)\big).$$

For $m \geq 3$, π has the form



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where some shaded regions are empty to avoid the indicated pattern and the π's are in their relative positions to avoid 3142. Hence, $G_m(x) = G_2(x)(xF_T(x))^{m-2}$.

Summing over $m \geq 0$, we obtain

$$F_T(x) = 1 + xF_T(x) + \frac{x^2F_T(x) + \frac{x}{1-x}(F_T(x) - 1 - xF_T(x))}{1 - xF_T(x)},$$

which has the desired solution. ☐

3.5 Case 235

The three representative triples T are:
{1423,1432,2143} (Theorem 23)
{1423,1432,3142} (Theorem 24)
{1234,1243,2314} (Theorem 27)

3.5.1 $\mathbf{T = \{1423, 1432, 2143\}}$

Let $a(n; i_1, i_2, \ldots, i_k)$, $a(n)$ and $\mathcal{T}_{i,j}$ be as in the second class in case 229 above. Note here that $a(n;n) = a(n;n-1) = a(n-1)$ for $n \geq 2$. It is convenient to consider separately the case of a permutation starting $i, j, j+2$, where $j \leq i-3$. Define $f(n;i,j) = a(n;i,j,j+2)$ for $4 \leq i \leq n$ and $1 \leq j \leq i-3$. The arrays $a(n;i,j)$ and $f(n;i,j)$ are determined recursively as follows.

Lemma 20. *We have*

$$a(n;i,i+2) = a(n-1;i,i+2) + a(n-1;i+1,i) + \sum_{j=1}^{i-1} a(n-1;i,j), \qquad 1 \le i \le n-2, \quad (37)$$

$$a(n;i,i-1) = a(n-1;i,i-1) + \sum_{j=1}^{i-2} a(n-1;i-1,j), \qquad 2 \le i \le n-1, \quad (38)$$

$$a(n;i,i-2) = a(n-1;i,i-2)+a(n-1;i-1,i-2)+\sum_{j=1}^{i-3} a(n-1;i-2,j), \qquad 3 \le i \le n-1, \quad (39)$$

$$a(n;i,j) = a(n-1;i-1,j) + f(n;i,j) + \sum_{\ell=1}^{j-1} a(n-1;j,\ell), \qquad 1 \le j \le i-3, \quad (40)$$

and

$$f(n;i,j) = f(n-1;i-1,j) + a(n-2;j+1,j) + \sum_{\ell=1}^{j-1} a(n-2;j,\ell), \qquad 1 \le j \le i-4, \quad (41)$$

with $f(n;i,i-3) = a(n-1;i-1,i-3)$ *for* $4 \le i \le n$, $a(n;i,i+1) = a(n-1;i)$ *for* $1 \le i \le n-1$, *and* $a(n;i,j) = 0$ *for* $1 \le i \le j-3 \le n-3$.

Proof: The formulas for $f(n;i,i-3)$ and $a(n;i,i+1)$, and for $a(n;i,j)$ when $i \le j-3$, follow from the definitions. In the cases that remain, let x denote the third letter of a T-avoiding permutation. For (37), first note that members of $\mathcal{T}_{i,i+2}$ where $i < n-2$ must have $x = i+3$, $x = i+1$ or $x < i$, lest there be an occurrence of 1423 or 1432. The letter $i+2$ can be deleted in the first case, while the letter i can in the second, giving $a(n-1;i,i+2)$ and $a(n-1;i+1,i)$ possibilities, respectively. If $x < i$, then i, x imposes a stricter requirement on subsequent letters than does $i+2, x$, whence $i+2$ may be deleted in this case. This gives $\sum_{j=1}^{i-1} a(n-1;i,j)$ possibilities, which implies (37) when $i < n-2$. Equation (37) is also seen to hold when $i = n-2$ since then there is no $x = i+3$ case with $a(n-1;i,i+2) = 0$ accordingly. For (38), note that members of $\mathcal{T}_{i,i-1}$ where $i < n$ must have $x = i+1$ or $x < i-2$ so as to avoid 2143. This yields $a(n-1;i,i-1)$ and $\sum_{j=1}^{i-2} a(n-1;i-1,j)$ possibilities, respectively, which implies (38). For (39), note that members of $\mathcal{T}_{i,i-2}$ where $i < n$ must have $x = i+1$, $x = i-1$ or $x < i-2$, yielding $a(n-1;i,i-2)$, $a(n-1;i-1,i-2)$ and $\sum_{j=1}^{i-3} a(n-1;i-2,j)$ possibilities, respectively.

To show (40), first observe that members of $\mathcal{T}_{i,j}$ where $j \le i-3$ must have $x = j+1$, $x = j+2$ or $x < j$, lest there be an occurrence of 1423 or 1432. If $x = j+1$, then there are $a(n-1;i-1,j)$ possibilities since the letter $j+1$ is extraneous and may be deleted. If $x = j+2$, then there are $f(n;i,j)$ possibilities, by definition. If $x < j$, then the letter i may be deleted, which gives the last term on the right-hand side of (40). Finally, to show (41), let y denote the fourth letter of a permutation enumerated by $f(n;i,j)$ where $j < i-3$. Then we must have $y = j+3$, $y = j+1$ or $y < j$. If $y = j+3$, then y may be deleted, yielding $f(n-1;i-1,j)$ possibilities, by definition. If $y = j+1$, then the occurrence of $j+2, j+1$ is seen to impose a stricter requirement on subsequent letters than does i, j with regard to 2143, with $j+1$ also making j redundant concerning 1423 or 1432. Thus, both i and j may be deleted in this case, giving $a(n-2; j+1, j)$ possibilities. Finally, if $y < j$, then both the i and $j+2$ may be deleted and thus there are $\sum_{\ell=1}^{j-1} a(n-2; j, \ell)$ possibilities, which implies (41) and completes the proof. $\square$

To aid in solving the recurrences of the prior lemma, we define the following auxiliary functions: $b(n;i) = \sum_{j=1}^{i-1} a(n;i,j)$, $c(n;i) = a(n;i,i-2)$, $d(n;i) = a(n;i,i-1)$ and $e(n;i) = a(n;i,i+2)$. Assume functions are defined on the natural range for i, given n, and are zero otherwise. For example, $c(n;i)$ is defined for $3 \le i \le n$, with $c(n;1) = c(n;2) = 0$. Let $f(n;i) = \sum_{j=1}^{i-3} f(n;i,j)$ for $4 \le i \le n$.

The recurrences in the previous lemma may be recast as follows.

Lemma 21. *We have*

$$a(n;i) = a(n-1;i) + b(n;i) + e(n;i), \qquad 1 \le i \le n-1, \tag{42}$$

$$\begin{aligned} b(n;i) &= c(n;i) + d(n;i) + b(n-1;i-1) - d(n-1;i-1) + f(n;i) \\ &\quad + \sum_{j=1}^{i-3} b(n-1;j), \qquad 2 \le i \le n-1, \end{aligned} \tag{43}$$

$$c(n;i) = b(n-1;i-2) + c(n-1;i) + d(n-1;i-1), \qquad 3 \le i \le n-1, \tag{44}$$

$$d(n;i) = b(n-1;i-1) + d(n-1;i), \qquad 2 \le i \le n-1, \tag{45}$$

$$e(n;i) = b(n-1;i) + d(n-1;i+1) + e(n-1;i), \qquad 1 \le i \le n-2, \tag{46}$$

and

$$f(n;i) = c(n-1;i-1) + f(n-1;i-1) + \sum_{j=1}^{i-4} b(n-2;j) + \sum_{j=1}^{i-4} d(n-2;j+1), \qquad 4 \le i \le n. \tag{47}$$

Proof: For (42), note that by the definitions, we have

$$\begin{aligned} a(n;i) &= \sum_{i=1, i\neq j}^{n} a(n;i,j) = \sum_{j=1}^{i-1} a(n;i,j) + a(n;i,i+1) + a(n;i,i+2) \\ &= b(n;i) + a(n-1;i) + e(n;i). \end{aligned}$$

For (43), note that by summing (40) over j and the definitions, we have

$$\begin{aligned}
b(n;i) &= a(n;i,i-2) + a(n;i,i-1) + \sum_{j=1}^{i-3} a(n;i,j) \\
&= a(n;i,i-2) + a(n;i,i-1) + \sum_{j=1}^{i-3} a(n-1;i-1,j) + \sum_{j=1}^{i-3} f(n;i,j) + \sum_{j=1}^{i-3} b(n-1;j) \\
&= c(n;i) + d(n;i) + (b(n-1;i-1) - d(n-1;i-1)) + f(n;i) + \sum_{j=1}^{i-3} b(n-1;j).
\end{aligned}$$

Next, observe that formulas (44), (45) and (46) follow directly from the definitions and recurrences (39), (38) and (37), respectively. Finally, formula (47) follows from summing (41) over $1 \le j \le i-4$ and noting $f(n;i,i-3) = c(n-1;i-1)$. $\square$

Define

$$\begin{aligned}
a_n(u) &= \sum_{i=1}^{n} a(n;i)u^i \text{ for } n \ge 1, & b_n(u) &= \sum_{i=2}^{n-1} b(n;i)u^i \text{ for } n \ge 3, \\
c_n(u) &= \sum_{i=3}^{n-1} c(n;i)u^i \text{ for } n \ge 4, & d_n(u) &= \sum_{i=2}^{n-1} d(n;i)u^i \text{ for } n \ge 3, \\
e_n(u) &= \sum_{i=1}^{n-2} e(n;i)u^i \text{ for } n \ge 3, & f_n(u) &= \sum_{i=4}^{n} f(n;i)u^i \text{ for } n \ge 4.
\end{aligned}$$

Assume all functions take the value zero if n is such that the sum in question is empty. Note that $a_1(u) = u$, with $b_3(u) = d_3(u) = u^2$.

Multiplying both sides of (42) by u^i, and summing over $1 \le i \le n-1$, yields

$$a_n(u) = a(n-1)u^n + a_{n-1}(u) + b_n(u) + e_n(u), \qquad n \ge 2. \tag{48}$$

Note that, by the definitions,

$$f(n;n) = \sum_{j=1}^{n-3} f(n;n,j) = \sum_{j=1}^{n-3} a(n-1;j,j+2) = \sum_{j=1}^{n-3} e(n-1;j) = e_{n-1}(1), \qquad n \ge 4,$$

and

$$b(n;n-1) = a(n;n-1) - a(n;n-1,n) = a(n-1) - a(n-2), \qquad n \ge 2.$$

By recurrence (43), we then have

$$\begin{aligned}
b_n(u) &= c_n(u) + d_n(u) + u(b_{n-1}(u) - d_{n-1}(u)) + f_n(u) - f(n;n)u^n + \sum_{j=1}^{n-3} b(n-1;j) \sum_{i=j+3}^{n-1} u^i \\
&= c_n(u) + d_n(u) + u(b_{n-1}(u) - d_{n-1}(u)) + f_n(u) - e_{n-1}(1)u^n \\
&\quad + \frac{u^3}{1-u}(b_{n-1}(u) - (a(n-2) - a(n-3))u^{n-2}) \\
&\quad - \frac{u^n}{1-u}(b_{n-1}(1) - (a(n-2) - a(n-3))) \\
&= c_n(u) + d_n(u) + u(b_{n-1}(u) - d_{n-1}(u)) + f_n(u) - e_{n-1}(1)u^n \\
&\quad + \frac{u}{1-u}(u^2 b_{n-1}(u) - u^{n-1} b_{n-1}(1)) + (a(n-2) - a(n-3))u^n, \qquad n \geq 3. \qquad (49)
\end{aligned}$$

From recurrence (44), we get

$$\begin{aligned}
c_n(u) &= u^2(b_{n-1}(u) - b(n-1;n-2)u^{n-2}) + c_{n-1}(u) + c(n-1;n-1)u^{n-1} + ud_{n-1}(u) \\
&= u^2 b_{n-1}(u) - u^n a(n-2) + u^{n-1}(1+u)a(n-3) + c_{n-1}(u) + ud_{n-1}(u), \qquad n \geq 4. \quad (50)
\end{aligned}$$

By (45) and (46), we have

$$d_n(u) = a(n-3)u^{n-1} + ub_{n-1}(u) + d_{n-1}(u), \qquad n \geq 3, \qquad (51)$$

and

$$e_n(u) = a(n-3)u^{n-2} + b_{n-1}(u) + \frac{1}{u}d_{n-1}(u) + e_{n-1}(u), \qquad n \geq 3. \qquad (52)$$

Finally, multiplying both sides of (47) by u^i, and summing over $4 \leq i \leq n$, yields

$$\begin{aligned}
f_n(u) &= u(c(n-1;u) + a(n-3)u^{n-1}) + uf_{n-1}(u) \\
&\quad + \sum_{j=1}^{n-3} b(n-2;j) \sum_{i=j+4}^{n} u^i + \sum_{j=1}^{n-4} d(n-2;j+1) \sum_{i=j+4}^{n} u^i \\
&= a(n-3)u^n + uc_{n-1}(u) + uf_{n-1}(u) \\
&\quad + \frac{u^3}{1-u}(ub_{n-2}(u) + d_{n-2}(u) - u^{n-2}(b_{n-2}(1) + d_{n-2}(1))), \qquad n \geq 4. \qquad (53)
\end{aligned}$$

Define the generating functions $a(x;u) = \sum_{n\geq1} a_n(u)x^n$, $b(x;u) = \sum_{n\geq3} b_n(u)x^n$, $c(x;u) = \sum_{n\geq4} c_n(u)x^n$, $d(x;u) = \sum_{n\geq3} d_n(u)x^n$, $e(x;u) = \sum_{n\geq3} e_n(u)x^n$ and $f(x;u) = \sum_{n\geq4} f_n(u)x^n$. Recall that $a(n) = a_n(1)$ for $n \geq 1$, with $a(0) = 1$. Rewriting recurrences (48)–(53) in terms of generating functions yields the following system of functional equations.

Lemma 22. *We have*

$$(1-x)a(x;u) = xu(1 + a(xu;1)) + b(x;u) + e(x;u), \qquad (54)$$

$$(1-xu)b(x;u) = -x^3u^3 + c(x;u) + (1-xu)d(x;u) - xue(xu;1) + f(x;u) + x^2u^2(1-xu)a(xu;1) + \frac{xu}{1-u}(u^2b(x;u) - b(xu;1)), \quad (55)$$

$$(1-x)c(x;u) = x^3u^3 - x^2u^2(1-x-xu)a(xu;1) + xu^2b(x;u) + xud(x;u), \quad (56)$$

$$(1-x)d(x;u) = x^3u^2(1+a(xu;1)) + xub(x;u), \quad (57)$$

$$(1-x)e(x;u) = x^3u(1+a(xu;1)) + xb(x;u) + \frac{x}{u}d(x;u), \quad (58)$$

and

$$(1-xu)f(x;u) = x^3u^3a(xu;1) + xuc(x;u) + \frac{x^2u^3}{1-u}(ub(x;u) + d(x;u) - b(xu;1) - d(xu;1)). \quad (59)$$

We now determine the generating function $F_T(x)$.

Theorem 23. *Let $T = \{1423, 1432, 2143\}$. Then $y = F_T(x)$ satisfies the equation*

$$y = 1 - x + xy + x(1-2x)y^2 + x^2y^3.$$

Proof: By solving (54), (57) and (58) with $u = 1$ for $b(x;1)$, $d(x;1)$ and $e(x;1)$, we obtain

$$\begin{aligned} b(x;1) &= \frac{1-4x+5x^2-3x^3}{1-x+x^2}a(x;1) - \frac{x(1-2x+2x^2)}{1-x+x^2}, \\ d(x;1) &= \frac{x(1-x)^3}{1-x+x^2}a(x;1) - \frac{x^2(1-x)^2}{1-x+x^2}, \\ e(x;1) &= \frac{x(1-x)^2}{1-x+x^2}a(x;1) - \frac{x^2(1-x)}{1-x+x^2}. \end{aligned}$$

Define $K(x;u) = u^2(1-u) - xu(2-u^2) + x^2(1+2u-2u^2) - x^3$. Substituting the expressions for $b(x;1)$, $d(x;1)$ and $e(x;1)$ into (55)–(59), and then solving for $b(x/u;u)$, $c(x/u;u)$, $d(x/u;u)$, $e(x/u;u)$ and $f(x/u;u)$, yields

$$\begin{aligned} K(x;u)b(x/u;u) &= x(-u^2 + 2xu(u+1) - x^2(u^2+3u+1) + x^3(2u+1))a(x;1) \\ &\quad + x^2u(2x^2 - xu - x + u), \\ K(x;u)e(x/u;u) &= x^2(1-x)(x-u)a(x;1) + x^3(1-x). \end{aligned}$$

Multiplying both sides of (54) by $K(x;u)$, and then substituting the expressions of $K(x;u)b(x/u;u)$ and $K(x;u)e(x/u;u)$, gives

$$\begin{aligned} (1-x/u)K(x;u)a(x/u;u) &= x(x-u)(u^2 + x(1-u-u^2) + x^2(2u-1))a(x;1) \\ &\quad + x(1-u)(u^2 - xu(2+u) + x^2(2+3u) - 2x^3). \end{aligned}$$

To solve this last equation, we let $u = u_0 = u_0(x)$ such that $K(x; u_0(x)) = 0$. Then

$$F_T(x) = 1 + a(x;1) = \frac{(1-x)(x^2 - xu_0 + u_0^2)}{(u_0 - x)(x(1-x) - x(1-2x)u_0 + (1-x)u_0^2)}.$$

Using the fact that $u_0^3 = u_0^2(1-u_0) - xu_0(2-u_0^2) + x^2(1+2u_0-2u_0^2)$, we obtain

$$\begin{aligned} &1 - x - (1-x)F_T(x) + x(1-2x)F_T^2(x) + x^2F_T^3(x) \\ &= \frac{(1-x)^2K(x;u_0)V(x;u_0)}{(x-u_0)^3(x(1-x) - x(1-2x)u_0 + (1-x)u_0^2)^3} = 0, \end{aligned}$$

where

$$\begin{aligned} V(x;u) = &-x^5(2x^4 + 7x^2(1-x) - 5x + 2) + x^2(x+1)(4x^4 - 7x^3 + 8x^2 - 5x + 1)(u-x) \\ &- (7x^6 + 2x^5(1-x^2) - x^4 - 37x^3(1-x) + 24x^2(1-x^2) - 8x + 1)(u-x)^2. \end{aligned}$$

Hence, the generating function $F_T(x)$ satisfies

$$F_T(x) = 1 - x + xF_T(x) + x(1-2x)F_T^2(x) + x^2F_T^3(x),$$

as desired. □

3.5.2 $\mathbf{T = \{1423, 1432, 3142\}}$

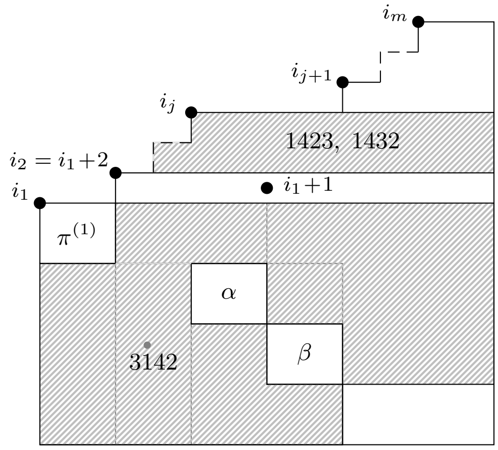
Theorem 24. *Let $T = \{1423, 1432, 3142\}$. Then $y = F_T(x)$ satisfies the equation*

$$y = 1 - x + xy + x(1-2x)y^2 + x^2y^3.$$

Proof: Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ with $m \geq 2$ left-right maxima. Since π avoids 1423 and 1432, we have that either $i_2 = i_1 + 1$ or $i_2 = i_1 + 2$.

- The case $i_2 = i_1 + 1$. Since π avoids 3142, we see that there is no element between the minimal element of $\pi^{(1)}$ and i_1 in $\pi^{(2)}\pi^{(3)}\cdots\pi^{(m)}$. Thus, the contribution in this case is $xF_T(x)G_{m-1}(x)$, where $xF_T(x)$ accounts for the section $i_1\pi^{(1)}$ and $G_{m-1}(x)$ for $i_2\pi^{(2)}\cdots i_m\pi^{(m)}$.

- The case $i_2 = i_1 + 2$. Let j be the index with $i_1 + 1 \in \pi^{(j)}$. Then π has the form



with $\pi^{(1)} > \alpha > \beta$ to avoid 3142, where $i_1\pi^{(1)}\alpha\beta$ spans an interval of integers, also to avoid 3142, and the other shaded regions are empty to avoid the indicated patterns.

Thus, for given j, we have a bijection between such permutations and triples

$$(\pi^{(1)}, \alpha, i_j\beta i_{j+1}\pi^{(j+1)}\cdots i_m\pi^{(m)}),$$

where $\pi^{(1)}$ and α avoid T, and $i_j\beta i_{j+1}\pi^{(j+1)}\cdots i_m\pi^{(m)}$ avoids T with exactly $m-(j-1)$ left-right maxima. Hence, the contribution in this case is given by $x^jF_T^2(x)G_{m-j+1}(x)$, where $j = 2, 3, \ldots, m$.

By adding all the contributions, we get

$$G_m(x) = xF_T(x)G_{m-1} + \sum_{j=2}^{m} x^jF_T^2(x)G_{m-j+1}(x), \quad m \geq 2,$$

which implies

$$G_m(x) - xG_{m-1}(x) = xF_T(x)G_{m-1} - x^2F_T(x)G_{m-2}(x) + x^2F_T^2(x)G_{m-1}(x)$$

with $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. By summing this recurrence over all $m \geq 2$, we have

$$F_T(x) - 1 - xF_T(x) - x(F_T(x) - 1) = xF_T(x)(F_T(x) - 1) - x^2F_T^2(x) + x^2F_T^2(x)(F_T(x) - 1),$$

which leads to

$$F_T(x) = 1 - x + xF_T(x) + x(1-2x)F_T^2(x) + x^2F_T^3(x),$$

as required. □

3.5.3 $\mathbf{T = \{1234, 1243, 2314\}}$

To enumerate the members of $S_n(T)$, we categorize them by their first letter and the position of the leftmost ascent. More precisely, given $1 \le j \le i \le n$, let $a(n; i, j)$ be the number of T-avoiding permutations of length n starting with the letter i whose leftmost ascent is at index j. For example, we have $a(4; 3, 2) = 3$, the enumerated permutations being 3124, 3142 and 3241. If $1 \le i \le n$, then let $a(n; i) = \sum_{j=1}^{i} a(n; i, j)$ and let $a(n) = \sum_{i=1}^{n} a(n; i)$ for $n \ge 1$, with $a(0) = 1$. The array $a(n; i, j)$ satisfies the following recurrence relations.

Lemma 25. *If $n \ge 3$, then*

$$a(n; i, j) = \sum_{\ell=1}^{n-i} \sum_{k=j}^{i} a(n-\ell; i, k), \qquad 1 \le j \le i \le n-2. \tag{60}$$

If $2 \le j \le n-1$, then $a(n; n-1, j) = \sum_{i=j-1}^{n-2} a(n-1; i, j-1)$, with $a(n; n-1, 1) = a(n-2)$ for $n \ge 2$. If $2 \le j \le n$, then $a(n; n, j) = \sum_{i=j-1}^{n-1} a(n-1; i, j-1)$, with $a(n; n, 1) = \delta_{n,1}$ for $n \ge 1$.

Proof: Let $A_{n,i,j}$ denote the subset of $S_n(T)$ enumerated by $a(n; i, j)$. First note that removing the initial letter n from members of $A_{n,n,j}$ for $2 \le j \le n$ defines a bijection with $\cup_{i=j-1}^{n-1} A_{n-1,i,j-1}$ (where $A_{n,n,n}$ is understood to be the singleton set consisting of the decreasing permutation $n(n-1)\cdots 1$). This implies the formula for $a(n; n, j)$ for $j > 1$, with the condition $a(n; n, 1) = \delta_{n,1}$ following from the definitions. Similarly, removing $n-1$ from members of $A_{n,n-1,j}$ when $j > 1$ implies the formula for $a(n; n-1, j)$ in this case. That $a(n; n-1, 1) = a(n-2)$ follows from the fact that one may safely delete both $n-1$ and n from members of $S_n(T)$ starting with these letters.

To show (60), we first consider the possible values of π_{j+1} within $\pi = \pi_1\pi_2\cdots\pi_n \in A_{n,i,j}$ where $i < n-1$. Note that if $\pi_{j+1} < n-1$, then π would contain either 1234 or 1243, as witnessed by the subsequences $\pi_j\pi_{j+1}(n-1)n$ or $\pi_j\pi_{j+1}n(n-1)$, which is impossible. Thus, we must have $\pi_{j+1} = n-1$ or n. If $\pi_{j+1} = n-1$, consider further the sequence of letters $\pi_{j+1}\pi_{j+2}\cdots\pi_r$, where r is such that $\pi_r = n$. If $r > j+2$, then each letter π_s for $j+2 \le s \le r-1$ must satisfy $\pi_s > i$, for otherwise π would contain 2314 (with the subsequence $i(n-1)xn$ for some $x < i$). Furthermore, if $r > j+2$ and $\pi_{j+2} < n-2$, then $i\pi_{j+2}$ would be the first two letters in an occurrence of 1234 or 1243, which is impossible. Thus, we must have $\pi_{j+2} = n-2$. Similarly, by an inductive argument, we get $\pi_{j+1}\pi_{j+2}\cdots\pi_{r-1}\pi_r = (n-1)(n-2)\cdots(n-r+j+1)n$. Note that each of these ℓ letters, where $\ell = r-j$, is seen to be extraneous concerning avoidance of T and thus may be deleted. The remaining letters comprise a member of $A_{n-\ell,i,k}$ for some $k \in [j, i]$ and hence there are $\sum_{k=j}^{i} a(n-\ell; i, k)$ possibilities for these letters. Since each letter of the section $\pi_{j+1}\cdots\pi_r$ belongs to $[i+1, n]$, its length ℓ can range from 1 to $n-i$, with the contents of the section determined by its length. Allowing ℓ to vary implies formula (60) and completes the proof. $\square$

Let $a_{n,i}(v) = \sum_{j=1}^{i} a(n; i, j)v^j$ for $1 \le i \le n$ and $a_n(u, v) = \sum_{i=1}^{n} a_{n,i}(v)u^i$ for $n \ge 1$. Multiplying both sides of (60) by v^j, and summing over $1 \leq j \leq i$, gives

$$\begin{aligned}
a_{n,i}(v) &= \sum_{j=1}^{i} v^j \sum_{\ell=1}^{n-i}\sum_{k=j}^{i} a(n-\ell;i,k) = \sum_{\ell=1}^{n-i}\sum_{k=1}^{i} a(n-\ell;i,k)\left(\frac{v-v^{k+1}}{1-v}\right) \\
&= \frac{v}{1-v}\sum_{\ell=1}^{n-i}(a_{n-\ell,i}(1) - a_{n-\ell,i}(v)), \qquad 1 \leq i \leq n-2,
\end{aligned}$$

with

$$\begin{aligned}
a_{n,n-1}(v) - a(n-2)v &= \sum_{j=2}^{n-1} v^j \sum_{i=j-1}^{n-2} a(n-1;i,j-1) = \sum_{i=1}^{n-2}\sum_{j=2}^{i+1} a(n-1;i,j-1)v^j \\
&= v\sum_{i=1}^{n-2} a_{n-1,i}(v) = v(a_{n-1}(1,v) - a_{n-1,n-1}(v)) \\
&= v(a_{n-1}(1,v) - va_{n-2}(1,v)), \qquad n \geq 2,
\end{aligned}$$

and

$$a_{n,n}(v) = \sum_{i=1}^{n-1}\sum_{j=2}^{i+1} a(n-1;i,j-1)v^j = v\sum_{i=1}^{n-1} a_{n-1,i}(v) = va_{n-1}(1,v), \qquad n \geq 1.$$

The preceding equations then imply

$$\begin{aligned}
a_n(u,v) &= \frac{v}{1-v}\sum_{i=1}^{n-2} u^i \sum_{\ell=1}^{n-i}(a_{n-\ell,i}(1) - a_{n-\ell,i}(v)) + u^{n-1}a_{n,n-1}(v) + u^n a_{n,n}(v) \\
&= \frac{v}{1-v}\sum_{\ell=1}^{n-1}\sum_{i=1}^{n-\ell}(a_{n-\ell,i}(1) - a_{n-\ell,i}(v))u^i - \frac{u^{n-1}v}{1-v}(a_{n-1,n-1}(1) - a_{n-1,n-1}(v)) \\
&\quad + u^{n-1}a_{n,n-1}(v) + u^n a_{n,n}(v) \\
&= \frac{v}{1-v}\sum_{\ell=1}^{n-1}(a_\ell(u,1) - a_\ell(u,v)) - \frac{u^{n-1}v}{1-v}(a(n-2) - va_{n-2}(1,v)) \\
&\quad + u^{n-1}v(a_{n-1}(1,v) - va_{n-2}(1,v)) + u^{n-1}va(n-2) + u^n va_{n-1}(1,v) \\
&= \frac{v}{1-v}\sum_{\ell=1}^{n-1}(a_\ell(u,1) - a_\ell(u,v)) + u^{n-1}v(1+u)a_{n-1}(1,v) \\
&\quad - \frac{u^{n-1}v^2}{1-v}(a(n-2) - va_{n-2}(1,v)), \qquad n \geq 2, \qquad (61)
\end{aligned}$$

with $a_0(u,v) = 1$ and $a_1(u,v) = uv$.

Let $a(x;u,v) = \sum_{n\geq 1} a_n(u,v)x^n$. Multiplying both sides of (61) by x^n, and summing over $n \geq 2$, yields the following functional equation.

Lemma 26. *We have*

$$a(x;u,v)=xuv(1-xv)+\frac{xv}{(1-x)(1-v)}(a(x;u,1)-a(x;u,v))+xv(1+u)a(xu;1,v)$$
$$-\frac{x^2uv^2}{1-v}(a(xu;1,1)-va(xu;1,v)). \tag{62}$$

We now determine the generating function $F_T(x)$.

Theorem 27. *Let $T=\{1234,1243,2314\}$. Then $y=F_T(x)$ satisfies the equation*

$$y=1-x+xy+x(1-2x)y^2+x^2y^3.$$

Proof: In the current notation, we need to determine $1+a(x;1,1)$. Letting $u=1$ in (62), and rearranging, gives

$$\left((1-v)(1-2xv)-x^2v^3+\frac{xv}{1-x}\right)a(x;1,v)=xv(1-v)(1-xv)+\left(\frac{xv}{1-x}-x^2v^2\right)a(x;1,1). \tag{63}$$

Setting $v=v_0$ in (63) such that

$$1-x-(1-2x^2)v_0+2x(1-x)v_0^2=x^2(1-x)v_0^3,$$

and solving for $a(x;1,1)$, implies

$$1+a(x;1,1)=\frac{x+(1-x)v_0-x(1-x)v_0^2}{1-x(1-x)v_0}.$$

Let $f(v)=1-x-(1-2x^2)v+2x(1-x)v^2-x^2(1-x)v^3$ and $h(v)=f(v)g(v)$, where

$$g(v)=(1-x)(1+x^3-x(2-3x+2x^2)v-2x^3(1-x)v^2+x^3(1-x)v^3).$$

Then $y=1+a(x;1,1)$ is a solution of the equation

$$1-x-(1-x)y+x(1-2x)y^2+x^2y^3=0$$

if and only if $h(v)=0$ at $v=v_0$, which is the case since $f(v_0)=0$, by definition. This implies $F_T(x)$ is a solution of the equation stated above, as desired. □

3.6 Case 238

The five representative triples T are:
{1423,2413,3142} (Theorem 28)
{2134,2143,2413} (Theorem 29)
{1234,1342,1423} (Theorem 32)
{1324,1342,1423} (Theorem 33)
{1243,1342,1423} (Theorem 34)

3.6.1 $\mathbf{T=\{1423,2413,3142\}}$

Theorem 28. *Let* $T=\{1423,2413,3142\}$. *Then*

$$F_T(x)=\frac{3-2x-\sqrt{1-4x}-\sqrt{2-16x+4x^2+(2+4x)\sqrt{1-4x}}}{2(1-\sqrt{1-4x})}.$$

Proof: We say that a permutation π has (m,k) *left-right maxima*, $1\leq k\leq m$, if it has m left-right maxima $i_1,i_2,\ldots,i_m$ of which the last k are consecutive, that is,

$$i_1<\cdots<i_{m-k}<i_{m-k+1}=n-k+1<i_{m-k+2}=n-k+2<\cdots<i_{m-1}=n-1<i_m=n,$$

where n is maximal letter of π. Let $G_{m,k}(x)$ be the generating function for T-avoiders with (m,k) left-right maxima. Define $G_{0,0}(x)=1$. To find an equation for $G_{m,k}(x)$, $1\leq k\leq m$, let $\pi=i_1\pi^{(1)}\cdots i_m\pi^{(m)}$ be a permutation that avoids T with (m,k) left-right maxima. If $k=m$, then it is easy to see that $\pi^{(1)}>\pi^{(2)}>\cdots>\pi^{(m)}$, where each $\pi^{(j)}$ avoids T. Thus, $G_{m,m}(x)=(xF_T(x))^m$.

So suppose $1\leq k\leq m-1$. Since π avoids 1423, all the letters in $I=\{i_{m-k}+1,\ldots,n-k\}$ appear in decreasing order in π . Since π avoids 2413, only left-right maxima can appear between letters that belong to I. If $I=\emptyset$, then the contribution is given by $G_{m,k+1}(x)$. Otherwise, there exists a largest $s\in[n-k+1,n]$ such that $\pi^{(s)}$ contains at least one letter from I. By the preceding observations,

$$\pi^{(n-k+1)}\cdots\pi^{(s)}=(n-k)(n-k-1)\cdots(i_{m-k}+1)\pi'^{(s)},$$

where $i_{m-k}>\pi'^{(s)}$. We can now safely delete the left-right maxima $n-k+2,n-k+3,\ldots,s$ and all elements of I. The deleted left-right maxima contribute $x^{s-(n-k)-1}$, the deleted $i_{m-k}+1\in I$ (necessarily present) contributes x, and the other elements of I, which amount to distributing an arbitrary number of balls (possibly none) among the $s-(n-k)$ boxes $\pi^{(n-k+1)},\ldots,\pi^{(s)}$, contribute $1/(1-x)^{s-(n-k)}$. After the deletion, we have a T-avoider with $m-(s-n+k-1)$ left-right maxima of which the last $n-s+2$ are guaranteed consecutive, and so it contributes $G_{m+1-s+n-k,n+2-s}(x)$. Hence, the contribution for a given s equals

$$\frac{x^{s-(n-k)}}{(1-x)^{s-(n-k)}}G_{m+1-s+n-k,n+2-s}(x)\,.$$

By summing over all $s=n-k+1,\ldots,n$, we see that the contribution for the case $I\neq\emptyset$ is given by

$$\sum_{j=1}^{k}\frac{x^j}{(1-x)^j}G_{m+1-j,k+2-j}(x)\,.$$

Combining all the contributions, we obtain for $1\leq k<m$,

$$G_{m,k}(x)=G_{m,k+1}(x)+\frac{x}{1-x}\sum_{j=0}^{k-1}\frac{x^j}{(1-x)^j}G_{m-j,k+1-j}(x)\,,$$

with $G_{m,m}(x)=(xF_T(x))^m$.

In order to determine an equation for $F_T(x)$, we define $G(t,u) = 1 + \sum_{m\geq 1}\sum_{k=1}^{m} G_{m,k}(x)u^{k-1}t^m$. By multiplying the above recurrence by $t^m u^{k-1}$ and summing over $k = 1, 2, \ldots, m-1$ and $m \geq 1$, we find

$$G(t,u) = 1 + \frac{xF_T(x)}{1 - tuxF_T(x)} + \frac{G(t,u) - G(t,0)}{u} + \frac{x(G(t,u) - G(t,0))}{u(1-x-xut)}.$$

Note that $G(1,0) = 1 + \sum_{m\geq 0} G_{m,1}(x) = F_T(x)$. Hence,

$$G(1,u) = 1 + \frac{xF_T(x)}{1 - uxF_T(x)} + \frac{G(1,u) - F_T(x)}{u} + \frac{x(G(1,u) - F_T(x))}{u(1-x-xu)}.$$

To solve this functional equation, we apply the kernel method and take $u = C(x)$, which is seen to cancel out $G(1,u)$. Thus,

$$0 = 1 + \frac{xF_T(x)}{1 - xC(x)F_T(x)} - \frac{F_T(x)}{C(x)} - \frac{xF_T(x))}{C(x)(1-x-xC(x))},$$

which, using the identity $C(x) = 1 + xC^2(x)$, is equivalent to

$$F_T(x) = 1 + \frac{xF_T(x)}{1 - xC(x)F_T(x)}.$$

Solving this last equation completes the proof. □

3.6.2 $\mathbf{T = \{2134, 2143, 2413\}}$

Theorem 29. *Let* $T = \{2134, 2143, 2413\}$. *Then*

$$F_T(x) = \frac{3 - 2x - \sqrt{1-4x} - \sqrt{2 - 16x + 4x^2 + (2+4x)\sqrt{1-4x}}}{2(1 - \sqrt{1-4x})}.$$

Proof: Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now let us write an equation for $G_m(x)$. If π is a permutation that avoids T with m left-right maxima, then, to avoid 2134, π has the form

$$\pi = i_1 i_2 \cdots i_{m-1}\pi' i_m \pi''$$

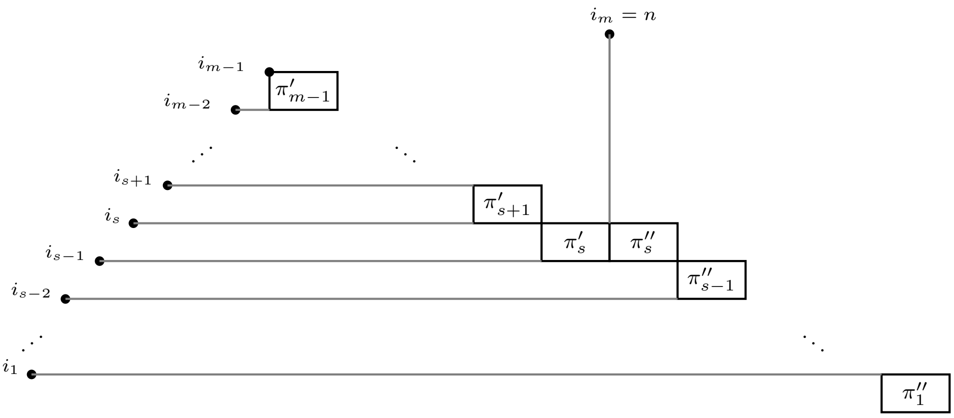
with $i_1 < i_2 < \cdots < i_m = n$ (n is the maximal letter of π), $i_{m-1} > \pi'$, and $i_m > \pi''$.

If π' is empty, then since π avoids 2413, we see that π'' can be decomposed as $\pi''_m \pi''_{m-1} \cdots \pi''_1$, where $\pi''_j > i_{j-1} > \pi''_{j-1}$, $j = 2, \ldots, m$, and π''_j avoids T.

If π' is not empty, then with $i_0 = 0$, there is a maximal integer s such that $i_{s-1} < \pi'$. Since π avoids 2413, we see that $\pi' = \pi'_{m-1} \cdots \pi'_{s+1}\pi'_s$ and $\pi'' = \pi''_s \cdots \pi''_1$, where

$$\pi'_{m-1} > i_{m-2} > \pi'_{m-2} > \cdots > i_{s+1} > \pi'_{s+1} > i_s > \pi'_s\pi''_s > i_{s-1} > \pi''_{s-1} > \cdots > i_1 > \pi''_1.$$

This means that π has the following diagrammatic shape.



Decomposition of T-avoider, case $\pi' \neq \emptyset$

Furthermore, π'_j avoids 213 for $j = m-1, m-2, \ldots, s+1$ or else n is the 4 of a 3124; $\pi'_s n \pi''_s$ avoids T and, since π'_s is not empty, it does not start with its largest letter; π''_j avoids T for $j = s-1, \ldots, 1$.

Hence, the contribution in the case π' is empty is $x^m F_t^m(x)$; otherwise, the contribution for given $s,\ 1 \leq s \leq m$, is

$$x^{m-1} C^{m-1-s}(x)(F_T(x) - 1 - xF_T(x))F_T^{s-1}(x)\,.$$

Combining all the contributions, we obtain

$$\begin{aligned}
F_T(x) &= 1 + \sum_{j\geq 1}(x^j F_T^j(x)) + (F_T(x) - 1 - xF_T(x)) \sum_{m\geq 2} \sum_{s=1}^{m-1} x^{m-1} C^{m-1-s}(x) F_T^{s-1}(x) \\
&= 1 + \sum_{j\geq 1}(x^j F_T^j(x)) + (F_T(x) - 1 - xF_T(x)) \sum_{m\geq 2} x^{m-1} \frac{C^{m-1}(x) - F_T^{m-1}(x)}{C(x) - F_T(x)},
\end{aligned}$$

and, using $C(x) = 1 + xC^2(x)$, we find that

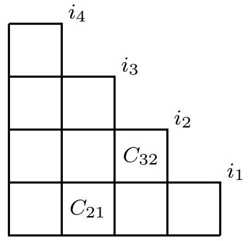
$$F_T(x) = 1 - x^2 C^2(x) F_T(x) + xC(x) F_T^2(x),$$

which yields the stated generating function. $\square$

For the remaining three cases, we consider (right-left) cell decompositions. So suppose

$$\pi = \pi^{(m)} i_m \pi^{(m-1)} i_{m-1} \cdots \pi^{(1)} i_1 \in S_n$$

has $m \geq 2$ right-left maxima $n = i_m > i_{m-1} > \cdots > i_1 \geq 1$. The right-left maxima determine a *cell decomposition* of the matrix diagram of π as illustrated in the figure below for $m = 4$. There are $\binom{m+1}{2}$ cells $C_{ij},\ i, j \geq 1,\ i + j \leq m + 1$, indexed by (x, y) coordinates, for example, C_{21} and C_{32} are shown.



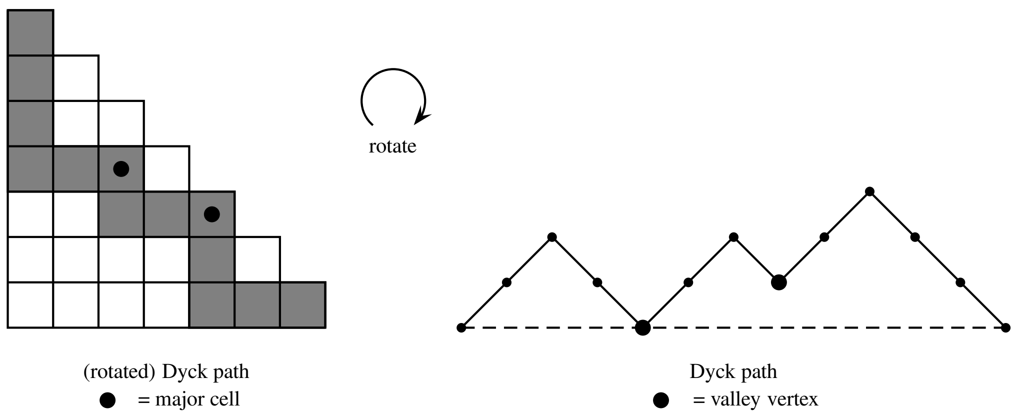
Cell decomposition

Cells with $i = 1$ or $j = 1$ are *boundary* cells, the others are *interior*. A cell is *occupied* if it contains at least one letter of π, otherwise it is *empty*. Let α_{ij} denote the subpermutation of entries in C_{ij}.

We now consider $R = \{1342, 1423\}$, a subset of the pattern set in the remaining three cases. The reader may check the following characterization of R-avoiders in terms of the cell decomposition. A permutation π is an R-avoider if and only if

1. For each occupied cell C, all cells that lie both strictly east and strictly north of C are empty.
2. For each pair of occupied cells C, D with D directly north of C (same column), all entries in C lie to the right of all entries in D.
3. For each pair of occupied cells C, D with D directly east of C (same row), all entries in C are larger than all entries in D.
4. α_{ij} avoids R for all i, j.

Condition (1) imposes restrictions on occupied cells as follows. A *major* cell for π is an interior cell C that is occupied and such that all cells directly north or directly east of C are empty. The set of major cells (possibly empty) determines a (rotated) Dyck path of semilength $m - 1$ with valley vertices at the major cells as illustrated in the figure below. (If there are no major cells, the Dyck path covers the boundary cells and has no valleys.)



If π avoids R, then condition (1) implies that all cells not on the Dyck path are empty, and condition (4) implies St(α_{ij}) is an R-avoider for all i, j. Conversely, if $n = i_m > i_{m-1} > \cdots > i_1 \geq 1$ are given and we have a Dyck path in the associated cell diagram, and an R-avoider π_C is specified for each cell C on the Dyck path, with the additional proviso $\pi_C \neq \emptyset$ for valley cells, then conditions (2) and (3) imply that an R-avoider with this Dyck path is uniquely determined.

It follows that an R-avoider π avoids the pattern τk where $\tau \in S_{k-1}$ if and only if all the subpermutations α_{ij} avoid R and τ. We use this observation in the next two results. As an immediate consequence, we have

Proposition 30. *Let τ and τ' be two patterns in S_{k-1}. If $F_{\{1342,1423,\tau\}}(x) = F_{\{1342,1423,\tau'\}}(x)$, then $F_{\{1342,1423,\tau k\}}(x) = F_{\{1342,1423,\tau' k\}}(x)$.* □

We can now find a recurrence for avoiders of the pattern set $R \cup \{12\cdots k\}$.

Proposition 31. *Let $T_k = \{1342, 1423, 12\cdots k\}$. Then*

$$F_{T_k}(x) = \frac{1+(x-2)F_{T_{k-1}}(x)+\sqrt{\big(1+xF_{T_{k-1}}(x)\big)^2-4xF^2_{T_{k-1}}(x)}}{2\big(1-F_{T_{k-1}}(x)\big)}.$$

Proof: For brevity, set $F_k = F_{T_k}(x)$. So, for m right-left maxima and an associated Dyck path of semilength $m-1$, the contribution to F_k is x^m for the right-left maxima, $F_{k-1}-1$ for each valley vertex, and F_{k-1} for every other vertex. Let ℓ denote the number of peaks in the Dyck path, so that $\ell-1$ is the number of valleys. Recall that the Narayana number $N_{m,\ell} = \frac{1}{m}\binom{m}{\ell}\binom{m}{\ell-1}$ counts Dyck paths of semilength m with ℓ peaks (see [26, Seq. A001263]). Hence, summing over m,

$$\begin{aligned}
F_k &= 1 + xF_{k-1} + \sum_{m\geq 2} x^m \sum_{\ell=1}^{m-1} N_{m-1,\ell}\,(F_{k-1}-1)^{\ell-1}F_{k-1}^{2m-\ell} \\
&= 1 + xF_{k-1} + \frac{xF_{k-1}^2}{F_{k-1}-1}\sum_{m\geq 1}\sum_{\ell=1}^{m} N_{m,\ell}\,\big(xF_{k-1}^2\big)^m\left(1-\frac{1}{F_{k-1}}\right)^{\ell} \\
&= 1 + xF_{k-1} + \frac{xF_{k-1}^2}{F_{k-1}-1}N\big(xF_{k-1}^2, 1-1/F_{k-1}\big),
\end{aligned}$$

where $N(x,y) := \sum_{m\geq 1}\sum_{\ell=1}^{m} N_{m,\ell}x^m y^\ell$ is the generating function the Narayana numbers. It is known that

$$N(x,y) = \frac{1-x(1+y)-\sqrt{(1-x(1+y))^2-4x^2y}}{2x}$$

and the theorem follows. □

3.6.3 $\mathbf{T = \{1234, 1342, 1423\}}$

Theorem 32. *Let $T = \{1234, 1342, 1423\}$. Then*

$$F_T(x) = \frac{3-2x-\sqrt{1-4x}-\sqrt{2-16x+4x^2+(2+4x)\sqrt{1-4x}}}{2(1-\sqrt{1-4x})}.$$

Proof: Since $F_{\{1342,1423,123\}}(x) = F_{\{123\}}(x) = C(x)$, we get by Proposition 31 that

$$F_T(x) = 1 + xC(x) + \frac{xC^2(x)}{C(x)-1} N\big(xC^2(x), 1 - 1/C(x)\big),$$

which, after some algebraic manipulation, agrees with the desired expression. □

3.6.4 $\mathbf{T = \{1324, 1342, 1423\}}$

Theorem 33. *Let* $T = \{1324, 1342, 1423\}$. *Then*

$$F_T(x) = \frac{3 - 2x - \sqrt{1-4x} - \sqrt{2 - 16x + 4x^2 + (2+4x)\sqrt{1-4x}}}{2(1-\sqrt{1-4x})}.$$

Proof: Since $F_{\{1342,1423,132\}}(x) = F_{\{132\}}(x) = C(x)$ and $F_{\{1342,1423,123\}}(x) = F_{\{123\}}(x) = C(x)$, we get by Proposition 30 with $\tau = 132$ and $\tau' = 123$ that $F_{\{1342,1423,1324\}}(x) = F_{\{1342,1423,1234\}}(x)$. Now apply Theorem 32. □

3.6.5 $\mathbf{T = \{1243, 1342, 1423\}}$

Theorem 34. *Let* $T = \{1243, 1342, 1423\}$. *Then*

$$F_T(x) = \frac{3 - 2x - \sqrt{1-4x} - \sqrt{2 - 16x + 4x^2 + (2+4x)\sqrt{1-4x}}}{2(1-\sqrt{1-4x})}.$$

Proof: A permutation $\pi \in S_T(n)$ with $m \geq 2$ right-left maxima avoids R and so the cell decomposition of π has an associated Dyck path that covers all occupied cells. To also avoid 1243, all the Dyck path cells except those incident with a right-left maximum, that is, cells C_{ij} with $i + j = m + 1$, must avoid 12 for otherwise some two right-left maxima would form the 43 of a 1243. Other cells need only avoid 1243. The cells C_{ij} with $i + j = m + 1$ consist of the extremities C_{1m} and C_{m1} together with all the low valleys in the Dyck path (a low valley is one incident with ground level, the line joining the path's endpoints). Suppose the Dyck path has ℓ low valleys and h high valleys. The contribution of the right-left maxima is x^m. Since $F_{\{12\}}(x) = 1/(1-x)$, the contributions of the $2m-1$ Dyck path cells are as follows. The two extremities contribute $F_T^2(x)$, the ℓ low valleys contribute $(F_T(x)-1)^\ell$, the h high valleys contribute $\big(\frac{1}{1-x} - 1\big)^h = \big(\frac{x}{1-x}\big)^h$, and the remaining cells contribute $\big(\frac{1}{1-x}\big)^{2m-3-\ell-h}$.

Let $M_{m,\ell,h}$ denote the number of Dyck paths of semilength m containing ℓ low valleys and h high valleys, with generating function $M(x,y,z) = \sum_{m,\ell,h\geq 0} M_{m,\ell,h} x^m y^\ell z^h$. Then, by the first return decomposition of the Dyck paths, we obtain

$$M(x,1,z) = 1 + xM(x,1,z) + xzM(x,1,z)(M(x,1,z)-1)$$

and

$$M(x,y,z) = 1 + xM(x,1,z) + xyM(x,1,z)(M(x,y,z)-1)\,.$$

Thus,

$$M(x,y,z)=\frac{y-2z-1+x(1-y)(1-z)+(1-y)\sqrt{1-2x(1+z)+x^2(1-z)^2}}{(1-x)y+(xy-2)z-y\sqrt{1-2x(1+z)+x^2(1-z)^2}}.$$

Hence, summing over m and over all Dyck paths gives

$$F_T(x)=1+xF_T(x)+\sum_{m\geq 2}\sum_{\ell,h\geq 0}M_{m-1,\ell,h}x^{m+h}F_T^2(x)\big(F_T(x)-1\big)^{\ell}\frac{1}{(1-x)^{2m-3-\ell}}.$$

After several algebraic steps and solving for $F_T(x)$, one obtains the desired formula. □

The preceding theorem can be extended to the case $T_k=\{1342,1423,\tau k(k-1)\}$ with $k\geq 4$ as follows.

Theorem 35. *Let $k\geq 4$ and $\tau\in S_{k-2}$. Let $T_k=\{1342,1423,\tau k(k-1)\}$ and $T'_k=\{1342,1423,\tau\}$. Then*

$$F_{T_k}(x)=\frac{(2-x)(1-t)-x^2F^2_{T'_k}(x)(1+(x-2)F_{T'_k}(x))+\sqrt{2x(a-bt)}}{2(1-xF^2_{T'_k}(x)+x^2F^3_{T'_k}(x)-t)},$$

where

$$\begin{aligned}
t&=\sqrt{(1-xF^2_{T'_k}(x))^2-x^2F^3_{T'_k}(x)(2-2xF^2_{T'_k}(x)+x^2F^3_{T'_k}(x))},\\
a&=(x-4)(1+x^4F^6_{T'_k}(x))+2xF^2_{T'_k}(x)(1+(1-x)F^2_{T'_k}(x)+x^2F^3_{T'_k}(x))+x^3F^4_{T'_k}(x),\\
b&=(4-x)(1+x^2F^3_{T'_k}(x))+x(2-x)F^2_{T'_k}(x).
\end{aligned}$$

Proof: The proof follows along the same lines as in the preceding theorem except that $F_T(x)$ is replaced by $F_{T_k}(x)$ and $F_{\{12\}}(x)$ by $F_{T'_k}(x)$. The details are left to the reader. □

4 Concluding remarks

No one technique seemed to have solved all of the $(4,4,4)$-cases, with more than one technique often required for triples in the same symmetry class. This is frequently the case for the problem of avoidance on permutations and also other discrete structures. It is well known that there is no general procedure for counting T-avoiding permutations, which is why it might be described as an art. The often complementary methods seen here included use of initial letters, left-right maxima, lattice paths (in conjunction with the cell decomposition used in Case 238), combinatorial statistics and auxiliary arrays. The last method, which was applied in such cases as 229B and 235A above, could rightfully be termed the method of multiple arrays. For it entailed defining several arrays enumerating various subsets of the class of avoiders in question and then finding recurrences for these arrays, which were often intertwined. These recurrences would then lead to a system of functional equations satisfied by the corresponding generating functions, which was often easier to solve than the single functional equation for the entire class (if it could even be found).

An integer sequence $\{a_n\}$ is called polynomially recursive, or P-recursive (or D-finite), if it satisfies a nontrivial linear recurrence relation of the form $q_0(n)a_n+q_1(n)a_{n-1}+\cdots+q_k(n)a_{n-k}=0$ for some polynomials $q_1(x), \ldots, q_k(x)$ having integer coefficients. A D-finite function is one satisfying a linear differential equation having polynomial coefficients. In 1996, Noonan and Zeilberger [24] conjectured that the sequence $|S_n(T)|$ is P-recursive in n for any set of permutation patterns T. In 2015, this conjecture was disproved by Garrabrant [9, Chapter 2] in his thesis. Thus, there is interest as to whether or not a sequence $|S_n(T)|$ is P-recursive, or if the corresponding generating function $F_T(x) = \sum_{n\geq 0} |S_n(T)|x^n$ is algebraic or not. Note that every algebraic function is D-finite (Abel's theorem). In this paper, we have demonstrated that the generating function $F_T(x)$ is algebraic for any set T of three patterns in S_4, where T belongs to a large Wilf class. But in general this is not the case, see [9].

We have made use of software from [14] in computing the initial terms of the sequence $\{|S_n(T)|\}_{n\geq 1}$ on which we based our assumptions for the various equivalences prior to proving them. One might wonder how many terms of this sequence were required to distinguish the various Wilf classes. Let us say that the Wilf classification of all sets of k patterns in S_4 has a depth $d = d(k)$ if sets T and T' each containing k patterns belong to the same Wilf class if and only if $|S_n(T)| = |S_n(T')|$ for all $n = 1, \ldots, d$. As a preliminary step, we created the sequences $\{|S_n(T)|\}_{n=1}^{16}$ for all sets T containing three patterns in S_4. After doing so, we noticed that $d(3) = 9$. Below are the values we have found for $1 \leq k \leq 23$:

k	1	2	3	4	5	6	7	8	9	10	11	12
$d(k)$	7	8	9	10	10	11	11	11	10	10	10	10
k	13	14	15	16	17	18	19	20	21	22	23	
$d(k)$	9	9	9	8	8	7	7	7	7	6	5	

Tab. 2: Values of $d(k)$.

The values $d(10), d(11), \ldots, d(23)$ above follow from the work in [18], while $d(8)$ and $d(9)$ follow from [19]. The values for $d(4)$ through $d(7)$ are conjectural and will be an object of forthcoming work. Once the Wilf classification for sets of patterns in S_4 of size four through seven is settled, so will be the complete classification of all subsets of S_4. This would represent a significant extension of the comparable result of Simion and Schmidt [25] for S_3.

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