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S-restricted compositions revisited

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An S-restricted composition of a positive integer n is an ordered partition of n where each summand is drawn from a given subset S of positive integers. There are various problems regarding such compositions which have received attention in recent years. This paper is an attempt at finding a closed-form formula for the number of S-restricted compositions of n. To do so, we reduce the problem to finding solutions to corresponding so-called interpreters which are linear homogeneous recurrence relations with constant coefficients. Then, we reduce interpreters to Diophantine equations. Such equations are not in general solvable. Thus, we restrict our attention to those S-restricted composition problems whose interpreters have a small number of coefficients, thereby leading to solvable Diophantine equations. The formalism developed is then used to study the integer sequences related to some well-known cases of the S-restricted composition problem.

Keywords: closed-form formula, Diophantine equations, homogeneous recurrence relations, restricted compositions

1 Introduction

A composition, or an ordered partition, of a positive integer n is a tuple of positive integers whose sum equals n. Integer compositions have been of high interest as they have formed part of the solution in many problems Wilcoxon (1945); Cipra (1987); Actor (1994); Blecher et al. (2013). Finding the number of compositions of n where summands belong to a subset S of positive integers, known as S-restricted compositions of n, is a challenging problem. This problem is referred to as the S-restricted composition problem in this paper.

There are closed-form solutions to the S-restricted composition problem for some particular subsets S of positive integers. Moreover, for some classes of S, it has been shown that the number of S-restricted compositions can be obtained by solving specific recursive equations. Deriving a closed-form solution for the general S-restricted composition problem through recursive equations and fractional generating functions will lead to solving polynomial equations. Moreover, the use of exponential generating functions for this problem leads to Diophantine equations. Neither polynomial nor Diophantine equations are in general solvable. That is why there is no closed-form formula for the general problem.

As stated above, solving a linear homogeneous recurrence relation with constant coefficients (LHRC) through classical methods requires solving a polynomial equation. The degree of such a polynomial equation is equal to the largest offset of the recurrence relation. Instead of solving such polynomial equations,

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which is impossible in most cases, we derive Diophantine equations whose number of variables is the number of offsets of the recurrence relation. Such a transformation is quite useful for the S-restricted composition problems which lead to LHRCs with a small number of offsets. To arrive at appropriate LHRCs with acceptable number of offsets, we give two equivalent LHRCs, which we call the interpreters of the S-restricted composition problem. The proposed interpreters lead to different Diophantine equations from which we choose the one with smaller number of variables. To illustrate the usefulness of our technique, we then apply it to some classes of the problem. It is worth noting that the interpreters are not limited to those given in this paper and one may find other interpreters that are more appropriate for other classes of the problem not studied in this paper.

To explain our technique, we first introduce some notation. We denote the set of all S-restricted compositions of n by $\mathcal{R}(S,n)$ and the number of such compositions, i.e., $|\mathcal{R}(S,n)|$, by $R(S,n)$. Moreover, $\mathcal{C}(n)$ is a shorthand for $\mathcal{R}(\mathbb{Z}^+,n)$. We denote the empty tuple by ϵ and define $\mathcal{R}(S,0)=\{\epsilon\}$ for $S\subset\mathbb{Z}^+$, although it is conventionally undefined. For $c\in\mathcal{R}(S,n)$, by first(c), we mean the first part of c. The tuple obtained from c by removing its first part is denoted by $f^-(c)$, which will obviously be equal to ϵ if c includes exactly one part. Moreover, the tuple obtained from prepending a positive integer a to c is denoted by $a;c$. The tuple $a;c$ is equal to a if $c=\epsilon$. Moreover, the sum and the product of all elements of a tuple T are represented by $\sigma(T)$ and $\pi(T)$, respectively. For $(i_1,\ldots,i_k)\in\mathcal{R}(S,n)$ and a vector $\mathbf{c}$, the tuple $t=(c_{i_1},\ldots,c_{i_k})$ is called an (S,n)-tuple of $\mathbf{c}$ in which c_{i_j} is the i_jth element of $\mathbf{c}$. Moreover, when $\mathbf{c}$ is known from the context, $(i_1,\ldots,i_k)$ is denoted by $\lambda(t)$ and is called the index tuple of t. Furthermore, $\mathcal{R}_{\mathbf{c}}(S,n)$ is the family of all (S,n)-tuples of $\mathbf{c}$.

Let A be a finite set of q positive integers. The elements of A can naturally be arranged as $a_1<a_2<\ldots<a_q$. Then, the row vector $[a_1,\ a_2,\ \ldots,\ a_q]$ is denoted by $\mathbf{v}^A$. For a positive integer n, we also define $A_{\leq n}$ as $A\cap[1:n]$. The jth element of a vector $\mathbf{x}$ is noted x_j. The length of a vector $\mathbf{x}$, noted $|\mathbf{x}|$, is defined to be the number of elements of $\mathbf{x}$. The inner product of vectors $\mathbf{x}$ and $\mathbf{y}$ is denoted by $\mathbf{x}\cdot\mathbf{y}$. Finally, for two vectors $\mathbf{x}$ and $\mathbf{y}$ of the same length, by $\mathbf{x}\geq\mathbf{y}$, we mean that $x_j\geq y_j$ for every j.

The S-restricted composition problem can be solved through solving a Diophantine equation. Consider the Diophantine equation

$$\mathbf{v}^S\cdot\mathbf{x}=n \tag{1}$$

in which $|\mathbf{x}|=|\mathbf{v}^S|=|S|$, that is, the number of variables equals the cardinality of S. Every solution $\mathbf{x}$ to (1) satisfying $\mathbf{x}\geq\mathbf{0}$ represents those S-restricted compositions of n in which v_i^S occurs x_i times where $i=1,\ldots,|S|$. The number of such compositions is

$$C(\mathbf{x})=\frac{\left(\sum_{i=1}^{|\mathbf{x}|}x_i\right)!}{\prod_{i=1}^{|\mathbf{x}|}x_i!}=\binom{\sum_{i=1}^{|\mathbf{x}|}x_i}{x_1,x_2,...,x_{|\mathbf{x}|}}.$$

Therefore, the number of all S-restricted compositions of n is $\sum_{\mathbf{x}\in D}C(\mathbf{x})$ where D is the set of all solutions to (1) satisfying $\mathbf{x}\geq\mathbf{0}$. The number of variables in the above equation is $|S|$. Finding a closed-form solution for a Diophantine equation with a large number of variables is a challenging problem. This makes it difficult to derive a closed-form formula for the number of S-restricted compositions by directly reducing the problem to a Diophantine equation similar to (1) if $|S|$ is large. It is also difficult to extract a closed-form solution for the S-restricted composition problem via reducing it to an LHRC and solving the LHRC via its characteristic equation. To clarify this point, assume that solving the S-restricted composition problem can be accomplished through solving the LHRC

$$f(n) = \sum_{i=1}^{l} k_i f(n - a_i) \tag{2}$$

in which $k_i \neq 0$ and $a_i \in \{1, 2, ..., M\}$ for some $M \in \mathbb{Z}^+$. In such a case, Equation (2) is referred to as an interpreter for the S-restricted composition problem. Solving (2) leads to solving a polynomial equation of degree $\max\{a_i\}$. Such a polynomial equation is not in general solvable for $\max\{a_i\} \geq 5$ Galois (1846); Kulkarni (2008); Benjamin et al. (2011); Klein (1884); Hermite (1858) making it impossible to solve most well-known cases of the S-restricted composition problem through this approach. This motivates us to give a method for solving (2) via finding solutions for Diophantine equations with $|\{a_i\}|$ variables. This simplifies solving LHRCs in which $|\{a_i\}|$ is small. Solving any case of the S-restricted composition problem for which an interpreter similar to (2) can be derived is consequently simplified provided that $|\{a_i\}|$ is small .

We derive two interpreters for the S-restricted composition problem which lead to closed-form solutions for the cases studied in this paper. That is, the cases in which the set S is of the form $\{a_1, a_2\}$, $\{a \mid a = r\ (mod\ m)\}$, $[1 : m]$, $[1 : n] \setminus \{m\}$, or $[1 : n] \setminus [1 : m]$ where $[i : j]$ is the subset $\{i, i+1, i+2, \ldots, j\}$ of $\mathbb{Z}^+$. One may, however, envision other interpreters that are more appropriate in other cases. Finding appropriate interpreters can indeed be thought of as a step towards solving the problem.

The rest of this paper is organized as follows: Section 2 discusses previous research on related problems. Section 3 derives the interpreters. Section 4 attempts to derive a closed-form solution for the LHRC of Section 3 and shows that it can be derived from the solution of a Diophantine equation whose number of variables is less than $|S|$. Section 4 also gives closed-form solutions for LHRCs (including the interpreters derived in Section 3) using Diophantine equations. Section 5 employs the results of Section 4 to solve some instances of the S-restricted composition problem and finds the elements of some well-known integer sequences. Section 6 concludes the paper.

2 Related Work

Many applied problems involve variants of the integer composition problem where different kinds of constraints are placed on the structure or parts of the composition. We briefly review earlier research on integer compositions with such constraints. As the first example, we can refer to those considering palindromic compositions which read the same from the left and the right Hoggatt and Bicknell (1975). Another example is the problem of locally restricted compositions in which every k successive parts meet particular constraints Bender et al. (2012); Bender and Canfield (2009, 2005). Compositions whose parts are from a given set and successive parts of a given length avoid particular patterns are also investigated in the literature Heubach and Mansour (2006); Savage and Wilf (2006). The research on the same has also led to the generating functions that yield a closed-form solution to the problem in special cases Heubach and Kitaev (2010). Generating functions have also been derived for the number of compositions where the swap of two parts of the composition is not relevant Heubach et al. (2009). Compositions avoiding some partially ordered patterns have also been studied Heubach et al. (2007). Partially ordered patterns define a partial order relation on parts of a composition.

A great deal of research has been dedicated to S-restricted compositions of a given positive integer n with constraints on the value and the number of summands and the way they can be arranged in the composition. Research on the number of such compositions has even been augmented by investigating their probabilistic, statistical, or asymptotic properties Shapcott and Schmutz (2013); Bóna and Knopfmacher (2010). In most cases, however, a closed-form formula is far from reach, and thus, the research has not gone beyond generating functions or recursive equations that yield the the number of compositions if solvable. It is for this reason that some researchers have estimated Ratsaby (2008) and even conjectured the number of compositions Banderiera and Hitczenko (2012). The following is a short overview of the earlier work on the problem of S-restricted composition.

In one line of research, it has been tried to derive generating functions for the number of S-restricted compositions. Among the attempts made, we can mention Heubach and Mansour (2004) in which a generating function is derived for the number of palindromic S-restricted compositions. We may also refer to the generating functions presented in Banderiera and Hitczenko (2012) for the number of compositions of n consisting of a specific number of parts. The functions are further used to calculate the probability that two independently-selected random compositions of n have the same number of parts.

Another line of research in this area has focused on solving the general problem of S-restricted composition, where the only constraint is that parts belong to S, by deriving recursive equations. An example is the recursive equation derived in Jaklic et al. (2010) for the general problem. The inherent difficulty with such equations is that they are in general unsolvable. The same paper solves the recursive equation in the special case where S is a bounded set of consecutive positive integers.

The impossibility of finding a general solution to the problem of S-restricted compositions has motivated researchers to give solutions in useful and well-known special cases. A case is where S consists of two elements, namely 1 and 2 Alladi and Hoggatt (1975), 1 and K Chinn and Heubach (2003a), and 2 and 3 Bisdorff and Marichal (2008), to mention a few. In another case, S is the set of powers of a specific number Chinn and Niederhausen (2004); Krenn and Wagner (2014). The number of compositions in case the summands are from a given sequence of numbers has also been investigated Knopfmacher and Robbins (2003). The same has also been studied when the summands are congruent to k in modulo m for some particular integers k and m. For example, it has been shown that for $k = 1$ and $m = 2$, the number of compositions of n is the nth Fibonacci number Grimaldi (2000). For $k \in \{1, 2\}$ and $m = 4$, the number of compositions of a given integer is an element of the Padovan sequence Medina and Straub (2016). The problem has also been studied in the case $S = \mathbb{Z}^+ \setminus \{m\}$ Chinn and Heubach (2003b); Grimaldi (2001); Chinn and Heubach (2003c).

Another approach to the problem of S-restricted compositions is to reduce it to some other known problems. For example, it has been shown that some cases of the problem are in bijective correspondence with some classes of restricted binary strings and pattern-avoiding permutations Baril and Do (2013). It has also been shown that the number of S-restricted compositions equals specific Fibonacci numbers for some specific sets S Sills (2011).

The S-restricted composition problem can also be studied as a special case of the problems weighted compositions Eger (2013, 2015, 2016); Whitney and Moser (1961), restricted words Janjić (2016), and colored compositions Shapcott (2013, 2012). The solutions presented for these problems, however, defer the problem to finding a closed-form formula for so-called extended binomial coefficients.

Having said all this, there is no closed-form formula for the number of S-restricted compositions. We do not solve the general problem either, but we take this area of research one step further. To do so, we reduce the general problem to linear homogeneous recursive equations and Diophantine equations so that we can find exact solutions in new families of cases.

3 Deriving Interpreters

In this section, we derive interpreters for the S-restricted composition problem. We begin with the fact that for any $c \in \mathcal{R}(S,n)$, there exists $s \in S$ such that first$(c) = s$ and $f^{-}(c) \in \mathcal{R}(S, n-s)$. It is also noted that for any $s \in S$ and $c \in \mathcal{R}(S, n-s)$, we have $s; c \in \mathcal{R}(S,n)$. Thus, the number of S-restricted compositions of n satisfies

$$R(S,n) = \sum_{s \in S_{\leq n}} R(S, n-s) \tag{3}$$

whenever $S_{\leq n}$ is not empty.

We refer to Equation (3) as the first interpreter of the S-restricted composition problem. It is obvious that for $n \geq \max(S)$, we should have the values of $R(S,i)$ for $i \in [1 : \max(S)]$ as initial conditions to solve the above equation. Let us examine how to obtain the initial conditions. Let $\mathbf{v}^S = [s_1, s_2, \ldots, s_{|S|}]$. $R(S,n)$ is equal to 0 for $n < s_1$ and $R(S, s_1)$ equals 1. One can calculate the mth initial value, $m = s_1+1, s_1+2, ..., \max(S)$, in their respective order using (3) where n is replaced with m.

The number of terms in the right side of (3) is clearly equal to $|S_{\leq n}|$. Moreover, in Section 4, we show that recursive equations of the form $f(n) = \sum_{i \in I} a_i f(n-i)$ with $a_i \in \mathbb{R}$ are solvable if we can solve linear Diophantine equations with $|I|$ variables. Since Diophantine equations with a small number of variables are solvable, we attempt to derive another interpreter (referred to as the second interpreter) in this section. We show in Section 5 that some instances of the S-restricted composition problem are easier to solve using the first interpreter and others through the second interpreter.

In order to derive the second interpreter, we first propose a procedure to produce $\mathcal{R}(S,i)$ using $\mathcal{R}(S, i-s)$ where $s \in \{s | s \in S_{\leq i}, s-1 \notin S\}$. The following lemma introduces the procedure.

Lemma 1. *For every positive integer i, the following procedure derives $\mathcal{R}(S,i)$ from the sets $\mathcal{R}(S,i-s)$ where $s \in \{s | s \in S_{\leq i}, s-1 \notin S\}$.*

1. *Set $R = \emptyset$.*
2. *If $i \neq 1$, for each composition c of $\mathcal{R}(S, i-1)$, add $($first$(c)+1); f^{-}(c)$ to R provided first$(c)+1 \in S$.*
3. *For every s in $S_{\leq i}$, if $s-1 \notin S$, for each composition c in $\mathcal{R}(S, i-s)$, add $s; c$ to R.*
4. *Set $\mathcal{R}(S,i) = R$.*

Proof: It is obvious that $R \subseteq \mathcal{R}(S,i)$. Thus, we only need to show that $\mathcal{R}(S,i) \subseteq R$. There are two possible cases for the first part s of a given composition c of i. If $s-1 \in S$, then $c = ($first$(c')+1); f^{-}(c')$ for some $c' \in \mathcal{R}(S, i-1)$. If $s-1 \notin S$, $c = s; c'$ for some $c' \in \mathcal{R}(S, i-s)$. □

The following theorem presents the second interpreter.

Theorem 1. *For every positive integer $n > 1$, $R(S,n)$ satisfies the following LHRC.*

$$R(S,n) = R(S,n-1) - \sum_{\substack{s-1 \in S_{\leq n}, \\ s \notin S}} R(S,n-s) + \sum_{\substack{s \in S_{\leq n}, \\ s-1 \notin S}} R(S,n-s) \tag{4}$$

Proof: In the right side of (4), the first two terms clearly count the compositions created by the second step and the second term counts those created by the third step of the procedure given in Lemma 1. □

The number of terms in the right side of the first interpreter is equal to $T_1 = |S_{\leq n}|$ while that of the second interpreter is equal to $T_2 = |S'|$ where

$$S' = \{1\} \cup \{s | s-1 \in S_{\leq n}, s \notin S\} \cup \{s | s \in S_{\leq n}, s-1 \notin S\}.$$

Solving the S-restricted composition problem through the first interpreter is easier if $T_1 < T_2$ and the second interpreter will lead to a simpler resolvent if $T_1 > T_2$.

4 Solving LHRCs through Diophantine Equations

In this section, we present a technique for solving LHRCs based on Diophantine equations. We use the technique to derive closed-form solutions for LHRCs with two or three terms because they are useful in S-restricted composition problems studied in Section 5. The same approach can be used to solve LHRCs with more terms.

Every LHRC can be written in the form of (2) in which k_is and a_is are called coefficients and offsets, respectively. The vectors $\mathbf{a} = [a_1, a_2, \ldots, a_l]$ and $\mathbf{k} = [k_1, k_2, \ldots, k_l] = [\kappa(a_1), \kappa(a_2), \ldots, \kappa(a_l)]$ are referred to as the offset and the coefficient vectors in which $\kappa(a_i)$ is the coefficient of the term with offset a_i. The function κ that maps the offsets to the coefficients of (2) is also referred to as the representative of (2). Moreover, $A = \{a_i | i \in [1:l]\}$ and $K = \{k_i | i \in [1:l]\}$ are called the offset and coefficient sets of (2), respectively. It is obvious that $\mathbf{a} = \mathbf{v}^A$. Using the representative function, (2) can be rewritten as follows.

$$f(n) = \sum_{i=1}^{l} \kappa(a_i) f(n-a_i) \tag{5}$$

By one-step expansion of (2), we mean expanding every individual term in the right side by (2) itself. The LHRC obtained from repeating one-step expansions is called an expanded version of (2). Moreover, by solving (2), we mean finding an expanded version of it where the right side only consists of $f(i)$s for $i \in [0 : a_l - 1]$. Such an expanded version is of the form

$$f(n) = \sum_{i=0}^{a_l - 1} V(n,i) f(i) \tag{6}$$

and is a solution to (2) because $f(i)$s are already known for $i \in [0 : a_l - 1]$ as initial conditions. The only remaining problem is to find the coefficients $V(n,i)$. The following lemma is the first step to this end.

Lemma 2. *The following procedure produces* $R = \bigcup_{i \in [n-(\max S - 1):n]} \mathcal{R}(S, i)$.

1. *Set* $m_1 = min(S)$, $m_2 = \max(S)$, $Q = \{(s) \,| s \in S\}$, *and* $R = \emptyset$.
2. *Repeat the following step* $\lfloor \frac{n}{m_1} \rfloor$ *times.*
3. *For every element* q *of* Q, *if* $\sigma(q) \geq n - (m_2 - 1)$, *add* q *to* R. *Otherwise, remove* q *from* Q, *set* $h = q$, *and for every element* s *of* S, *add* $s; h$ *to* Q.

Proof: The above procedure clearly does not allow $\sigma(q)$ to get smaller than $n-(m_2-1)$ or greater than n. Thus, R will be a subset of $\bigcup_{i\in[n-(m_2-1):n]}\mathcal{R}(s,i)$. On the other hand, $\bigcup_{i\in[n-(m_2-1):n]}\mathcal{R}(s,i)$ is a subset of R because R includes every tuple consisting of a permutation of elements of S with a total sum of $i\in[n-(m_2-1):n]$. $\square$

The following lemma is another step towards calculating $V(n,i)$s.

Lemma 3. *The coefficients $V(n,i)$ in the solution* (6) *for* (2) *can be obtained from*

$$V(n,i)=\sum_{\substack{z\,\in\,\mathcal{R}_{\mathbf{k}}(A,\,n\,-\,i),\\ \mathit{first}(\lambda(z))\,\geq\,a_l\,-\,i}}\pi(z) \tag{7}$$

Proof: Let us examine the process of expanding (2) till reaching a solution in the form of (6). Now, consider the offset a_i and its corresponding coefficient $\kappa(a_i)=k_i$ in (2) as tuples $(\kappa(a_i))$ and (a_i). Expanding every term in the right side of (2) represented by $(o_i,c_i)=(a_i,\kappa(a_i))$ replaces the term by $\{(\kappa(a_j)o_i,c_i+a_j)|j\in[1:l]\}$. It is immediate that if $n-(a_j+c_i)\in[0:a_l-1]$, the pair $(\kappa(a_j)o_i,c_i+a_j)$ represents $\kappa(a_j)o_if(n-(c_i+a_j))$ which should not be expanded any more because $f(n-(c_i+a_j))$ is an initial value. Thus, during successive steps of expanding individual terms of the equation, every offset will be formed as $\sigma(z_A)$ where z_A is a tuple consisting of elements of A. The corresponding coefficient will also be $\pi(z_K)$ in which z_K is a tuple consisting of elements of K. It can easily be shown through induction that $\lambda(z_K)$ will be equal to z_A for every term in every expansion step. Lemma 2 states that the index tuple of every z_K (like its corresponding z_A) will be an A-restricted composition of an integer $n-(a_l-1)\leq i\leq n$ after $\tau=\lfloor\frac{n}{a_1}\rfloor$ expansion steps provided that initial terms (terms with offsets $n-(a_l-1)\leq O\leq n$) are not further expanded. Thus, after τ steps of expansions, we reach an equation in which only initial terms $f(i):i\in[0:l-1]$ appear in the right side. We refer to the latter equation as the fully-expanded equation. Lemma 2 also states that for every composition $c\in\bigcup_{i\in[n-(m_2-1):n]}\mathcal{R}(s,i)$, there will be a term with coefficient $\pi(z_K)$ and offset $\sigma(z_A)$ in the fully-expanded equation where $\lambda(z_K)=z_A=c$ if first$(c)\geq a_l-1$. The condition first$(c)\geq a_l-i$ guarantees that terms with offsets $n-(a_l-1)\leq O\leq n$ are not further expanded. $\square$

Theorem 2. *The following is the solution to* (2)

$$f(n)=\sum_{\alpha\in A\setminus\{a_1\}}\sum_{i=a_l-\alpha}^{a_l-a_1-1}\kappa(\alpha)f_R(n-i-\alpha+(a_l-a_1))f(i)+ \tag{8}$$
$$\sum_{i=a_l-a_1}^{a_l-1}f_R(n-i+(a_l-a_1))f(i),$$

where

$$f_R(q)=\sum_{z\in\mathcal{R}_{\mathbf{k}}(A,q-(a_l-a_1))}\pi(z). \tag{9}$$

Proof: For every $z\in\mathcal{R}_{\mathbf{k}}(A,n-i)$, $\lambda(z)$ is an A-restricted composition of $n-i$. Therefore, first$(\lambda(z))$ is obviously greater than or equal to a_1, the least element of A. Thus, in (7), since $a_1\geq a_l-i$, first$(\lambda(z))$

will certainly be greater than or equal to $a_l - i$ for $i \geq a_l - a_1$. Thus, we can rewrite (7) as

$$V(n,i) = \sum_{z \in \mathcal{R}_{\mathbf{k}}(A,n-i)} \pi(z) = f_R(n - i + (a_l - a_1)). \tag{10}$$

for $i \geq a_l - a_1$. For $i < a_l - a_1$, $\text{first}(\lambda(z)) \geq a_l - i$, or equivalently, $i \geq a_l - \text{first}(\lambda(z))$. In this case, (7) can be rewritten as

$$\begin{aligned} V(n,i) &= \sum_{\alpha \in A \setminus \{a_1\}} \left(\kappa(\alpha) \sum_{z \in \mathcal{R}_{\mathbf{k}}(A,n-i-\alpha)} \pi(z) \right) \\ &= \sum_{\alpha \in A \setminus \{a_1\}} \kappa(\alpha) f_R(n - i - \alpha + (a_l - a_1)). \end{aligned} \tag{11}$$

By feeding (10) and (11) into (6), we derive (8). □

It can easily be shown that

$$\sum_{z \in \mathcal{R}_{\mathbf{k}}(A,q-(a_l-a_1))} \pi(z) = \sum_{\{\mathbf{x} | \mathbf{x} \cdot \mathbf{v}^A = q - (a_l - a_1),\ \mathbf{x} \geq \mathbf{0}\}} P_o(\mathbf{k}, \mathbf{x}) C(\mathbf{x}) \tag{12}$$

in which

$$P_o(\mathbf{k}, \mathbf{x}) = \prod_{j=1}^{|\mathbf{x}|} k_j^{x_j}. \tag{13}$$

Thus, given A and $\mathbf{k}$, in order to calculate $f_R(q)$'s, we should solve the Diophantine equation $\mathbf{x} \cdot \mathbf{v}^A = q - (a_l - a_1)$ with the constraint $\mathbf{x} \geq \mathbf{0}$, which we call the resolvent of (2).

Now, we apply our technique to two-term and three-term LHRCs.

Corollary 1. *The solution to the two-term LHRC*

$$f(n) = \kappa(a_1) f(n - a_1) + \kappa(a_2) f(n - a_2) \tag{14}$$

is

$$\begin{aligned} f(n) = \kappa(a_2) \sum_{i=0}^{a_2 - a_1 - 1} & f_R(n - i - a_1) f(i) + \\ & \sum_{i = a_2 - a_1}^{a_2 - 1} f_R(n - i + (a_2 - a_1)) f(i), \end{aligned} \tag{15}$$

where

$$\begin{aligned}
&f_R(q) = \sum_{t=L}^{U} P_o(\mathbf{k}, \mathbf{x^{(t)}})C(\mathbf{x^{(t)}}),\\
&\mathbf{k} = [\kappa(a_1),\ \kappa(a_2)],\\
&\mathbf{x^{(t)}} = [rq'/g + t, sq'/g - ta_1/g],\\
&q' = q - (a_2 - a_1),\\
&L = \lceil -rq'/a_2 \rceil,\\
&U = \lfloor sq'/a_1 \rfloor,\\
&g = \gcd(a_1, a_2),
\end{aligned}$$

and r and s are Bezout coefficients for a_1 and a_2.

Proof: The resolvent of (14) is the Diophantine equation $a_1x_1 + a_2x_2 = q-(a_2-a_1)$ with the constraint $x_1, x_2 \geq 0$. The result is then immediate from Theorem 2. □

Three-term LHRCs are of special interest to researchers Gonoskov (2014); Liu (2010). The following corollary shows how to solve these equations by the technique described above.

Corollary 2. *The solution to the three-term LHRC*

$$f(n) = \kappa(a_1)f(n-a_1) + \kappa(a_2)f(n-a_2) + \kappa(a_3)f(n-a_3) \tag{16}$$

is

$$\begin{aligned}
f(n) =&\kappa(a_2)\sum_{i=a_3-a_2}^{a_3-a_1-1} f_R(n-i-a_2+(a_3-a_1))f(i)+\\
&\kappa(a_3)\sum_{i=0}^{a_3-a_1-1} f_R(n-i-a_1)f(i)+\\
&\sum_{i=a_3-a_1}^{a_3-1} f_R(n-i+(a_3-a_1))f(i),
\end{aligned} \tag{17}$$

where

$$
\begin{aligned}
&f_R(q) = \sum_{x_3=0}^{\lfloor q'/a_3 \rfloor} \sum_{t=L}^{U} P_o(\mathbf{k}, \mathbf{x}^{(\mathbf{t})}) C(\mathbf{x}^{(t)}),\\
&\mathbf{k} = [\kappa(a_1), \kappa(a_2), \kappa(a_3)],\\
&\mathbf{x}^{(\mathbf{t})} = [rq'/g + t, sq'/g - ta_1/g, x_3],\\
&q' = q - (a_3 - a_1),\\
&\widehat{q} = q' - a_3 x_3,\\
&L = \lceil -rq'/a_2 \rceil,\\
&U = \lfloor sq'/a_1 \rfloor,\\
&g = \gcd(a_1, a_2),
\end{aligned}
$$

and r and s are Bezout coefficients for a_1 and a_2.

Proof: The resolvent of (16) is $a_1x_1 + a_2x_2 + a_3x_3 = q - (a_3 - a_1)$ with the constraint $x_1, x_2, x_3 \geq 0$. It can be solved through solving the Diophantine equations $a_1x_1 + a_2x_2 = q - (a_3 - a_1) - a_3x_3$ for every possible x_3 in $[0 : \lfloor q/a_3 \rfloor]$. Theorem 2, then, implies (17). □

5 Relevant Problems

In this section, we derive closed-form solutions for some well-known cases of the S-restricted composition problem. We divide these problems into two classes. The first class consists of those problems that are directly solved through corresponding interpreters. The second class is comprised of the problems that can be solved through finding the general term of related integer sequences.

5.1 Relevant S-restricted Composition Problems

We employ the first and second interpreters, i.e., (3) and (4), to derive closed-form solutions for some well-known S-restricted composition problems. We begin with a well-known case of the S-restricted composition problem where S consists of two positive integers a_1 and a_2.

Corollary 3. *The number of compositions of n into two distinct positive integers a_1 and a_2 is obtained from*

$$
\begin{aligned}
R(\{a_1, a_2\}, n) = &\sum_{h=0}^{\lfloor \frac{a_2 - a1 - 1}{a_1} \rfloor} f_R(n - a_1(h+1)) + \\
&\sum_{h=\lfloor \frac{a_2 - a1 - 1}{a_1} \rfloor + 1}^{\lfloor \frac{a_2 - 1}{a_1} \rfloor} f_R(n - a_1(h+1) + a_2),
\end{aligned}
\tag{18}
$$

where

$$\begin{aligned}
&f_R(q) = \sum_{t=L}^{U} C(\mathbf{x}^{(\mathbf{t})}),\\
&\mathbf{x}^{(\mathbf{t})} = [rq'/g + t, sq'/g - ta_1/g],\\
&q' = q - (a_2 - a_1),\\
&L = \lceil -rq'/a_2 \rceil,\\
&U = \lfloor sq'/a_1 \rfloor,\\
&g = \gcd(a_1, a_2),
\end{aligned}$$

and r and s are Bezout coefficients for a_1 and a_2.

Proof: In this case, it is evident that the first interpreter has fewer terms than the second interpreter. Indeed, the first interpreter is $R(\{a_1, a_2\}, n) = R(\{a_1, a_2\}, n-a_1)+R(\{a_1, a_2\}, n-a_2)$ for $n > a_2-1$. Moreover, $R(\{a_1, a_2\}, i)$ is equal to 1 for $0 \le i \le a_2 - 1$ if $n \mod a_1 = 0$ and equals 0 otherwise. Thus, the solution is (18). □ Equation (18) gives a closed-form solution to the general problem whose special cases have been studied in Alladi and Hoggatt (1975); Chinn and Heubach (2003a); Bisdorff and Marichal (2008). The same equation can also be used to calculate the number of compositions of n into the positive integers which are congruent to a given integer r modulo the given integer m. In doing so, it suffices to calculate the number of compositions of $n - r$ into r and m. In this way, (18) gives a closed-form solution to the general problems whose special cases have been studied in Grimaldi (2000); Medina and Straub (2016).

Corollary 4. *The number of compositions of a positive integer n into the positive integers which are less than or equal to m is obtained from*

$$R([1:m], n) = (2)^{m-1} f_R(n) - f_R(n-1) - \sum_{i=1}^{m-1} (2)^{i-1} f_R(n-i-1), \tag{19}$$

where

$$\begin{aligned}
&f_R(q) = \sum_{t=L}^{U} P_o(\mathbf{k}, \mathbf{x}^{(\mathbf{t})}) C(\mathbf{x}^{(\mathbf{t})}),\\
&\mathbf{x}^{(\mathbf{t})} = [-mq' + (m+1)t,\ q' - t],\\
&\mathbf{k} = [2, -1],\\
&L = \lceil mq'/(m+1) \rceil,\\
&U = q',\\
&q' = q - m.
\end{aligned}$$

Proof: In this case, the second interpreter is preferred. It is

$$R([1:m], n) = 2R([1:m], n-1) - R([1:m], n-(m+1))$$

with initial conditions $R([1:m],0) = 1$ and $R([1:m],i) = 2^{i-1}$ for $1 \leq i \leq m-1$. Thus, in order to calculate $R([1:m])$, n), we put $a_1 = 1, a_2 = m+1, \kappa(a_1) = 2, \kappa(a_2) = -1$ and apply corollary 3. It leads to (19). $\square$

Corollary 5. *The number of compositions of a positive integer n into positive integers greater than or equal to m satisfies*

$$R([m:n],n) = f_R(n-m) + \sum_{i=0}^{m-2} f_R(n-m-i-1)$$

in which

$$\begin{aligned}
f_R(q) &= \sum_{t=L}^{U} C(\mathbf{x^{(t)}}),\\
\mathbf{x^{(t)}} &= [-(m-1)q' + tm, q' - t],\\
L &= \lceil (m-1)q'/m \rceil,\\
U &= q',\\
q' &= q - (m-1).
\end{aligned}$$

Proof: It is known that the number of compositions of n into parts greater than or equal to m is equivalent to the number of compositions of $n-m$ into 1 and m. $\square$

The following corollary shows how we can calculate the number of compositions of n in which the integer m does not appear. This problem has already been studied in Chinn and Heubach (2003b), but no closed-form solution has yet been presented.

Corollary 6. *For an integer $m > 1$, the number of compositions of n where no part is m is obtained from*

$$R([1:n] \setminus \{m\}, n) = \left(2^{m-1} - 1\right) f_R(n) + \left(2^{m-2}\right) f_R(n-m) - \sum_{i=2}^{m-1} \left(2^{i-2}\right) f_R(n-i),$$

where

$$\begin{aligned}
f_R(q) &= \sum_{x_3=0}^{\lfloor q'/(m+1) \rfloor} \sum_{t=L}^{U} P_o(\mathbf{k}, \mathbf{x^{(t)}}) C(\mathbf{x^{(t)}}),\\
q' &= q - m,\\
\mathbf{x^{(t)}} &= [-(m-1)\widehat{q} + tm, \widehat{q} - t, x_3],\\
\widehat{q} &= q' - (m+1)x_3,\\
L &= \lceil (m-1)\widehat{q}/m \rceil,\\
U &= \widehat{q}.
\end{aligned}$$

Proof: For this case, the second interpreter is

$$\begin{aligned} R([1:n]\setminus\{m\},n) =& 2R([1:n]\setminus\{m\},n-1)- \\ & R([1:n]\setminus\{m\},n-m)+ \\ & R([1:n]\setminus\{m\},n-(m+1)). \end{aligned}$$

For $m>1$, we have $k_{a_1}=2, k_{a_2}=-1, k_{a_3}=1, a_1=1, a_2=m$, and $a_3=m+1$ with initial conditions $R([1:n]\setminus\{m\},0)=R([1:n]\setminus\{m\},1)=1$, $R([1:n]\setminus\{m\},n)=2^{n-1}$ for $1<n\leq m-1$, and $R([1:n]\setminus\{m\},m)=2^{m-1}-1$. It has previously been shown that $R([1:n]\setminus\{1\},n)$ is equal to the $(n-1)$th Fibonacci numberGrimaldi (2001). □

For $m=1$, the equation in the proof of Corollary 6 is converted to $R([2:n],n)=R([2:n],n-1)+R([2:n],n-2)$ with initial conditions $R([2:n],0)=1$ and $R([2:n],1)=0$. In this case, the solution is

$$R([2:n],n)=\sum_{t=\lceil\frac{n-1}{2}\rceil}^{n-1}\binom{t}{n-1-t}$$

for $n\geq 2$ which is equal to the $(n-1)$th Fibonacci number Grimaldi (2001); Hoggatt and Lind (1969).

Corollary 7. *The number of compositions of n in which no part is in $[m_1:m_2]$ satisfies*

$$\begin{aligned} R([1:n]\setminus[m_1:m_2],n) =& f_R(n-1)+R([1:m_1-1],m_2)f_R(n)+ \\ & \sum_{i=1}^{m_1-1}(2^{i-1})f_R(n-i-1)+ \\ & \sum_{i=m_1}^{m_2-1}R([1:m_1-1],i)f_R(n-i-1)- \\ & \sum_{i=m_2-m_1+1}^{m_1-1}(2^{i-1})f_R(n-i-m_1+m_2)+ \\ & \sum_{i=m_1}^{m_2-1}R([1:m_1-1],i)f_R(n-i-m_1+m_2), \end{aligned}$$

where

$$\begin{aligned} & f_R(q)=\sum_{x_3=0}^{\lfloor q'/(m_2+1)\rfloor}\sum_{t=L}^{U}P_o(\mathbf{k},\mathbf{x}^{(\mathbf{t})})C(\mathbf{x}^{(\mathbf{t})}), \\ & q'=q-m_2, \\ & L=\lceil(m_1-1)\widehat{q}/m\rceil, \\ & U=\widehat{q}, \\ & \mathbf{x}^{(\mathbf{t})}=[-(m_1-1)\widehat{q}+tm_1,\widehat{q}-t,x_3], \\ & \widehat{q}=q'-(m_2+1)x_3, \\ & \mathbf{k}=[2,-1,1]. \end{aligned}$$

Proof: In this case, the second interpreter is

$$\begin{aligned} R([1:n]\setminus[m_1:m_2],n) =& 2R([1:n-1]\setminus[m_1:m_2],n-1)- \\ & R([1:n-m_1]\setminus[m_1:m_2],n-m_1)+ \\ & R([1:n-(m_2+1)]\setminus[m_1:m_2],n-(m_2+1)). \end{aligned} \tag{20}$$

If $m_2 = m_1 + 1$, (20) is solved in the same way as the second interpreter of compositions without m where m is replaced with m_1. If $m_2 \geq m_1 + 2$ (which results in $m_2 - 2 > m_1 - 1$), we have $\mathbf{k} = [2, -1, 1]$ and $\mathbf{a} = [1, m_1, m_2]$. The initial conditions are also $R([1 : n] \setminus [m_1 : m_2], 0) = 1$, $R([1 : n] \setminus [m_1 : m_2], i) = 2^{i-1}$ for $1 \leq i \leq m_1 - 1$, and $R([1 : n] \setminus [m_1 : m_2], i) = R([1 : m - 1], i)$ for $m_1 \leq i \leq m_2$. $\square$

5.2 Relevant Sequences

This section is devoted to finding the general term of a number of known sequences by applying the technique developed in this paper. We begin with the m-Fibonacci sequence, which is known as the Fibonacci, the Tribonacci, and the Tetranacci sequence for $m = 2$, $m = 3$, and $m = 4$, respectively.

Corollary 8. *The nth element of the m-Fibonacci sequence defined by the LHRC $F^m(n) = \sum_{i=1}^{m} F^m(n-i)$ with the initial conditions $F^m(i) = 0$ for $0 \leq i \leq m-2$ and $F^m(m-1) = 1$ is*

$$F^m(n) = f_R(n) + f_R(n-m), \tag{21}$$

where

$$\begin{aligned} f_R(q) =& \sum_{t=L}^{U} P_o(\mathbf{k}, \mathbf{x^{(t)}})C(\mathbf{x^{(t)}}), \\ & \mathbf{x^{(t)}} = [-mq' + (m+1)t, q' - t], \\ & L = \lceil mq'/(m+1) \rceil, \\ & U = q', \\ & q' = q - m, \\ & \mathbf{k} = [2, -1]. \end{aligned}$$

Proof: The nth element of the sequence satisfies

$$F^m(n) - F^m(n-1) = F^m(n-1) - F^m(n-(m+1))$$

or, equivalently,

$$F^m(n) = 2F^m(n-1) - F^m(n-(m+1)). \tag{22}$$

We have solved two-term LHRCs such as (22) in Section 4. Thus, we can use (15) where the initial conditions are $f(i) = 0$ for $0 \leq i \leq m-2$ and $f(m-1) = f(m) = 1$. The result is then (21). $\square$

As an instance, $F^2(n) = f_R(n) + f_R(n-2)$ represents the nth element of the well-known Fibonacci sequence where $f_R(q) = \sum_{t=L}^{U} P_o(\mathbf{k}, \mathbf{x^{(t)}})C(\mathbf{x^{(t)}})$, $L = \lceil 2q'/3 \rceil$, $U = q'$, $\mathbf{x}^{(t)} = [-2q' + 3t, q' - t]$ and $q' = q - 2$.

Corollary 9. *The nth element of the Lucas sequence defined by*

$$L_u(n) = L_u(n-1) + L_u(n-2); \qquad L_u(0) = 1, L_u(1) = 3$$

is obtained from

$$L_u(n) = F_{n-1} + 3F_n = 3\sum_{t=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-t}{t} + \sum_{t=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-2-t}{t}, \tag{23}$$

where $F_n = F^2(n) = \sum_{t=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-t}{t}$ is the nth Fibonacci number.

Proof: The LHRC $f(n) = f(n-1) + f(n-2)$ is solved as follows.

$$f(n) = f(0)\sum_{t=\lceil\frac{n-2}{2}\rceil}^{n-2}\binom{t}{n-2-t} + f(1)\sum_{t=\lceil\frac{n-1}{2}\rceil}^{n-1}\binom{t}{n-1-t}.$$

Since the nth Fibonacci number is equal to $F_n = \sum_{t=\lceil\frac{n-1}{2}\rceil}^{n-1}\binom{t}{n-1-t}$, the above equation can be rewritten as

$$f(n) = F_{n-1}f(0) + F_nf(1). \tag{24}$$

As $f(0) = L_u(0) = 1$ and $f(1) = L_u(1) = 3$, (24) implies (23). □

Corollary 10. *The nth element of the Padovan sequence represented by the LHRC $P_n^d = P_{n-2}^d + P_{n-3}^d$ with the initial conditions $P_0^d = 0$, $P_1^d = 1$, and $P_2^d = 0$ is*

$$P_n^d = \sum_{t=\lceil\frac{(n-1)}{3}\rceil}^{\lfloor\frac{(n-1)}{2}\rfloor}\binom{t}{n-1-2t}.$$

Moreover, the nth element of the Perrin sequence represented by the LHRC $P_n^r = P_{n-2}^r + P_{n-3}^r$ with the initial conditions $P_0^r = 3$, $P_1^r = 0$, and $P_2^d = 2$ is equal to $2P_{n-1}^d + 3P_{n-2}^d$.

Proof: The LHRC $f(n) = f(n-2) + f(n-3)$ satisfied by the nth element of the Padovan and Perrin sequences is solved as follows:

$$\begin{aligned} f(n) =& f(0)\sum_{t=\lceil\frac{(n-3)}{3}\rceil}^{\lfloor\frac{(n-3)}{2}\rfloor}\binom{t}{n-3-2t} + f(1)\sum_{t=\lceil\frac{(n-1)}{3}\rceil}^{\lfloor\frac{(n-1)}{2}\rfloor}\binom{t}{n-1-2t} + \\ & f(2)\sum_{t=\lceil\frac{(n-2)}{3}\rceil}^{\lfloor\frac{(n-2)}{2}\rfloor}\binom{t}{n-2-2t} \end{aligned}$$

In the Padovan sequence, $f(0) = 0$, $f(1) = 1$ and $f(2) = 0$. Thus, the nth term of the Padovan sequence can be calculated as $P_n^d = \sum_{t=\lceil(n-1)/3\rceil}^{\lfloor(n-1)/2\rfloor}\binom{t}{n-1-2t}$. Furthermore, the solution to $f(n) = f(n-2) + f(n-3)$ is $f(n) = P_{n-2}^d f(0) + P_n^d f(1) + P_{n-1}^d f(2)$. By putting $f(0) = P_0^r = 3$, $f(1) = P_1^r = 0$ and $f(2) = P_2^r = 2$ in this equation, we obtain $P_n^r = 2P_{n-1}^d + 3P_{n-2}^d$. □

Corollary 11. *The nth element of the Pell sequence represented by the LHRC $P_n^l = 2P_{n-1}^l + P_{n-2}^l$ with the initial conditions $P_0^l = 0$ and $P_1^l = 1$ is*

$$P_n^l = \sum_{t=\lceil (n-1)/2 \rceil}^{n-1} 2^{-n+1+2t} \binom{t}{n-1-t}.$$

Moreover, the nth element of the Pell-Lucas sequence represented by the LHRC $P_n^L = 2P_{n-1}^L + P_{n-2}^L$ with the initial conditions $P_0^L = P_1^L = 2$ is equal to $2P_n^l + 2P_{n-1}^l$.

Proof: Both Pell and Pell-Lucas sequences satisfy $f(n) = 2f(n-1) + f(n-2)$ which is solved as follows:

$$\begin{aligned} f(n) = & f(0) \sum_{t=\lceil \frac{n-2}{2} \rceil}^{n-2} 2^{-n+2+2t} \binom{t}{n-2-t} + \\ & f(1) \sum_{t=\lceil \frac{n-1}{2} \rceil}^{n-1} 2^{-n+1+2t} \binom{t}{n-1-t}. \end{aligned}$$

The initial conditions are $P_0^l = 0$ and $P_1^l = 1$ for the Pell sequence and $P_0^L = P_1^L = 2$ for the Pell-Lucas sequence. □

Finally, we refer to a number of proven bijections between the elements of the sequences studied in this section and solutions to some cases of the S-restricted composition problem. They indeed indicate that the study of such sequences is quite helpful in finding the number of S-restricted compositions of a given integer.

1. The number of compositions of n into 2 and 3 is the $(n-2)$th term of the Padovan sequence.
2. The number of compositions of n in which no part is 2 is the $(n-2)$th term of the Padovan sequence.
3. The number of compositions of n with summands congruent to 2 modulo 3 is the $(n-4)$th term of the Padovan sequence.
4. The number of compositions of n with odd summands greater than 1 is the $(n-5)$th term of the Padovan sequence.
5. The number of compositions of n into odd parts is the nth term of the Fibonacci sequence.

6 Conclusion

We propose a technique for solving LHRCs through solving Diophantine equations. The technique can be applied to the LHRCs having a small number of terms. We also show that the S-restricted composition problem is reduced to the problem of solving particular LHRCs which we call interpreters. Some well-known cases of the S-restricted composition problem are studied and it is demonstrated that deciding on an appropriate interpreter, like the ones proposed in this paper, can lead to a closed-form solution to the problem. By using the same technique, we also find closed-form formulas for the general term of a number of well-known integer sequences. Deriving appropriate interpreters for the cases not studied in this paper deserves further research.

References

A. A. Actor. Infinite products, partition functions, and the Meinardus theorem. *Journal of Mathematical Physics*, 35(11):5749–5764, 1994.

K. Alladi and V. E. Hoggatt. Compositions with ones and twos. *The Fibonacci Quarterly*, 13(3):233–239, 1975.

C. Banderiera and P. Hitczenko. Enumeration and asymptotics of restricted compositions having the same number of parts. *Discrete Applied Mathematics*, 160(18):2542–2554, 2012.

J.-L. Baril and P.-T. Do. ECO-generation for some restricted classes of compositions. *The Computer Science Journal of Moldova*, 21(1):97–119, 2013.

E. A. Bender and E. R. Canfield. Locally restricted compositions I: Restricted adjacent differences. *The Electronic Journal of Combinatorics*, 12(1):1–27, 2005.

E. A. Bender and E. R. Canfield. Locally restricted compositions II: General restrictions and infinite matrices. *The Electronic Journal of Combinatorics*, 16(1):1–36, 2009.

E. A. Bender, E. R. Canfield, and Z. Gao. Locally restricted compositions IV: Nearly free large parts and gap-freeness. *The Electronic Journal of Combinatorics*, 19(4):1–29, 2012.

A. T. Benjamin, H. Derks, and J. J. Quinn. The combinatorialization of linear recurrences. *The Electronic Journal of Combinatorics*, 18(2):12–18, 2011.

R. Bisdorff and J.-L. Marichal. Counting non-isomorphic maximal independent sets of the n-cycle graph. *Journal of Integer Sequences*, 11(5):1–16, 2008.

A. Blecher, C. Brennan, and T. Mansour. Compositions as non-alternating sequences of partitions. *Applicable Analysis and Discrete Mathematics*, 7(2):211–224, 2013.

M. Bóna and A. Knopfmacher. On the probability that certain compositions have the same number of parts. *Annals of Combinatorics*, 14(3):291–306, 2010.

P. Chinn and S. Heubach. $(1, k)$-compositions. *Congressus Numerantium*, 164(1):183–194, 2003a.

P. Chinn and S. Heubach. Compositions of n with no occurrence of k. *Congressus Numerantium*, 164(1): 33–51, 2003b.

P. Chinn and S. Heubach. Integer sequences related to compositions without 2's. *Journal of Integer Sequences*, 6(2):1–12, 2003c.

P. Chinn and H. Niederhausen. Compositions into powers of 2. *Congressus Numerantium*, 168(1):215–220, 2004.

B. A. Cipra. An introduction to the Ising model. *American Mathematical Monthly*, 94(10):937–959, 1987.

S. Eger. Restricted weighted integer compositions and extended binomial coefficients. *Journal of Integer Sequences*, 16(1):1–25, 2013.

S. Eger. Some elementary congruences for the number of weighted integer compositions. *Journal of Integer Sequences*, 18(4):1–18, 2015.

S. Eger. Identities for partial Bell polynomials derived from identities for weighted integer compositions. *Aequationes Mathematicae*, 90(2):299–306, 2016.

E. Galois. Lettre de Galois à M. Auguste Chevalier. *Journal des Mathématiques Pures et Appliquées*, 11 (1):381–444, 1846.

I. Gonoskov. Closed-form solution of a general three-term recurrence relation. *Advances in Difference Equations*, 196(1):1–12, 2014.

R. P. Grimaldi. Compositions with odd summands. *Congressus Numerantium*, 142(1):113–127, 2000.

R. P. Grimaldi. Compositions without the summand 1. *Congressus Numerantium*, 152(1):33–43, 2001.

C. Hermite. Sur la résolution de l'équation du cinquième degré. *Comptes Rendus de l'Académie des Sciences Paris*, 46(1):508–515, 1858.

S. Heubach and S. Kitaev. Avoiding substrings in compositions. *Congressus Numerantium*, 202(1):87–95, 2010.

S. Heubach and T. Mansour. Compositions of n with parts in a set. *Congressus Numerantium*, 168(1): 127–143, 2004.

S. Heubach and T. Mansour. Avoiding patterns of length three in compositions and multiset permutations. *Advances in Applied Mathematics*, 36(2):156–174, 2006.

S. Heubach, S. Kitaev, and T. Mansour. Partially ordered patterns and compositions. *Pure Mathematics and Applications*, 17(1):1–12, 2007.

S. Heubach, T. Mansour, and A. Munagi. Avoiding permutation patterns of type $(2,1)$ in compositions. *Online Journal of Analytic Combinatorics*, 4(1):1–9, 2009.

V. E. Hoggatt and M. Bicknell. Palindromic compositions. *The Fibonacci Quarterly*, 13(4):350–356, 1975.

V. E. Hoggatt and D. A. Lind. Compositions and Fibonacci numbers. *The Fibonacci Quarterly*, 7(3): 253–266, 1969.

G. Jaklic, V. Vitrih, and E. Zagar. Compositions of n with parts in a set. *Bulletin of the Australian Mathematical Society*, 81(2):289–297, 2010.

M. Janjić. Binomial coefficients and enumeration of restricted words. *Journal of Integer Sequences*, 19 (7):1–18, 2016.

F. Klein. *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*. Teubner Verlag, 1884.

A. Knopfmacher and N. Robbins. On binary and Fibonacci compositions. *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae, Sectio Computatorica*, 22(1):193–206, 2003.

D. Krenn and S. Wagner. The number of compositions into powers of b. In *25th Int. Conf. Probabilistic, Combinatorial, and Asymptotic Methods for the Analysis of Algorithms (AofA'14)*, volume BA of *DMTCS-HAL Proceedings Series*, pages 241–252. 2014.

R. G. Kulkarni. Extracting the roots of septics by polynomial decomposition. *Lecturas Matemáticas*, 29 (1):5–12, 2008.

L. L. Liu. Positivity of three-term recurrence sequences. *The Electronic Journal of Combinatorics*, 17(1): 1–9, 2010.

L. A. Medina and A. Straub. On multiple and infinite log-concavity. *Annals of Combinatorics*, 20(1): 125–138, 2016.

J. Ratsaby. Estimate of the number of restricted integer-partitions. *Applicable Analysis and Discrete Mathematics*, 2(2):222–233, 2008.

C. D. Savage and H. S. Wilf. Pattern avoidance in compositions and multiset permutations. *Advances in Applied Mathematics*, 36(2):194–201, 2006.

C. Shapcott. C-color compositions and palindromes. *The Fibonacci Quarterly*, 50(4):297–303, 2012.

C. Shapcott. New bijections from n-color compositions. *Journal of Combinatorics*, 4(3):373–385, 2013.

C. Shapcott and E. Schmutz. Part products of s-restricted integer compositions. *Applicable Analysis and Discrete Mathematics*, 7(1):51–71, 2013.

A. Sills. Compositions, partitions, and Fibonacci numbers. *The Fibonacci Quarterly*, 49(4):348–354, 2011.

E. L. Whitney and W. Moser. Weighted compositions. *Canadian Mathematical Bulletin*, 4(1):39–43, 1961.

F. Wilcoxon. Individual comparisons by ranking methods. *Biometrics Bulletin*, 1(6):80–83, 1945.