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Reconstruction Thresholds on Regular Trees

James B. Martin

CNRS, LIAFA, Univ. Paris 7, 2 pl. Jussieu, 75251 Paris 05, France.
martin@liafa.jussieu.fr

We consider the model of broadcasting on a tree, with binary state space, on the infinite rooted tree $T^k$ in which each node has $k$ children. The root of the tree takes a random value 0 or 1, and then each node passes a value independently to each of its children according to a $2 \times 2$ transition matrix $P$. We say that reconstruction is possible if the values at the $d$th level of the tree contain non-vanishing information about the value at the root as $d \to \infty$. Extending a method of Brightwell and Winkler, we obtain new conditions under which reconstruction is impossible, both in the general case and in the special case $p_{11} = 0$. The latter case is closely related to the hard-core model from statistical physics; a corollary of our results is that, for the hard-core model on the $(k+1)$-regular tree with activity $\lambda = 1$, the unique simple invariant Gibbs measure is extremal in the set of Gibbs measures, for any $k \geq 2$.

Keywords: broadcasting on a tree, reconstruction, hard-core model, Gibbs measure, extremality

1 Introduction

1.1 Broadcasting on a tree

We consider a model of a broadcasting on the rooted tree $T^k$, in which every node has $k$ children.

Let $P = \{p_{ij}, i, j = 0, 1\}$ be a $2 \times 2$ stochastic matrix, which we regard as a transition matrix on the set $\{0, 1\}$. Each node $u \in T^k$ will carry a value $\phi(u) \in \{0, 1\}$, generated as follows. The root takes value 0 with probability $\pi_0 = p_{00}/(p_{00} + p_{01})$ and value 1 with probability $\pi_1 = 1 - \pi_0$. Thereafter the configuration on $T^k$ is generated recursively; if a node has value $i \in \{0, 1\}$, each of its $k$ children takes the value 0 with probability $p_{i0}$ and the value 1 with probability $p_{i1}$, all choices being made independently.

We write $\phi = \{\phi(u), u \in T^k\}$ for a configuration on the whole tree, and denote by $\mu$ the probability measure on $\{0, 1\}^{T^k}$ resulting from this broadcasting construction.

For a node $u \in T^k$, let $T^k(u)$ be the subtree consisting of $u$ and all its descendants. By the choice of $\pi_0$, we have a translation invariance property for $\mu$: namely that $\mu(\phi(u) = 0) = \pi_0$ for every $u \in T^k$, and so for any $u, v \in T^k$, the configurations on $T^k(u)$ and $T^k(v)$ have the same distribution, under a natural mapping between the subtrees $T^k(u)$ and $T^k(v)$.

We are interested in the following question of reconstruction: for $d \geq 1$, how much information about the value at node $u$ is given by the values of the $d$th generation of its descendants?

Questions of this sort arise in several contexts – for example genetics, communication theory and statistical physics – and have been quite widely studied in the last few years; see Mossel [Mos03] for a survey, and [EKPS00, BRZ95, Iof96, KMP01, Mos01, MP03, BW03, JM03] for a variety of approaches to this
sort of model (which can of course be considerably generalised from our particular setting of a binary state space and a regular tree).

The question above can be made precise in several (often equivalent) ways. We use the following formulation.

Let $\mathbb{W}_d(u)$ be the set of descendants of $u$ at distance exactly $d$ from $u$. For a set $S \subseteq \mathbb{T}^k$, write $\sigma(S)$ for the $\sigma$-algebra of events which depend only on the values $\{\phi(u), u \in S\}$.

Define the random variable

$$A(d, u) = \mu(\phi(u) = 0 | \sigma(\bigcup_{d' = d}^{\infty} \mathbb{W}_{d'}(u)))$$

that is, the conditional probability that the value at $u$ is 0, given only the information from the $d$th generation of its descendants.

From the independence structure given by the broadcasting construction, additional knowledge of any information from nodes beyond the $d$th generation does not change the conditional distribution of the value of $u$; that is,

$$A(d, u) = \mu(\phi(u) = 0 | \sigma(\bigcup_{d' = d}^{\infty} \mathbb{W}_{d'}(u)))$$

Of course, if $d_1 > d_2$, then

$$\sigma(\bigcup_{d' = d_1}^{\infty} \mathbb{W}_{d'}(u)) \subseteq \sigma(\bigcup_{d' = d_2}^{\infty} \mathbb{W}_{d'}(u)),$$

so by the backwards martingale convergence theorem (see e.g. Section 14.4 of [Wil91]), we have that $A(d, u) \to A(u)$ a.s. as $d \to \infty$, where

$$A(u) = \mu(\phi(u) = 0 | T(u))$$

here $T(u)$ is the tail $\sigma$-algebra of descendants of $u$, defined by

$$T(u) = \bigcap_{d = 1}^{\infty} \sigma(\bigcup_{d' = d}^{\infty} \mathbb{W}_{d'}(u)).$$

By the translation invariance property above, the random variable $A(u)$ has the same distribution for all $u \in \mathbb{T}^k$.

**Definition:** We say that reconstruction is impossible (for a given $P$ and $k$) if the random variable $A(u)$ is almost surely constant, and otherwise that reconstruction is possible.

A complete answer to the question of when reconstruction is possible is currently only known for the case where $P$ is symmetric. Then let $p_{00} = p_{11} = 1 - \varepsilon$; reconstruction is possible if and only if

$$k(1 - 2\varepsilon)^2 > 1$$

(see for example [BRZ95, EKPS00, Iof96]).

In general, however, there are gaps between the best known necessary and sufficient conditions for reconstruction to be possible. In this paper we give new conditions on $P$ under which we show that reconstruction is impossible.

In Proposition 4.1 of [MP03], Mossel and Peres show that reconstruction is impossible whenever

$$\frac{(p_{00} - p_{10})^2}{\min\{p_{00} + p_{10}, p_{01} + p_{11}\}} \leq \frac{1}{k}.$$  \hspace{1cm} (1)

We improve the bound to give the following condition:
Theorem 1. Reconstruction is impossible whenever

\[ \left( \sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}} \right)^2 \leq \frac{1}{k}. \]  

A calculation (see Section 4) shows that the LHS of (2) is always less than or equal to that of (1), with equality in the following special cases: (i) \( P \) is symmetric; (ii) \( p_{ij} = 0 \) for some \( i, j \); (iii) \( p_{00} = p_{10}, p_{01} = p_{11} \). Note that for symmetric \( P \), (2) becomes the condition that \( k(1 - 2\epsilon)^2 \leq 1 \), and our proof of Theorem 1 gives another proof that reconstruction is impossible under this condition.

We then focus on the special case where \( p_{11} = 0 \) (of course, the case \( p_{00} = 0 \) is analogous). This case is closely related to the hard-core model from statistical physics, and has been recently studied by Brightwell and Winkler [BW03] and Rozikov and Suhov [RS03]. Certain specific properties in this case allow a more sophisticated argument which gives a much better condition than is obtained by putting \( p_{11} = 0 \) in Theorem 1.

1.2 Hard-core model

In this section we state our result for the case \( p_{11} = 0 \) and explain the correspondence with the hard-core model on a regular tree.

Following [BW03], we parametrise \( P \) by the quantity \( w > 0 \), setting

\[ P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+w} & \frac{w}{1+w} \\ 0 & 0 \end{pmatrix}, \]

or equivalently by the quantity \( \lambda = w(1 + w)^k > 0 \), whose significance we explain later; note that the correspondence between \( \lambda \) and \( w \) is one-to-one and monotonic.

Let \( \lambda_c = \lambda_c(k) \) be the infimum of the set of \( \lambda \) such that reconstruction is possible. If follows from Proposition 12 of [Mos01] that in fact reconstruction is possible for any \( \lambda > \lambda_c \) (so that \( \lambda_c \) is also the supremum of the set of \( \lambda \) such that reconstruction is impossible).

Brightwell and Winkler [BW03] show that, as \( k \to \infty \),

\[ \frac{1 + o(1)}{\ln k} \leq \lambda_c(k) \leq (\ln k)^2 \left( 1 + o(1) \right). \]

We improve the lower bound to give the following:

Theorem 2. \( \lambda_c(k) > e - 1 \) for all \( k \).

(For the equivalent threshold value \( w_c \) with \( w_c(1 - w_c)^k = \lambda_c \), one can deduce that \( w_c(k) > (\ln k - \ln \ln k) / k \) for all \( k \).)

We will now describe the correspondence between the broadcasting model and the hard-core model, and explain (without proofs) the significance of Theorem 2 for the hard-core model on the \( (k+1) \)-regular tree. For more details on the correspondence between the two models, see also [BW03] and its references.

We denote the \( (k+1) \)-regular tree by \( \bar{T}^k \). We can still regard \( \bar{T}^k \) as a rooted tree, in which the root node has \( k+1 \) children and every other node has \( k \) children. We can then carry out the broadcasting construction on \( \bar{T}^k \) in exactly the same way as we did on \( T^k \); now the root has \( k+1 \) rather than \( k \) children, but the values at these \( k+1 \) children are chosen i.i.d. according to the value at the root and the transition matrix \( P \) just as before. We will write \( \bar{\mu} \) for the probability measure on \( \{0, 1\}^{\bar{T}^k} \) resulting from this construction.
The independence structure of the random walk implies that the measure \( \tilde{\mu} \) is simple, by which we mean that, for any \( u \), the configurations

\[ \{ \phi(v), v \in C_1(u) \}, \ldots, \{ \phi(v), v \in C_{k+1}(u) \} \]

are mutually independent given \( \phi(u) \), where the \( C_i \) are the connected components of \( \tilde{T}^k \setminus \{ u \} \). Although we have defined \( \tilde{\mu} \) in an asymmetric way, it’s also the case that it is invariant, in the sense that it is preserved by any automorphism of \( \tilde{T}^k \). In particular, the choice of the root is not important.

To introduce the hard-core model, we first consider the case of a finite graph with node-set \( S \) (and some neighbour relation).

We can identify a configuration \( \phi \) with the subset \( I_{\phi} \): \( u \in S : \phi(u) = 1 \) of \( S \).

A set \( I \subseteq S \) is called an independent set if no two neighbours in the graph are both members of \( I \).

The hard-core measure on \( S \) with activity \( \lambda > 0 \) is the probability measure \( \nu \) on \( \{0,1\}^S \) such that

\[ \nu(I_{\phi} \mid \phi(u) = 1 \text{ for all } v \neq u) = \frac{\lambda}{1 + \lambda} 1 \{ I_{\phi_0} \cup \{ u \} \text{ is independent} \} \quad (5) \]

The condition (5) makes sense equally when \( S \) is infinite, except that (since conditional probabilities are only well defined up to almost sure equality) we should now only demand the condition holds for \( \nu \)-almost all \( \phi_0 \). Putting \( S = \tilde{T}^k \), we say that a probability measure \( \nu \) satisfying (5) (for all \( u \in \tilde{T}^k \) and \( \nu \)-almost all \( \phi_0 \)) is a Gibbs measure for the hard-core model on \( \tilde{T}^k \) with activity \( \tilde{\lambda} \).

It is quite straightforward to show that the measure \( \tilde{\mu} \) defined above by the broadcasting construction with \( P \) as in (3) is a Gibbs measure for the hard-core model with activity \( \lambda \). However, now that the state space is infinite, it’s no longer the case that such a measure need be unique. In fact, there is a critical point \( \lambda_c' = \lambda_c'(k) = k^2 / (k - 1)(k+1) \) (identified by Kelly [Kel85]); for \( \lambda \leq \lambda_c' \), \( \tilde{\mu} \) is the only Gibbs measure, whereas for \( \lambda > \lambda_c' \), there are others. Nevertheless, for any \( \lambda \), the measure \( \tilde{\mu} \) is the only simple invariant Gibbs measure; (this can be deduced, for example, from Theorem 4.1 of [Zac83] – see also Section 5 of that paper for relevant discussion).

The set of Gibbs measures forms a simplex; that is, any mixture of Gibbs measures is also a Gibbs measure, and in particular there is a set of extremal Gibbs measures such that every Gibbs measure is expressible in a unique way as a mixture of extremal measures. For \( \lambda > \lambda_c' \), we can therefore ask whether the measure \( \tilde{\mu} \) is extremal (equivalently, not expressible as a mixture of other Gibbs measures).
It turns out that $\tilde{\mu}$ is extremal at activity $\lambda$ if and only if reconstruction is impossible for the corresponding broadcasting model on $T_k$ with transition matrix $P$. (This is a consequence of the general fact that a Gibbs measure is extremal iff it is trivial on the tail $\sigma$-algebra, and of the independence structure given by the broadcasting constructions of $\mu$ and $\tilde{\mu}$). Hence the reconstruction threshold $\lambda_c$, defined after (3) is also the extremality threshold for $\tilde{\mu}$; Theorem 2 therefore shows that whenever $\lambda \leq e - 1$, the unique simple invariant Gibbs measure $\tilde{\mu}$ for the hard-core model with activity $\lambda$ is extreme, for any $k$.

In particular, $\tilde{\mu}$ is extreme in the special case $\lambda = 1$ for any $k$.

1.3 Outline of proof

Our proof of Theorems 1 and 2 is in the same spirit as the proof by Brightwell and Winkler of the lower bound in (4) for the hard-core model [BW03].

We first develop a coupling between the distributions of the random variable $A(u)$ conditioned on two different events, with certain additional properties beyond those used in [BW03]. We then use this coupling to establish a recursion linking the distribution of $A(u)$ to those of $A(u_1), \ldots, A(u_k)$, where $u_1, \ldots, u_k$ are the children of $u$. (Of course, we already know from the translation invariance property described in Section 1.1 that all of these distributions are the same). If the recursion relation is contractive in a suitable sense, we obtain that $A(u)$ must be a.s. constant.

In Section 2, we first prove a lemma on conditional probabilities in a more general setting. Specialising to our context, we obtain the existence of a coupling of a pair of random variables $A_0, A_1$ with the following properties:

(i) The distribution of $A_0$ is the distribution of $A(u)$ under $\mu$ conditioned on the event $\{\phi(u) = 0\}$;
(ii) The distribution of $A_1$ is the distribution of $A(u)$ under $\mu$ conditioned on the event $\{\phi(u) = 1\}$;
(iii) With probability 1, either $A_0 = A_1$ or $A_1 = \pi_0 \leq A_0$;
(iv) If $A_0 = A_1$ with probability 1, then both are equal to $\pi_0$ with probability 1, and so also $A(u) = \pi_0$ a.s. under $\mu$.

We develop the recursion relations and complete the proofs in Section 3.

The full properties of the coupling are only needed in the hard-core case, where a particular convexity property holds for the recursion relations. The argument in the general case is not as powerful, and rather than all of property (iii) above, we use only that $A_1 \leq A_0$ with probability 1. Restricting the bound in Theorem 1 to the case $p_{11} = 0$ gives a much weaker bound than that in Theorem 2 (in fact, one obtains only the bound $\lambda_c \geq \lambda'_c$ where $\lambda'_c$ is the threshold for the uniqueness of the Gibbs measure; this bound is obvious in the context of the hard-core model since if the Gibbs measure is unique it is trivially extreme).

2 Conditioned conditional probabilities

We first consider the setting of a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $B \in \mathcal{F}$ be an event with probability $\pi_0 = 1 - \pi_1$, and suppose $0 < \pi_0 < 1$. Write $B^c$ for the complement of $B$. Let $\mathcal{G}$ be a sub-$\sigma$-algebra of $\mathcal{F}$.

We consider the random variable $\mathbb{P}(B|\mathcal{G})$ (which is the $\mathcal{G}$-measurable random variable, unique up to almost sure equality, such that for all $D \in \mathcal{G}$

$$\mathbb{P}(D \cap B) = \int_D \mathbb{P}(B|\mathcal{G})(\omega) d\mathbb{P}(\omega).$$

(6)
See for example Chapter 9 of [Wil91] for background on conditional probabilities).

**Lemma 3.** Suppose $0 \leq p_0 \leq p_1 \leq 1$, and that $D \in G$ with

$$\mathbb{P}(B|G)(\omega) \in [p_0, p_1] \text{ for all } \omega \in D.$$  \hfill (7)

Then

$$\frac{\pi_1 p_0}{\pi_0} \mathbb{P}(D|B^c) \leq \mathbb{P}(D|B) \leq \frac{\pi_1 p_1}{\pi_0} \mathbb{P}(D|B^c).$$

**Proof.** From (6) and (7) we have

$$p_0 \mathbb{P}(D) \leq \mathbb{P}(D \cap B) \leq p_1 \mathbb{P}(D)$$

and

$$(1 - p_1) \mathbb{P}(D) \leq \mathbb{P}(D \cap B^c) \leq (1 - p_0) \mathbb{P}(D).$$

Combining these we get

$$\frac{p_0}{1 - p_0} \mathbb{P}(D \cap B^c) \leq \mathbb{P}(D \cap B) \leq \frac{p_1}{1 - p_1} \mathbb{P}(D \cap B^c).$$

Since $\mathbb{P}(D|B) = \mathbb{P}(D \cap B)/\pi_0$ and $\mathbb{P}(D|B^c) = \mathbb{P}(D \cap B^c)/\pi_1$, the result follows.  \hfill \square

In particular, if $J$ is a subset of the interval $[0, \pi_0)$ then we can set $D = \{ \omega : \mathbb{P}(B|G)(\omega) \in J \}$ to obtain

$$\mathbb{P}\left\{ \mathbb{P}(B|G) \in J|B \right\} \leq \mathbb{P}\left\{ \mathbb{P}(B|G) \in J|B^c \right\},$$

while if $J \subseteq (\pi, 1]$ then the inequality is reversed. In each case equality holds only if both sides are 0. Hence:

**Corollary 4.** There exists a coupling of two random variables $Y_0$ and $Y_1$, such that $Y_0$ has the distribution of $\mathbb{P}(B|G)$ conditioned on $B$ occurring, such that $Y_1$ has the distribution of $\mathbb{P}(B|G)$ conditioned on $B$ not occurring, and such that:

(i) whenever $Y_0 < \pi_0$, then $Y_1 = Y_0$, and

(ii) whenever $Y_1 > \pi_0$, then $Y_1 = Y_0$.

Therefore either $Y_0 = Y_1$ or $Y_1 \leq \pi_0 \leq Y_0$.

Also the distributions of $Y_0$ and $Y_1$ are identical iff $Y_0 = Y_1 = \pi_0$ with probability 1, or equivalently iff $\mathbb{P}(B|G) = \pi_0$ with probability 1.

Applying this result with $B = \{ \phi(u) = 0 \}$, with $\mathcal{G} = T(u)$, with $\mathbb{P} = \mu$ and so with $A(u) = \mathbb{P}(B|G)$, we obtain the coupling of $A_0, A_1$ with the properties claimed in Section 1.3.
3 Recurrences for likelihood ratios

Let \( u \in T^k \) and let \( y \) be a configuration on the set \( \mathbb{W}_d(u) \) (the descendants of \( u \) at distance exactly \( d \)).

For \( S \subset T^k \), write \( \phi \mid S \) for the configuration \( \phi \) restricted to \( S \).

Define the “likelihood functions”

\[
q^{(0)}(d, y) = \mu(\phi \mid \mathbb{W}_d(u) = y | \phi(u) = 0)
\]

\[
q^{(1)}(d, y) = \mu(\phi \mid \mathbb{W}_d(u) = y | \phi(u) = 1).
\]

For \( i = 0, 1 \), the function \( q^{(i)}(d, y) \) gives the probability of observing the configuration \( y \) on the set of descendants of \( u \) at distance \( d \), given that the value at \( u \) itself is \( i \). (Note that because of the translation invariance property noted in Section 1.1, the choice of \( u \) is not important).

Define also the “likelihood ratio” function

\[
q(d, y) = \frac{q^{(0)}(d, y)}{q^{(1)}(d, y)}.
\]

Let \( d \geq 2 \) and let the children of \( u \) be \( u_1, \ldots, u_k \). A configuration \( y \) on \( \mathbb{W}_d(u) \) corresponds to a set of configurations \( y_1, \ldots, y_k \) on \( \mathbb{W}_{d-1}(u_1), \ldots, \mathbb{W}_{d-1}(u_k) \). We then have

\[
q^{(0)}(d, y) = \prod_{j=1}^k [p_{00}q^{(0)}(d-1, y_j) + p_{01}q^{(1)}(d-1, y_j)]
\]

\[
q^{(1)}(d, y) = \prod_{j=1}^k [p_{10}q^{(0)}(d-1, y_j) + p_{11}q^{(1)}(d-1, y_j)]
\]

and so

\[
q(d, y) = \prod_{j=1}^k \left\{ \frac{p_{00}q^{(0)}(d-1, y_j) + p_{01}q^{(1)}(d-1, y_j)}{p_{10}q^{(0)}(d-1, y_j) + p_{11}q^{(1)}(d-1, y_j)} \right\}
\]

\[
= \left( \frac{p_{00}}{p_{10}} \right)^k \prod_{j=1}^k \left\{ 1 + \frac{c_0 - c_1}{q(d-1, y_j) + c_1} \right\}, \tag{8}
\]

where we define

\[
c_0 = \frac{p_{01}}{p_{00}}, \quad c_1 = \frac{p_{11}}{p_{10}}.
\]

Define also \( a(d, y) = \mu(\phi(u) = 0 | \phi \mid \mathbb{W}_d(u) = y) \). The function \( a \) gives the conditional probability that the value at the node \( u \) is 0, given a configuration on the set of descendants of \( u \) at distance \( d \). We have

\[
a(d, y) = \frac{\pi_0 q^{(0)}(d, y)}{\pi_0 q^{(0)}(d, y) + \pi_1 q^{(1)}(d, y)}
\]

\[
= \frac{1}{1 + \frac{c_1}{\pi_0 q(d, y)}}. \tag{9}
\]
Returning to the random variable \( A(d, u) = \mu(\phi(u) = 0 \mid \mathcal{W}_d(u)) \) defined in Section 1, we have
\[
A(d, u) = a(d, \phi | \mathcal{W}_d(u)),
\]
that is, the function \( a(d, \cdot) \) applied to the actually observed values of the configuration \( \phi \) on \( \mathcal{W}_d(u) \).

Similarly define
\[
Q(d, u) = q(d, \phi | \mathcal{W}_d(u)).
\]

From (9), we have
\[
A(d, u) = \frac{1}{1 + \frac{\pi_1}{\pi_0 Q(d, u)}}, \quad Q(d, u) = \frac{\pi_1}{\pi_0} \left( \frac{1}{1 - A(d, u)} - 1 \right).
\]

Recalling \( A(d, u) \to A(u) \) a.s., we have \( Q(d, u) \to Q(u) \) a.s., where
\[
Q(u) = \frac{\pi_1}{\pi_0} \left( \frac{1}{1 - A(u)} - 1 \right).
\]

From (8) we get
\[
Q(d, u) = \left( \frac{p_{00}}{p_{10}} \right)^k \prod_{j=1}^k \left( 1 + \frac{c_0 - c_1}{Q(d - 1, u_j) + c_1} \right),
\]
and, taking \( d \to \infty \),
\[
Q(u) = \left( \frac{p_{00}}{p_{10}} \right)^k \prod_{j=1}^k \left( 1 + \frac{c_0 - c_1}{Q(u_j) + c_1} \right).
\]

Now put
\[
L(u, d) = \ln Q(u, d)
\]
and
\[
L(u) = \ln Q(u)
= \ln \left[ \frac{\pi_1}{\pi_0} \left( \frac{1}{1 - A(u)} - 1 \right) \right].
\]

We have \( L(u, d) \to L(u) \) a.s. as \( d \to \infty \), and
\[
L(u) = k \ln \left( \frac{p_{00}}{p_{10}} \right) + \sum_{j=1}^k \ln \left( 1 + \frac{c_0 - c_1}{\exp(L(u_j)) + c_1} \right). \tag{10}
\]

Since \( L(u) \) can be written as a strictly increasing function of \( A(u) \), with \( A(u) = \pi_0 \) corresponding to \( L(u) = 0 \), we can translate the coupling of \( A_0, A_1 \) described in Section 1.3 and proved in Section 2 into a coupling of two random variables \( L_0, L_1 \) with the following properties:
(i) The distribution of $L_0$ is the distribution of $L(u)$ conditioned on the event $\{\phi(u) = 0\}$;
(ii) The distribution of $L_1$ is the distribution of $L(u)$ conditioned on the event $\{\phi(u) = 1\}$;
(iii) With probability 1, either $L_0 = L_1$ or $L_1 \leq 0 \leq L_0$;
(iv) If $L_0 = L_1$ with probability 1, then both are equal to 0 with probability 1, and then also $A(u) = \pi_0$ with probability 1.

So to conclude that $A(u)$ is a.s. constant, it’s enough to show that $E[L_0 - L_1] = 0$.

Returning to (10), note that $\exp(L(u)) \geq 0$, and so the quantity inside the second logarithm is always at least $\min(c_0/c_1, 1) > 0$. Thus the distribution of $L(u)$ has compact support; hence the same is true for $L_0$ and $L_1$, and certainly $E[L_0] < \infty$, $E[L_1] < \infty$.

Again let $u_1, \ldots, u_k$ be the children of a node $u$. From the broadcasting construction we get the following information.

Conditional on $\phi(u) = 0$:

the $\phi(u_j), j = 1, \ldots, k$ are i.i.d. taking value 0 w.p. $p_{00}$ and value 1 w.p. $p_{01}$. Then the $L(u_j)$ are i.i.d., and the distribution of each is a mixture of the distribution of $L_0$ (with weight $p_{00}$) and the distribution of $L_1$ (with weight $p_{01}$).

Conditional on $\phi(u) = 1$:

the $\phi(u_j), j = 1, \ldots, k$ are i.i.d. taking value 0 w.p. $p_{10}$ and value 1 w.p. $p_{11}$. Then the $L(u_j)$ are i.i.d., and the distribution of each is a mixture of the distribution of $L_0$ (with weight $p_{10}$) and the distribution of $L_1$ (with weight $p_{11}$).

Hence from (10),

$E[L_0] = -k \ln \left( \frac{p_{00}}{p_{10}} \right) + k \left[ p_{00} E \ln \left( 1 + \frac{c_0 - c_1}{\exp(L_0) + c_1} \right) + p_{01} E \ln \left( 1 + \frac{c_0 - c_1}{\exp(L_1) + c_1} \right) \right].$

and

$E[L_1] = -k \ln \left( \frac{p_{00}}{p_{10}} \right) + k \left[ p_{10} E \ln \left( 1 + \frac{c_0 - c_1}{\exp(L_0) + c_1} \right) + p_{11} E \ln \left( 1 + \frac{c_0 - c_1}{\exp(L_1) + c_1} \right) \right].$

Subtracting and using the fact that $p_{00} - p_{10} = p_{11} - p_{01}$, we obtain

$E(L_0 - L_1) = k E [f(L_0) - f(L_1)],$

(11)

where

$f(x) = (p_{11} - p_{01}) \ln \left( 1 + \frac{c_0 - c_1}{e^x + c_1} \right)$

$= \frac{c_1 - c_0}{(1 + c_0)(1 + c_1)} \ln \left( 1 + \frac{c_0 - c_1}{e^x + c_1} \right).$

(12)
3.1 General case

If $c_0 = c_1$, then the function $f$ defined at (12) is constant. In that case, (11) shows that $\mathbb{E}(L_0 = L_1) = 0$, and reconstruction is impossible.

So assume that $c_0 \neq c_1$. Then $f$ is strictly increasing, and one obtains that

$$f'(x) = \frac{(c_1 - c_0)^2}{(1 + c_0)(1 + c_1)} \frac{e^x}{(e^x + c_0)(e^x + c_1)}. \tag{13}$$

Putting $y = e^x$ and taking the reciprocal, one can find the value of $x$ maximising (13) by finding the value of $y \geq 0$ minimising $(y + c_0)(y + c_1)y^{-1}$. One obtains $y = e^x = (c_0 c_1)^{1/2}$, and so

$$\sup_x f'(x) = \frac{(c_1 - c_0)^2}{(1 + c_0)(1 + c_1)} \frac{(c_0 c_1)^{1/2}}{((c_0 c_1)^{1/2} + c_0)((c_0 c_1)^{1/2} + c_1)}$$

$$= \frac{(c_1 - c_0)^2}{(1 + c_0)(1 + c_1)} \frac{1}{(\sqrt{c_1} + \sqrt{c_0})^2}$$

$$= \frac{\sqrt{c_1} - \sqrt{c_0}}{(1 + c_0)(1 + c_1)}$$

$$= \left(\frac{1}{1 + c_0} \frac{c_1}{1 + c_1} - \frac{c_0}{1 + c_0} \frac{1}{1 + c_1}\right)^2$$

$$= \left(\sqrt{p_{00} p_{11}} - \sqrt{p_{01} p_{10}}\right)^2. \tag{14}$$

Since we know that $L_1 \leq L_0$ with probability 1, we then have that

$$0 \leq f(L_0) - f(L_1) \leq \sup_x f'(x)(L_0 - L_1)$$

with equality on the RHS iff $L_0 = L_1$, with probability 1. Hence, from (11),

$$\mathbb{E}(L_0 - L_1) \leq k \sup_x f'(x) \mathbb{E}(L_0 - L_1),$$

with equality iff both sides are 0. So to show that $\mathbb{E}(L_0 - L_1) = 0$, and therefore that reconstruction is impossible, it’s enough to show that $k \sup_x f'(x) \leq 1$. Using (14), we see that (2) indeed implies that reconstruction is impossible, and the proof of Theorem 1 is done.

3.2 Hard-core case

Recall that

$$\begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+w} & \frac{w}{1+w} \\ 1 & 0 \end{pmatrix} ;$$

we have also that $\pi_0 = (1 + w)/(1 + 2w)$, $\pi_1 = w/(1 + 2w)$, and $c_0 = w$, $c_1 = 0$. We have also defined $\lambda = w(1 + w)^k$. Equation (12) now becomes

$$f(x) = -\frac{w}{1+w} \ln \left(1 + we^{-x}\right),$$
and now the function \( f \) is concave as well as strictly increasing. Hence in particular, if \( x_0 < x_1 \) and \( y_0 < y_1 \) with \( x_0 \leq y_0 \) and \( x_1 \leq y_1 \), then

\[
0 \leq \frac{f(y_1) - f(y_0)}{y_1 - y_0} \leq \frac{f(x_1) - f(x_0)}{x_1 - x_0}.
\]  

Recurrence (10) now becomes

\[
L(u) = -k \ln(1 + w) + \sum_{j=1}^{k} \ln \left( 1 + we^{-L(u)} \right),
\]

and so in particular \( L(u) \) is always greater than or equal to \(-k \ln(1 + w)\); the same is therefore true of \( L_0 \) and \( L_1 \) also. Combining this with property (iii) of the coupling described after (10), we have that, with probability 1, either \( L_0 = L_1 \) or \(-k \ln(1 + w) \leq L_1 \leq 0 \leq L_0 \). Thus, using (15), we obtain that with probability 1

\[
0 \leq f(L_0) - f(L_1) \leq (L_0 - L_1) \frac{f(0) - f\left(-k \ln(1 + w)\right)}{0 - \left(-k \ln(1 + w)\right)}
\]

\[
= (L_0 - L_1) \frac{w}{1 + w} \left( -\ln(1 + w) + \ln(1 + \lambda) \right)
\]

\[
= (L_0 - L_1) \frac{w}{k(1 + w)} \left( \frac{\ln(1 + \lambda)}{\ln(1 + w)} - 1 \right),
\]  

(16)

where we have used

\[
f(0) = -w \ln(1 + w) / (1 + w)
\]

and

\[
f(-k \ln(1 + w)) = -\frac{w}{1 + w} \ln \left( 1 + we^{k \ln(1 + w)} \right)
\]

\[
= -\frac{w}{1 + w} \ln \left( 1 + w(1 + w)^k \right)
\]

\[
= -\frac{w}{1 + w} \ln(1 + \lambda).
\]

Combining (11) and (16), we get \( 0 \leq \mathbb{E}(L_0 - L_1) \leq \rho \mathbb{E}(L_0 - L_1) \), where

\[
\rho = \frac{w}{1 + w} \left( \frac{\ln(1 + \lambda)}{\ln(1 + w)} - 1 \right).
\]

To obtain that \( \mathbb{E}(L_0 - L_1) = 0 \), and hence that reconstruction is impossible, it’s enough that \( \rho < 1 \). But

\[
\rho < \frac{w}{1 + w} \ln(1 + \lambda)
\]

\[
\leq \ln(1 + \lambda) \sup_{w > 0} \frac{w}{(1 + w) \ln(1 + w)}
\]

\[
= \ln(1 + \lambda),
\]
(since the quantity within the sup is decreasing in \( w \) and tends to 1 as \( w \downarrow 0 \)). So certainly if \( \lambda < e - 1 \), then \( \rho < 1 \) as desired. Also \( \rho \) is continuous as a function of \( \lambda \) (or \( w \)), so the threshold value \( \lambda_2(\hat{k}) \) is in fact strictly greater than \( e - 1 \), and the proof of Theorem 2 is done.

4  A calculation

For completeness, we here include details to show that the LHS of (2) is always less than or equal to that of (1). Define \( b_0 = \min(p_{00}, p_{11}) \leq \max(p_{00}, p_{11}) = b_1 \). We obtain

\[
\min\{p_{00} + p_{10}, p_{01} + p_{11}\} = \min\{b_0 + 1 - b_1, b_1 + 1 - b_0\} \\
= b_0 + 1 - b_1 \\
= (1 - b_0)(1 - b_1) + b_0b_1 + 2b_0(1 - b_1) \\
\leq (1 - b_0)(1 - b_1) + b_0b_1 + 2\sqrt{b_0b_1}(1 - b_1)(1 - b_0) \\
= \left[\sqrt{b_0b_1} + \sqrt{(1 - b_1)(1 - b_0)}\right]^2. 
\]

(The inequality in (17) follows since \( b_0 \leq b_1 \) and \( 1 - b_1 \leq 1 - b_0 \); equality holds if \( b_0 = b_1 \) or if one of \( b_0 \) and \( b_1 \) is 0 or 1).

Then, starting from the LHS of (2) and using (18),

\[
\left(\sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}}\right)^2 = \left[\sqrt{b_0b_1} - \sqrt{(1 - b_1)(1 - b_0)}\right]^2 \\
\leq \left[\sqrt{b_0b_1} - \sqrt{(1 - b_1)(1 - b_0)}\right]^2 \frac{\left[\sqrt{b_0b_1} + \sqrt{(1 - b_1)(1 - b_0)}\right]^2}{\min\{p_{00} + p_{10}, p_{01} + p_{11}\}} \\
= \frac{[b_0b_1 - (1 - b_1)(1 - b_0)]^2}{\min\{p_{00} + p_{10}, p_{01} + p_{11}\}} \\
= \frac{(b_0 - (1 - b_1))^2}{\min\{p_{00} + p_{10}, p_{01} + p_{11}\}} \\
= \frac{[\pm(p_{00} - p_{10})]^2}{\min\{p_{00} + p_{10}, p_{01} + p_{11}\}},
\]

which gives the LHS of (1) as required.

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References


