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The width of Galton-Watson trees conditioned by the size[†]

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It is proved that the moments of the width of Galton-Watson trees of size n and with offspring variance σ^2 are asymptotically given by $(\sigma\sqrt{n})^p m_p$ where m_p are the moments of the maximum of the local time of a standard scaled Brownian excursion. This is done by combining a weak limit theorem and a tightness estimate. The method is quite general and we state some further applications.

Keywords: Galton-Watson branching process, width, moment convergence theorem, Brownian excursion

1 Introduction

In this paper we are considering rooted trees which are family trees of a Galton-Watson branching process conditioned to have total progeny n. These trees are also called simply generated trees (see [35]). Without loss of generality we may assume that the offspring distribution ξ is given by

$$\mathbf{P}\{\xi = k\} = \frac{\tau^k \varphi_k}{\varphi(\tau)}, \tag{1}$$

where $(\varphi_k; k \geq 0)$ is a sequence of non-negative numbers such that $\varphi_0 > 0$ and $\varphi(t) = \sum_{k\geq 0} \varphi_k t^k$ has a positive or infinite radius of convergence R and τ is an arbitrary positive number within the circle of convergence of $\varphi(t)$. These conditions in particular imply that all moments of ξ exist and that $\tau < R$. Due to conditioning on the total progeny and finiteness of moments it is no restriction if we confine ourselves to studying only the critical case, that is, $\mathbf{E}\,\xi = 1$ which equivalently means that τ satisfies $\tau\varphi'(\tau) = \varphi(\tau)$. The variance of ξ can also be expressed in terms of $\varphi(t)$ and is given by

$$\sigma^2 = \frac{\tau^2 \varphi''(\tau)}{\varphi(\tau)}. \tag{2}$$

Note that the offspring distribution (1) can be interpreted as assigning weights to all trees defined by

$$\omega(T) = \prod_{k\geq 0} \varphi_k^{n_k(T)}$$

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for a tree T having n nodes, n_k of which have out-degree k, $k \geq 0$. Denote by $|T|$ the number of nodes of such a tree and let a_n be the (weighted) number of all trees with n nodes, i.e.

$$a_n = \sum_{T:|T|=n} \omega(T).$$

Then the corresponding generating function $a(z) = \sum_{n\geq 0} a_n z^n$ satisfies the functional equation

$$a(z) = z\varphi(a(z)). \tag{3}$$

Denote by $(L_n(t), t \geq 0)$ the sequence of the generation sizes of a Galton-Watson tree the total progeny of which is n. For non-integer t we define $L_n(t)$ by linear interpolation:

$$L_n(t) = (\lfloor t \rfloor + 1 - t)L_n(\lfloor t \rfloor) + (t - \lfloor t \rfloor)L_n(\lfloor t \rfloor + 1), \quad t \geq 0.$$

We are interested in the width of such a tree which is defined by

$$w_n = \max_{t\geq 0} L_n(t).$$

This quantity attracted the interest of many authors. First, Odlyzko and Wilf [37] became interested in this tree parameter when studying the bandwidth

$$\beta(T) = \min_f \left(\max_{(u,v)\in E(T)} |f(u) - f(v)| \right)$$

of a tree T, where f is an assignment of distinct integers to the vertices of the tree. They showed for a tree with n vertices and height $h(T)$ and width $w(T)$ that

$$\frac{n-1}{2h(T)} \leq \beta(T) \leq 2w(T) - 1$$

and furthermore they showed that there exist positive constants c_1 and c_2 such that the estimate

$$c_1\sqrt{n} < \mathbf{E}\, w_n < c_2\sqrt{n \log n} \tag{4}$$

holds. The exact order of magnitude was left as an open problem. Aldous conjectured [1, Conj. 4] that L_n (suitably normalized) converges to Brownian excursion local time. This was first proved in [15], later by different methods by Kersting [29] and Pitman [38]. More precisely, set

$$l_n(t) = \frac{2}{\sigma\sqrt{n}} L_n\left(\frac{2t}{\sigma}\sqrt{n}\right)$$

and

$$l(t) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^1 I_{[t,t+\varepsilon]}(W(s))\, ds,$$

where $(W(s), 0 \leq s \leq 1)$ is the standard scaled Brownian excursion. $l(t)$ is the local time (at time 1 and level t) of the normalized Brownian excursion. Then the above described limit theorem reads as follows:

Theorem 1 ([15]). *Let $\varphi(t)$ be the GF of a family of random trees. Assume that $\varphi(t)$ has a positive or infinite radius of convergence R. Furthermore suppose that the equation $t\varphi'(t) = \varphi(t)$ has a minimal positive solution $\tau < R$. Then we have*

$$(l_n(t), t \geq 0) \xrightarrow{w} (l(t), t \geq 0)$$

in $C[0,\infty)$, as $n \to \infty$.

Partial results go back to [9, 22, 27, 34, 41]. The density of the finite dimensional distributions of l was computed in [25]. A consequence of Theorem 1 is the following result which was proved directly by Takács [40].

Corollary 1 ([15]). *Under the assumptions of Theorem 1 we have*

$$\sup_{t\geq 0} l_n(t) \xrightarrow{w} \sup_{t\geq 0} l(t).$$

Thus this suggests (but does not imply) $\sqrt{n}$ as correct order of magnitude in (4).

Note that the maximum of local time is well studied (cf. [28, 8, 18, 3, 34]). We have

$$\sup_{t\geq 0} l(t) \stackrel{d}{=} 2 \sup_{0\leq t\leq 1} W(t),$$

moreover it is theta-distributed, i.e.,

$$\mathbf{P}\left\{\sup_{0\leq t\leq 1} l(t) \leq x\right\} = 1 - 2\sum_{k\geq 1}(x^2k^2 - 1)e^{-x^2k^2/2}, \quad x > 0,$$

and

$$\mathbf{E}\left[\left(\sup_{t\geq 0} l(t)\right)^p\right] = 2^{p/2}p(p-1)\Gamma\left(\frac{p}{2}\right)\zeta(p).$$

The purpose of this paper is to show that, in addition to the weak limit theorem above, we have a moment convergence theorem of $\sup_{t\geq 0} l_n(t)$ to $\sup_{t\geq 0} l(t)$, too. We formulate it in terms of the width $w_n = \max_{t\geq 0} L_n(t) = (\sigma/2)\sqrt{n}\sup_{t\geq 0} l_n(t)$.

Theorem 2. *Suppose that there exists a minimal positive solution $\tau < R$ of $t\varphi'(t) = \varphi(t)$. Then the width w_n satisfies*

$$\mathbf{E}\left(w_n^p\right) = \sigma^p 2^{-p/2} p(p-1)\Gamma\left(\frac{p}{2}\right)\zeta(p)\cdot n^{p/2}\cdot(1+o(1))$$

as $n \to \infty$.

It should be further mentioned that Chassaing and Marckert [7] used the relation of parking functions and rooted trees as well as the strong convergence theorem of Komlos, Major and Tusnady [33] to derive tight bounds for the moments of the width for Cayley trees. They showed (here and throughout the whole paper, $a \ll b$ denotes $a \leq Cb$ for some positive constant C)

Theorem 3 ([7]). *If $\varphi(t) = e^t$ and $p \geq 1$, then*

$$\left|\mathbf{E}\left(\frac{w_n}{\sigma\sqrt{n}}\right)^p - \mathbf{E}\left(\frac{1}{2}\sup_{t\geq 0} l(t)\right)^p\right| = \left|\mathbf{E}\left(\frac{w_n}{\sigma\sqrt{n}}\right)^p - \mathbf{E}\left(\sup_{t\geq 0} W(t)\right)^p\right| \ll n^{-p/4}\log n.$$

Remark. In fact, Chassaing and Marckert [7] showed an even stronger result: In some probability space there exist a sequence of copies of w_n and a sequence of theta-distributed random variables D_n such that for any $p \ge 1$

$$\left\| \frac{2w_n}{\sigma\sqrt{n}} - D_n \right\|_p = O\left(n^{-1/4}\sqrt{\log n}\right)$$

where the O-constant depends on p.

Recently, Chassaing, Marckert, and Yor [6] have used Theorems 1 and 3 in conjunction with results of Aldous [1] to obtain a weak limit theorem (without moments) for the joint law of height and width of simply generated trees. (For binary trees they present an elementary proof, too.)

2 Plan of the Proof of Theorem 2

In view of Corollary 1 the result of Theorem 2 is not unexpected. Nevertheless, it does not follow directly from Corollary 1 since convergence of moments is not automatically transfered via weak convergence (from Theorem 1).

In order to prove Theorem 2 we actually use the result of Theorem 1, that is, the normalized profile of Galton-Watson trees converges weakly to Brownian excursion local time: $(l_n(t), t \ge 0) \xrightarrow{w} (l(t), t \ge 0)$. However, we need some additional considerations: In [17] (see also [14]) Drmota and Marckert introduced the notion of so-called *polynomial convergence* (that is inspired by the notion of uniform integrability). The key property for our purposes is the following one. It generalizes the results of [17] (see also [14, Theorem 3.7]) that only apply for processes with compact support.

Theorem 4. *Let x_n be a sequence of stochastic processes in $C[0,\infty)$ which converges weakly to x. Assume that for any choice of fixed positive integers p and d there exist positive constants c_0, c_1, c_2, c_3 such that*

$$\sup_{n\ge 0} \mathbf{E}\,|x_n(t)|^p \le c_0 e^{-c_1 t} \text{ for all } t \ge 0, \tag{5}$$

and

$$\sup_{n\ge 0} \mathbf{E}\,|x_n(t+s) - x_n(t)|^{2d} \le c_2 e^{-c_3 t} s^d \text{ for all } s,t \ge 0. \tag{6}$$

Then x_n is polynomially convergent to x, that is, for every continuous functional $F : C[0,\infty) \to \mathbb{R}$ of polynomial growth (i.e. $|F(y)| \ll (1+\|y\|_\infty)^r$ for some $r \ge 0$) we have

$$\lim_{n\to\infty} \mathbf{E}\,F(x_n) = \mathbf{E}\,F(x).$$

We will show that l_n satisfies the assumptions (5) and (6) of Theorem 4 and thus taking $F(x) = \|x\|_\infty^r$ yields immediately Theorem 2.

The next section is devoted to the proof of Theorem 4. In sections 4 and 5 we prove (5) and (6). Finally in section 6 we provide some further applications of Theorem 4.

3 Proof of Theorem 4

Let us start with the following two observations.

Lemma 1. *Suppose that x_n satisfies (5). Then for every $p \geq 0$ we have*

$$\mathbf{E} \sup_{j \in \mathbb{N}} |x_n(j)|^p \ll 1$$

uniformly for all n.

Proof. Since $\mathbf{E}\,|x_n(t)|^{p+1} \ll e^{-c_1 t}$, uniformly in n, we have

$$\begin{aligned}
\mathbf{P}\left\{\sup_{j \in \mathbf{N}} |x_n(j)| \geq A\right\} &\leq \sum_{j \geq 0} \mathbf{P}\left\{|x_n(j)| \geq A\right\} \\
&\leq \frac{1}{A^{p+1}} \sum_{j \geq 0} \mathbf{E}\,|x_n(j)|^{p+1} \text{ by Markov's inequality} \\
&\ll \frac{1}{A^{p+1}} \sum_{j \geq 0} e^{-c_1 j} \ll \frac{1}{A^{p+1}}
\end{aligned}$$

Thus it follows that

$$\mathbf{E}\left(\sup_{j \in \mathbf{N}} |x_n(j)|^p\right) \ll 1 + p \int_1^\infty A^{p-1} \frac{1}{A^{p+1}}\, dA \ll 1.$$

□

Lemma 2. *Suppose that x_n satisfies (6). Then, for fixed p we have*

$$\mathbf{E}\left(\sup_{|s-t| \leq \delta} |x_n(s) - x_n(t)|^p\right) \ll \delta^{p/2}.$$

uniformly for δ with $0 < \delta < 1$ and for all n.

Proof. First we prove that for every integer $d > 1$ there exists a constant $K > 0$ such that for $\varepsilon > 0$ and $0 < \delta < 1$

$$\mathbf{P}\left\{\sup_{|s-t| \leq \delta} |x_n(s) - x_n(t)| \geq \varepsilon\right\} \leq K \frac{\delta^{d-1}}{\varepsilon^{2d}}. \tag{7}$$

Arguing as in [4, pp. 95] guarantees that there exists a constant $K_1 > 0$ such that for all $m \geq 0$

$$\mathbf{P}\left\{\sup_{|s-t| \leq \delta, m \leq s,t \leq m+2} |x_n(s) - x_n(t)| \geq \varepsilon\right\} \leq K_1 e^{-c_5 m} \frac{\delta^{d-1}}{\varepsilon^{2d}}.$$

Thus

$$\mathbf{P}\left\{\sup_{|s-t| \leq \delta} |x_n(s) - x_n(t)| \geq \varepsilon\right\} \leq \sum_{m=0}^{\infty} K_1 e^{-c_5 m} \frac{\delta^{d-1}}{\varepsilon^{2d}} \leq K \frac{\delta^{d-1}}{\varepsilon^{2d}}$$

for some constant $K > 0$.

Set

$$Z = \sup_{|s-t|\le\delta} |x_n(s) - x_n(t)|.$$

Then by applying (7) it follows that (if $2d \ge p+1$)

$$\begin{aligned}
\mathbf{E}\,Z^p &= p\int_0^\infty z^{p-1}\mathbf{P}[Z>z]\,dz \\
&= p\int_0^{(K\delta)^{(d-1)/d}} z^{p-1}\mathbf{P}[Z>z]\,dz + p\int_{(K\delta)^{(d-1)/d}}^\infty z^{p-1}\mathbf{P}[Z>z]\,dz \\
&\le (K\delta)^{p(d-1)/d} + pK\delta^{d-1}\int_{(K\delta)^{(d-1)/d}}^\infty z^{p-1-2d}\,dz \\
&\ll \delta^{p(d-1)/d} \le \delta^{p/2},
\end{aligned}$$

which proves the Lemma. □

The proof of Theorem 4 is now an easy task. Note that the results of Lemma 1 and 2 in conjunction with the triangular inequality imply

$$\sup_{n\ge0}\mathbf{E}\left(\sup_{t\ge0}|x_n(t)|^r\right) < \infty \text{ for all } r\ge 0.$$

Thus, if F is a continuous functional of polynomial growth we have for any $\varepsilon > 0$

$$\sup_{n\ge0}\mathbf{E}\,|F(x_n)|^{1+\varepsilon} < \infty.$$

By continuity of F we also obtain $F(x_n) \xrightarrow{w} F(x)$ and finally, by Billingsley [5, p. 338] it directly follows that

$$\lim_{n\to\infty}\mathbf{E}\,F(x_n) = \mathbf{E}\,F(x)$$

as desired. □

4 Moments for the Profile of Galton-Watson Trees

We start with a lemma on the growth of coefficients of powers of certain generating functions.

Lemma 3. *Let* $z_0 \ne 0$ *and* $\Delta = \{z : |z| < z_0 + \eta,\ |\arg(z - z_0)| > \vartheta\}$, *where* $\eta > 0$ *and* $0 < \vartheta < \pi/2$. *Suppose that* $f(z)$ *and* $g(z)$ *are analytic functions in* Δ *which satisfy*

$$\begin{aligned}
|f(z)| &\le \exp\left(-C\sqrt{\left|1-\frac{z}{z_0}\right|}\right), \quad z\in\Delta, \\
g(z) &= 1 - D\sqrt{1-\frac{z}{z_0}} + O\left(\left|1-\frac{z}{z_0}\right|\right), \quad z\in\Delta,
\end{aligned}$$

for some positive constants C, D. Then for any fixed ℓ there exists a constant $C' > 0$ such that

$$[z^n]\frac{f(z)^r}{(1-g(z))^\ell} = O\left(e^{-C'r/\sqrt{n}}n^{(\ell-2)/2}\right)$$

uniformly for all $r, n \geq 0$ (where $[z^n]F(z)$ denotes the coefficient of z^n of the function $F(z)$).

Proof. The only difference to [23, Lemma 3.5] is the factor $1/(1-g(z))^\ell$, but since its behavior in Δ is known and [20, Theorem 3] is applicable, the proof is analogous to that of [23, Lemma 3.5]. $\square$

By means of this lemma we can show

Lemma 4. *For every fixed integer $p > 0$ there exist positive constants c_0 and c_1 such that*

$$\sup_{n\geq 0} \mathbf{E}\, l_n(t)^p \leq c_0 e^{-c_1 t} \tag{8}$$

for all $t \geq 0$.

Proof. For technical simplicity we assume that $g = \gcd\{i \geq 1 : \varphi_i > 0\} = 1$. This assumption ensures that the tree function $a(z)$ defined by (3) has only one singularity $z_0 = 1/\varphi'(\tau)$ on the circle of convergence. If $g = \gcd\{i \geq 0 : \varphi_i > 0\} > 1$ then we can use the substitution $x = z^{1/g}$ to get $a(z) = xb(x)$ where $b(x)$ is analytic with only one singularity on the circle of convergence. Thus this case reduces to the case $g = 1$. The other possibility is to deal with the g singularities $z_0 e^{2\pi i j/g}$, $j = 0, 1, \ldots, g-1$, on the circle of convergence and add all contributions.

In particular, it is also well known that (if $g = 1$) $a(z)$ admits a representation of the following kind

$$a(z) = \tau - \frac{\tau\sqrt{2}}{\sigma}\sqrt{1-\frac{z}{z_0}} + O\left(\left|1-\frac{z}{z_0}\right|\right), \tag{9}$$

that is valid for $|z| < z_0 + \eta$ and $\arg(z - z_0) \neq 0$, where $\eta > 0$ is suitably small, compare with [35] and [13].

In what follows we will need the local expansion of $\alpha(z) = z\varphi'(a(z))$. From (9) we immediately get

$$\alpha(z) = 1 - \sigma\sqrt{2}\sqrt{1-\frac{z}{z_0}} + O\left(\left|1-\frac{z}{z_0}\right|\right) \tag{10}$$

for $|z| < z_0 + \eta$ and $\arg(z - z_0) \neq 0$.

Due to (10) there exists a constant $C > 0$ such that $|\alpha(z)| \leq \exp\left(-C\sqrt{|1-z/z_0|}\right)$ for $z \in \Delta$ (with Δ from Lemma 3). Furthermore, it follows that

$$\sup_{z\in\Delta} |\alpha(z)| = 1, \tag{11}$$

where we have to choose $\eta > 0$ and $0 < \vartheta < \pi/2$ in a proper way. First, since the power series of $\alpha(z)$ has only positive coefficients, we have $\max_{|z|\leq z_0} |\alpha(z)| = 1$. If we assume that $d = \gcd\{i \geq 1 : \varphi_i > 0\} = 1$ it also follows that

$$\max_{|z|\leq z_0, |z-z_0|\geq\varepsilon} |\alpha(z)| < 1$$

for every $\varepsilon > 0$. Now, in the vicinity of the singularity z_0, that is, for $|z - z_0| < \varepsilon$ we can again use (10) and get for $z = z_0(1 + te^{i\theta})$

$$\left|\alpha\left(1 + te^{i\theta}\right)\right| = \left|1 - \sigma\sqrt{2t}e^{\pm i(\pi-\theta)/2} + O(t)\right|, \tag{12}$$

where $\theta > \pi/2$. Hence we have $|\alpha(z)| \le 1$ for $|z - z_0| \le \varepsilon$ and $|\arg(z - z_0)| > \theta$. Finally, for $|z| \le z_0 + \eta$ and $|z - z_0| \ge \varepsilon$ we obtain the same inequality from (12) by a continuity argument (for some sufficiently small $\eta > 0$). This proves (11).

Now observe that by substituting $r = \lfloor t\sqrt{n} \rfloor$ in (8) we get

$$\mathbf{E}\, L_n(r)^p \le c_0 e^{-c_1 r/\sqrt{n}} n^{p/2}. \tag{13}$$

Furthermore note that it suffices to show (13) for the pth factorial moment

$$\mathbf{E}\,[L_n(r)]_p = \mathbf{E}\, L_n(r)(L_n(r) - 1)\cdots(L_n(r) - p + 1)$$

instead of the pth moment, which we can easily express in terms of the proper coefficient of a generating function. Indeed we have

$$\mathbf{E}\,[L_n(r)]_p = \frac{1}{a_n}[z^n]\,\left(\frac{\partial}{\partial u}\right)^p y_r(z, ua(z))\bigg|_{u=a(z)},$$

where

$$\begin{aligned} y_0(z,u) &= u \\ y_{i+1}(z,u) &= z\varphi(y_i(z,u)), \quad i \ge 0. \end{aligned} \tag{14}$$

In order to evaluate this coefficient we use Lemma 3 which translates the local behavior of the function near its singularity into an asymptotic estimate for the coefficients.

By [24, p. 287, equ. (22)] we have

$$\left(\frac{\partial}{\partial u}\right)^p y_r(z, ua(z))\bigg|_{u=a(z)} = O\left(a(z)^p |\alpha(z)^r| \left|\frac{1 - \alpha(z)^r}{1 - \alpha(z)}\right|^{p-1}\right). \tag{15}$$

From (11) we get

$$\max_{z\in\Delta}\left|\frac{1 - \alpha(z)^r}{1 - \alpha(z)}\right| \le r. \tag{16}$$

Moreover $a(z)^p$ behaves like a constant near the singularity and $\alpha(z)^r$ meets the condition in Lemma 3. Hence the last factor in (15) is bounded by r^{p-1} and hence contributes a factor $n^{(p-1)/2}$ to the order of magnitude of the pth factorial moment $\mathbf{E}\,[L_n(r)]_p$. Applying Lemma 3, which yields $\exp(-c_1 r/\sqrt{n})$, and normalizing by $a_n \sim \tau/\sigma z_0^n \sqrt{2\pi n^3}$ we get the desired result. □

5 Quantitative Tightness Estimates

With help of Lemma 3 we can prove the following quantitative *tightness estimate*.

Lemma 5. *For every fixed positive integer d there exist constants c_2, c_3 such that for every $s, t > 0$*

$$\mathbf{E}\,|l_n(t+s) - l_n(t)|^{2d} \le c_2 e^{-c_3 t} s^d. \tag{17}$$

Proof(Sketch). Observe that we can rewrite (17) as

$$\mathbf{E}\;|L_n(r) - L_n(r+h)|^{2d} \le c_2 e^{-c_3 r/\sqrt{n}} h^d n^{d/2} \tag{18}$$

which is quite similar to [15, Theorem 6.1]. From [15] it follows that

$$\mathbf{E}\;|L_n(r) - L_n(r+h)|^{2d} = \frac{1}{a_n}[z^n]H_{r,h}(z),$$

in which

$$H_{r,h}(z) = \left(u\frac{\partial}{\partial u}\right)^{2d} y_r(z, u y_h(z, u^{-1}a(z)))\Big|_{u=1}$$

and $y(z,u)$ is given by (14).

Evaluation of this coefficient is again done by Lemma 3. By [15, Proposition 6.1] it is easy to show that

$$H_{r,h}(z) = \alpha(z)^r \sum_{j=0}^{d} G_{j,rh}(z) \frac{(1-\alpha(z)^h)^j}{(1-\alpha(z))^{d-1+j}}, \tag{19}$$

where $G_{j,rh}(z)$ satisfy

$$\max_{z\in\Delta} |G_{j,rh}(z)| = O(1).$$

Eventually, an application of (16), with h instead of r, and Lemma 3 to (19) yields

$$[z^n]H_{r,h}(z) = O\left(\frac{h^d n^{(d-3)/2}}{z_0^n}\right)$$

and, thus, by $a_n \sim \tau/\sigma z_0^n \sqrt{2\pi n^3}$ the proof is complete. □

6 Extensions

6.1 Nodes of given degree

In [12] the number of nodes with fixed degree d in layers of random trees was investigated. In this case also limit theorems like Theorem 1 and Corollary 1 hold. In fact, we have

Theorem 5. *Let $L_n^{(d)}(k)$ denote the number of nodes with degree d in layer k in a random tree of total progeny n. Furthermore, set for any $t \ge 0$*

$$l_n^{(d)}(t) = \frac{2}{\sigma c_d \sqrt{n}} L_n^{(d)}\left(\frac{2t}{\sigma}\sqrt{n}\right),$$

where $c_d = \varphi_{d-1}\tau^{d-1}/\varphi(\tau)$. Then we have

1.
$$l_n^{(d)} \xrightarrow{w} l \text{ and } \sup_{t\geq 0} l_n^{(d)}(t) \xrightarrow{w} \sup_{t\geq 0} l(t)$$

2.
$$\mathbf{E}\left(\left(w_n^{(d)}\right)^p\right) = \mathbf{E}\left(\sup_{t\geq 0} l(t)\right)^p (1+o(1)),$$

where $w_n^{(d)} = \max_{k\geq 0} l_n^{(d)}(k)$.

Proof(Sketch).
Part 1 was proved in [12]. The proof of part 2 runs similarly to the proof of Theorem 2. The only crucial point is to get estimates as in Lemma 4 and Lemma 5, namely

$$\mathbf{E}\, L_n^{(d)}(r)^p \leq c_1 e^{-c_2 r/\sqrt{n}} n^{p/2}$$

and
$$\mathbf{E}\left|L_n^{(d)}(r) - L_n^{(d)}(r+h)\right|^{2d} \leq c_1 e^{-c_2 r/\sqrt{n}} h^d n^{d/2}. \tag{20}$$

Both inequalities can be proved in a similar manner, so let us look at the second one (the first is the easier one). The results in [12] imply

$$\mathbf{E}\left|L_n^{(d)}(r) - L_n^{(d)}(r+h)\right|^{2d} = \frac{2}{\sigma a_n}[z^n] H^{(d)}_{2r/\sigma, 2h/\sigma}(z)$$

with

$$\begin{aligned} H_{r,h}^{(d)}(z) = \left(u\frac{\partial}{\partial u}\right)^{2d} & y_r(z, z(u-1)\varphi_{d-1} y_{h-1}(z, z(u^{-1}-1)\varphi_{d-1} a(z)^{d-1} + a(z)) \\ & y_h(z, z(u^{-1}-1)\varphi_{d-1} a(z)^{d-1} + a(z)))\Big|_{u=1} \end{aligned}$$

and since the right-hand side of this equation can be expressed in a form similar to (19), we can easily prove (20). □

6.2 Strata of random mappings

A random mapping of size n is an element of the set F_n of all mappings of a set with n elements into itself, where F_n is equipped with the uniform distribution. These mappings can be represented by functional digraphs consisting of components which are cycles of trees, i.e., each component of this graph contains exactly one cycle and each vertex in this cycle is the root of a tree in which each edge is directed towards the root.

The set of points in distance r from a cycle is called the rth stratum of a random mapping. This parameter was previously studied in [2, 11, 16, 36, 39]. For general results on random mappings and literature see [32, 21]. Let $M_n(r)$ denote the number of nodes in the rth stratum of a random mapping of size n. Then in [16] we proved

Theorem 6. *Let $B(t)$ denote reflecting Brownian bridge, i.e., a process on the interval $[0,1]$ which is identical in law to $|\overline{W}(s)-s\overline{W}(1)|$ ($\overline{W}(t)$ is the standard Brownian motion), and $l^{(B)}(t)$ its local time, i.e.,*

$$l^{(B)}(t)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^1 I_{[t,t+\varepsilon]}(B(s))\,ds.$$

Then we have

$$(m_n(t),t\geq 0)=\left(\frac{2}{\sqrt{n}}M_n\left(2t\sqrt{n}\right),t\geq 0\right)\xrightarrow{w}(l^{(B)}(t),t\geq 0)$$

in $C[0,\infty)$, as $n\to\infty$. Thus we also have

$$\sup_{t\geq 0}m_n(t)\xrightarrow{w}\sup_{t\geq 0}l^{(B)}(t). \tag{21}$$

Here again the corresponding moment convergence theorem is not a consequence of (21). However, as before we can show

Theorem 7. *We have*

$$\mathbf{E}\left(\left(\sup_{t\geq 0}m_n(t)\right)^p\right)=\mathbf{E}\left(\sup_{t\geq 0}l^{(B)}(t)\right)^p(1+o(1)). \tag{22}$$

Proof(Sketch). Again the crucial point is to get proper estimates. From [16] it is an easy exercise to get

$$\mathbf{E}\,|M_n(r)-M_n(r+h)|^{2d}=\frac{2n!}{n^n}[z^n]H_{2r,2h}(z),$$

in which

$$H_{r,h}(z)=\left(u\frac{\partial}{\partial u}\right)^{2d}\frac{1}{1-y_r(z,uy_h(z,u^{-1}a(z)))}\Bigg|_{u=1}.$$

This function can be written in a form similar to (19) and thus we can easily prove

$$\mathbf{E}\,|M_n(r)-M_n(r+h)|^{2d}\leq c_1e^{-c_2r/\sqrt{n}}h^dn^{d/2}$$

and then (22). The corresponding bound for the moments, obtained in the same way, carries out even easier. □

6.3 Height of random trees

The same method can be used to re-derive the analogue for the height h_n of simply generated trees (see Flajolet and Odlyzko [19]).

Theorem 8. *Suppose that there exists a minimal positive solution $\tau<R$ of $t\varphi'(t)=\varphi(t)$. Then*

$$\mathbf{E}\,(h_n^p)=\left(\frac{\sqrt{2n}}{\sigma}\right)^p p(p-1)\Gamma\left(\frac{p}{2}\right)\zeta(p)(1+o(1))$$

as $n\to\infty$.

h_n is equal to the maximum of the traversal process $T_n(r)$, defined to be the distance between the root and the rth node during preorder traversal of the tree. Obviously, the same holds when we only traverse leaves (call the corresponding process $\hat{T}_n(r)$). It is well known (see [1]) that

$$(X_n(t), t \geq 0) = \left(\frac{1}{\sqrt{n}} T_n(2tn) t \geq 0 \right) \xrightarrow{w} \left(\frac{2}{\sigma} W(t) t \geq 0 \right)$$

The height of leaves was investigated by several authors (see [30, 31, 26, 10, 23]. Here a similar limit theorem holds: With $\hat{X}_n(t) = \hat{T}_n(tn)/\sqrt{n}$ we have (see [23])

$$\left(\hat{X}_n \left(\frac{\varphi_0}{\varphi(\tau)} t \right) t \geq 0 \right) \xrightarrow{w} \left(\frac{2}{\sigma} W(t) t \geq 0 \right).$$

In addition, in [23] the tightness estimate

$$P\{|\hat{X}_n(s) - \hat{X}_n(t)| \geq \varepsilon\} \leq C \frac{1}{\varepsilon^4 |s-t|} \exp\left(-D \frac{\varepsilon}{\sqrt{|s-t|}} \right)$$

for some positive constants C and D was shown. This can be used to derive moment estimates like in Lemma 5 and then one proceeds as in the previous section to re-derive Flajolet and Odlyzko's [19] result on the moments of the height.

Finally, we want to mention that it is also possible to obtain the moments of the height of a random mapping (this was done by Flajolet and Odlyzko [21]) by our method. One has to use the weak limit theorem by Aldous and Pitman [2] and derive a tightness estimate in a similar fashion as has been done in [16].

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