The b-chromatic number of power graphs
Brice Effantin, Hamamache Kheddouci

To cite this version:
Brice Effantin, Hamamache Kheddouci. The b-chromatic number of power graphs. Discrete Mathematics and Theoretical Computer Science, DMTCS, 2003, 6 (1), pp.45-54. <hal-00958987>
The b-chromatic number of some power graphs

Brice Effantin\(^1\) and Hamamache Kheddouci\(^2\)

LE2I FRE-CNRS 2309, Université de Bourgogne, B.P. 47870, 21078 Dijon Cedex, France
\(^1\)brice.effantin@u-bourgogne.fr
\(^2\)kheddouc@u-bourgogne.fr


Let \( G \) be a graph on vertices \( v_1, v_2, \ldots, v_n \). The b-chromatic number of \( G \) is defined as the maximum number \( k \) of colors that can be used to color the vertices of \( G \), such that we obtain a proper coloring and each color \( i \), with \( 1 \leq i \leq k \), has at least one representant \( x_i \) adjacent to a vertex of every color \( j \), \( 1 \leq j \neq i \leq k \). In this paper, we give the exact value for the b-chromatic number of power graphs of a path and we determine bounds for the b-chromatic number of power graphs of a cycle.

Keywords: b-chromatic number, coloring, cycle, path, power graphs

1 Introduction

We consider graphs without loops or multiple edges. Let \( G \) be a graph with a vertex set \( V \) and an edge set \( E \). We denote by \( d(x) \) the degree of the vertex \( x \) in \( G \), and by \( \text{dist}_G(x,y) \) the distance between two vertices \( x \) and \( y \) in \( G \). The \( p \)-th power graph \( G^p \) is a graph obtained from \( G \) by adding an edge between every pair of vertices at distance \( p \) or less, with \( p \geq 1 \). It is easy to see that \( G^1 = G \). In the literature, power graphs of several classes have been investigated \([2, 3, 8]\). In this note we study a vertex coloring of power graphs. The power graph of a path and the power graph of a cycle can be also considered as respectively subclasses of distance graphs and circulant graphs. The distance graph \( G(D) \) with distance set \( D = \{d_1, d_2, \ldots\} \) has the set \( Z \) of integers as vertex set, with two vertices \( i, j \in Z \) adjacent if and only if \( |i - j| \in D \). The circulant graph can be defined as follows. Let \( n \) be a natural number and let \( S = \{k_1, k_2, \ldots, k_r\} \) with \( k_1 < k_2 < \ldots < k_r \leq n/2 \). Then the vertex set of the circulant graph \( G(n,S) \) is \( \{0, 1, \ldots, n-1\} \) and the set of neighbors of the vertex \( i \) is \( \{i \pm k_j \mod n | j = 1, 2, \ldots, r\} \).

The study of distance graphs was initiated by Eggleton and al. \([4]\). Recently, the problem of coloring of this class of graphs has attracted considerable attention, see e.g. \([12, 13]\). Circulant graphs have been extensively studied and have a vast number of applications to multicomputer networks and distributed computation (see \([10, 11]\)). The special cases we consider are the distance graph \( G(D) \) with finite distance set \( D = \{1, 2, \ldots, p\} \) which is isomorphic to the \( p \)-th power graph of a path and the circulant graph \( G(n,S) \) with \( S = \{1, 2, \ldots, p\} \) which is isomorphic to the \( p \)-th power graph of a cycle.

1365–8050 © 2003 Discrete Mathematics and Theoretical Computer Science (DMTCS), Nancy, France
A $k$-coloring of $G$ is defined as a function $c$ on $V(G) = \{v_1, v_2, \ldots, v_n\}$ into a set of colors $C = \{1, 2, \ldots, k\}$ such that for each vertex $v_i$, with $1 \leq i \leq n$, we have $c(v_i) \in C$. A proper $k$-coloring is a $k$-coloring satisfying the condition $c_x \neq c_y$ for each pair of adjacent vertices $x, y \in V(G)$. A dominating proper $k$-coloring is a proper $k$-coloring satisfying the following property $P$: for each $i$, $1 \leq i \leq k$, there exists a vertex $x_i$ of color $i$ such that, for each $j$, with $1 \leq j \neq i \leq k$, there exists a vertex $y_j$ of color $j$ adjacent to $x_i$. A set of vertices satisfying the property $P$ is called a dominating system. Each vertex of a dominating system is called a dominating vertex. The $b$-chromatic number $\varphi(G)$ of a graph $G$ is defined as the maximum $k$ such that $G$ admits a dominating proper $k$-coloring.

The $b$-chromatic number was introduced in [7]. The motivation, similarly as for the previously studied achromatic number (cf. e.g. [3, 4]), comes from algorithmic graph theory. The achromatic number $\psi(G)$ of a graph $G$ is the largest number of colors which can be assigned to the vertices of $G$ such that the coloring is proper and every pair of distinct colors appears on an edge. A proper coloring of a graph $G$ using $k > \chi(G)$ colors could be improved if the vertices of two color classes could be recolored by a single color so as to obtain a proper coloring. The largest number of colors for which such a recoloring strategy is not possible is given by the achromatic number. A more versatile form of recoloring strategy would be to allow the vertices of a single color class to be redistributed among the colors of the remaining classes, so as to obtain a proper coloring. The largest number of colors for which such a recoloring strategy is not possible is given by $\varphi(G)$ (these recolorings are discussed in [2] and [11]). Thus $\psi(G) \leq \varphi(G)$ (also given in [2]). From this point of view, both complexity results and tight bounds for the $b$-chromatic number are interesting. The following bounds of $b$-chromatic number are already presented in [7].

**Proposition 1** Assume that the vertices $x_1, x_2, \ldots, x_n$ of $G$ are ordered such that $d(x_1) \geq d(x_2) \geq \ldots \geq d(x_n)$. Then $\varphi(G) \leq m(G) \leq \Delta(G) + 1$, where $m(G) = \max\{1 \leq i \leq n : d(x_i) \geq i - 1\}$ and $\Delta(G)$ is the maximum degree of $G$.

R. W. Irving and D. F. Manlove [7] proved that finding the $b$-chromatic number of any graph is an NP-hard problem, and they gave a polynomial-time algorithm for finding the $b$-chromatic number of trees. Kouider and Mahéo [9] gave some lower and upper bounds for the $b$-chromatic number of the cartesian product of two graphs. They gave, in particular, a lower bound for the $b$-chromatic number of the cartesian product of two graphs where each one has a stable dominating system. More recently in [11], the authors characterized bipartite graphs for which the lower bound on the $b$-chromatic number is attained and proved the NP-completeness of the problem to decide whether there is a dominating proper $k$-coloring even for connected bipartite graphs and $k = \Delta(G) + 1$. They also determine the asymptotic behavior for the $b$-chromatic number of random graphs.

In this paper, we present several exact values and determine bounds for the $b$-chromatic number of power graphs of paths and cycles.

Let $\text{Diam}(G)$ be the diameter of a graph $G$, defined as the maximum distance between any pair of vertices of $G$. Let us begin with the following observation.

**Fact 2** For any graph $G$ of order $n$, if $\text{Diam}(G) \leq p$, then $\varphi(G^p) = n$, with $p \geq 2$.

**Proof.** If $\text{Diam}(G) \leq p$, it is trivial to see that $G^p$ is a complete graph. So $\varphi(G^p) = n$. \qed

Let $G$ be a path or a cycle on vertices $x_1, x_2, \ldots, x_n$. We fix an orientation of $G$ (left to right if $G$ is a path and clockwise if $G$ is a cycle). For each $1 \leq i \leq n$, we denote by $x_i^+$ (resp. $x_i^-$) the successor (resp.
The b-chromatic number of some power graphs

predecessor) of \( x_i \) in \( G \) (if any). For \( 1 \leq i \neq j \leq n \), we define \([x_i, x_j \rangle_G \), \([x_i, x_j \rangle_G \) and \((x_i, x_j \rangle_G \) as the set of consecutive vertices on \( G \) from respectively \( x_i \) to \( x_j \), \( x_i \) to \( x_j^+ \) and \( x_i^+ \) to \( x_j^+ \), following the fixed orientation of \( G \). If there is no ambiguity, we denote \([x_i, x_j \rangle_G \), \([x_i, x_j \rangle_G \) and \((x_i, x_j \rangle_G \) by respectively \([x_i, x_j \rangle \), \([x_i, x_j \rangle \) and \((x_i, x_j \rangle \).

In all figures, the graph \( G \) is represented with solid edges. Edges added in a \( p \)-th power graph \( G^p \) are represented with dashed edges. In some figures, vertices are surrounded and represent a dominating system of the coloring. In any coloring of a graph \( G \), we will say that a vertex \( x \) of \( G \) is adjacent to a color \( i \) if there exists a neighbor of \( x \) which is colored by \( i \).

2 Power Graph of a Path

In this section, we determine the b-chromatic number of a \( p \)-th power graph of a path, with \( p \geq 1 \). First we give a lemma used in the proof of Theorem 4. Then the b-chromatic number of a \( p \)-th power graph of a path is computed.

**Lemma 3** For any \( p \geq 1 \), and for any \( n \geq p + 1 \), let \( P_n \) be the path on vertices \( x_1, x_2, \ldots, x_n \). For each integer \( k \), with \( p + 1 \leq k \leq \min(2p + 1, n) \), there exists a proper \( k \)-coloring on \( P_n^p \). Moreover each vertex \( x \), such that \( x \in \{ x_{k-1}, x_{k-p+1}, \ldots, x_{n-k+p+1} \} \), is adjacent to each color \( j \), with \( 1 \leq j \neq c_x \leq k \).

**Proof.** As \( k \geq p + 1 \), it is easy to see that if we put the set of colors \( \{1, 2, \ldots, k\} \) cyclically on \( V(P_n) \), then two adjacent vertices will not have the same color. The coloring is thus a proper \( k \)-coloring.

Let \( S = \{ x_{k-p}, x_{k-p+1}, \ldots, x_{n-k+p+1} \} \). First we show that each vertex of \( S \) is adjacent to at least \( k - 1 \) vertices. Observe that the vertex \( x_{k-p} \) is adjacent to \( (k - p - 1) + p = k - 1 \) vertices. And the vertex \( x_{n-k+p+1} \) is adjacent to \( n + (n + k + 1) = k - 1 \) vertices. Since each vertex \( x_i \), with \( k - p + 1 \leq i \leq n - k + p \), has a degree \( d(x_i) \geq d(x_{k-p}) \), then each vertex of \( S \) is adjacent to at least \( k - 1 \) other vertices.

Next, we can see by the construction that all the colors \( \{1, 2, \ldots, k\} \setminus \{c_{x_i}\} \) appear between the first and the last neighbor of \( x_i \). Therefore each vertex \( x_i \) of \( S \) is adjacent to each color \( j \), with \( 1 \leq j \neq c_{x_i} \leq k \) and \( k - p \leq i \leq n - k + p + 1 \). \( \Box \)

The b-chromatic number of a \( p \)-th power graph of a path is given by:

**Theorem 4** Let \( P_n \) be a path on vertices \( x_1, x_2, \ldots, x_n \). The b-chromatic number of \( P_n^p \), with \( p \geq 1 \), is given by:

\[
\phi(P_n^p) = \begin{cases} 
  n & \text{if } n \leq p + 1, \\
  p + 1 + \left\lceil \frac{n - p - 1}{2} \right\rceil & \text{if } p + 2 \leq n \leq 4p + 1, \\
  2p + 1 & \text{if } n \geq 4p + 2
\end{cases}
\]

**Proof.**

1. If \( n \leq p + 1 \), then \( \text{Diam}(P_n) \leq p \). So, by Fact 2, \( \phi(P_n^p) = n \).

2. We prove first that \( \phi(P_n^p) \geq p + 1 + \left\lceil \frac{n - p - 1}{2} \right\rceil \) for \( p + 2 \leq n \leq 4p + 1 \). Let \( k = p + 1 + \left\lceil \frac{n - p - 1}{2} \right\rceil \).

By Lemma 3, we give a proper \( k \)-coloring of \( P_n^p \). For example, Figure 6 shows a dominating proper 5-coloring of \( P_8^3 \).
Let $S'$ be the set of vertices $\{x_{k-p}, x_{k-p+1}, \ldots, x_{2k-p-1}\}$. Since $2k - p - 1 \leq n - k + p + 1$, then $S' \subseteq \{x_{k-p}, x_{k-p+1}, \ldots, x_{n-k+p+1}\}$. By Lemma 3, $S'$ is a dominating system. As the coloring is proper and has a dominating system, we obtain a dominating proper $k$-coloring. So, $\phi(P^p_n) \geq p + 1 + \left\lceil \frac{n-p-1}{3} \right\rceil$.

Next we prove that $\phi(P^p_n) \leq p + 1 + \left\lceil \frac{n-p-1}{3} \right\rceil$ for $p + 2 \leq n \leq 4p + 1$. The proof is by contradiction. Suppose that there exists a dominating proper $k'$-coloring such that

$\quad k' > p + 1 + \left\lceil \frac{n-p-1}{3} \right\rceil$. \hspace{1cm} (1)

Let $W = \{w_1, w_2, \ldots, w_{k'}\}$ be a dominating system of the coloring on $P^p_n$ (following the orientation of $P_n$, we meet $w_1, w_2, \ldots, w_{k'}$). The vertices $w_1$ and $w_{k'}$ are adjacent to, at most, $p$ different colors in $[w_1, w_{k'}]$. As $w_1$ (respectively $w_{k'}$) is a dominating vertex, it must be adjacent to at least $k' - 1$ different colors. Then, there are at least $k' - p - 1$ vertices on $[x_1, w_1]$ (respectively $(w_{k'}, x_n]$).

Therefore, $n - k' \geq n - \|[w_1, w_{k'}]\| \geq 2(k' - p - 1)$.

On the other hand by hypothesis $k' \geq p + 2 + \left\lceil \frac{n-p-1}{3} \right\rceil$, so that $n - k' \leq n - p - 2 - \left\lceil \frac{n-p-1}{3} \right\rceil$.

These two results give the following inequality,

$\quad 2(k' - p - 1) \leq n - k' \leq n - p - 2 - \left\lceil \frac{n-p-1}{3} \right\rceil$, 

$\quad k' \leq \frac{1}{2}(n + p - \left\lceil \frac{n-p-1}{3} \right\rceil)$. \hspace{1cm} (2)

By (1) and (2), we obtain,

$\quad \frac{1}{2}(n + p - \left\lceil \frac{n-p-1}{3} \right\rceil) \geq k' \geq p + 2 + \left\lceil \frac{n-p-1}{3} \right\rceil$, 

$\quad n - p - 4 \geq 3 \left\lceil \frac{n-p-1}{3} \right\rceil$, 

which is a contradiction. Hence such a coloring does not exist. Therefore, $\phi(P^p_n) \leq p + 1 + \left\lceil \frac{n-p-1}{3} \right\rceil$.

We deduce from these two parts that $\phi(P^p_n) = p + 1 + \left\lceil \frac{n-p-1}{3} \right\rceil$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Coloring of $P^3_8$}
\end{figure}
3. $Δ(P^p_n) = 2p$, so by Proposition 4, $ϕ(P^p_n) ≤ 2p + 1$. Lemma 5 gives a proper $(2p + 1)$-coloring and shows that each vertex $x$ of the set $\{x_{p+1}, x_{p+2}, \ldots, x_{3p+1}\}$ is adjacent to each color $j$ with $1 ≤ j ≠ c_x ≤ k$. So this set is a dominating system and $ϕ(P^p_n) = 2p + 1$. Therefore $ϕ(P^p_n) = 2p + 1$. For example, Figure 2 gives a dominating proper $7$-coloring of $P^3_{15}$.

3 Power Graph of a Cycle

In this section, we study the $b$-chromatic number of a $p$-th power graph of a cycle, with $p ≥ 1$. First we give two lemmas used in the proof of Theorem 7. Then we bound the $b$-chromatic number of a $p$-th power graph of a cycle.

Lemma 5 Let $C^p_n$ be a $p$-th power graph of a cycle $C_n$, with $p ≥ 2$. For any $2p + 3 ≤ n ≤ 4p$, let $k ≥ min(n - p - 1, p + 1 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor)$. Then $n ≤ 2k$.

Proof. The proof is by contradiction. Suppose $n ≥ 2k + 1$. We consider two cases. Firstly, $k ≥ n - p - 1$. So,

$$n ≥ 2k + 1 ≥ 2(n - p - 1) + 1,$$

$$n ≤ 2p + 1,$$

which is a contradiction. Secondly, $k ≥ p + 1 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor$. So,

$$n ≥ 2k + 1 ≥ 2(p + 1 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor) + 1,$$

$$n - 2p - 3 ≥ 2 \left\lfloor \frac{n - p - 1}{3} \right\rfloor,$$

which is a contradiction too. □

Lemma 6 For any $p ≥ 2$, and for any $2p + 3 ≤ n ≤ 4p$, let $C_n$ be the cycle on vertices $x_1, x_2, \ldots, x_n$. Let $k = min(n - p - 1, p + 1 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor)$. So there exists a proper $k$-coloring on $C^p_n$. Moreover each vertex $x$, such that $x ∈ \{x_{k-p}, x_{k-p+1}, \ldots, x_{2k-p-1}\}$, is adjacent to each color $j$, with $1 ≤ j ≠ c_x ≤ k$.

Proof. We put the set of colors $\{1, 2, \ldots, k\}$ cyclically on $V(C_n)$. As $k ≤ p + 1 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor$ and $n ≤ 4p$, then $k ≤ 2p + 1$. Moreover, by Lemma 5 we deduce that $2k ≥ n ≥ 2p + 3 ≥ k + 2$. So, the full set of colors $\{1, 2, \ldots, k\}$ appears consecutively at least once, and at most twice, in the cyclic coloring of $C^p_n$. As $2k ≥ n ≥ 2p + 3$, we have $k ≥ p + 1$. Furthermore, by definition of $k$ we have $n - k ≥ p + 1$. Thus, as $k ≥ p + 1$ and $n - k ≥ p + 1$, the coloring is proper.
Let $P_n$ be the subpath of $C_n$ induced by $x_1, x_2, \ldots, x_n$. Let $S = \{x_{k-p}, x_{k-p+1}, \ldots, x_{k+(k-p-1)}\}$. As $p+1 \leq k \leq 2p+1$ and $2k-p-1 \leq n-k+p+1$, then by Lemma 5, each vertex $x_i$ of $S$, with $k-p \leq i \leq 2k-p-1$, is adjacent to each color $q$, with $1 \leq q \neq c_i \leq k$, on $P_n^p$. Therefore each vertex $x_i$ of $S$ is adjacent to each color $q$, with $1 \leq q \neq c_i \leq k$, on $C_n^p$. □

Theorem 7 Let $C_n$ be a cycle on vertices $x_1, x_2, \ldots, x_n$. The $b$-chromatic number of $C_n^p$, with $p \geq 1$, is

$$\varphi(C_n^p) = \begin{cases} 
  n & \text{if } n \leq 2p+1, \\
  p+1 & \text{if } n = 2p+2, \\
  (\geq) \min(n-p-1, p+1 + \left\lceil \frac{n-p-1}{3} \right\rceil) & \text{if } 2p+3 \leq n \leq 3p \\
  p+1 + \left\lceil \frac{n-p-1}{3} \right\rceil & \text{if } 3p+1 \leq n \leq 4p \\
  2p+1 & \text{if } n \geq 4p+1
\end{cases}$$

Proof.

1. If $n \leq 2p+1$, then $Diam(C_n) \leq p$. So, by Fact 3, $\varphi(C_n^p) = n$.

2. To color the graph, we put the set of colors $\{1, 2, \ldots, p+1\}$ cyclically twice. One can easily see that this coloring is a proper $(p+1)$-coloring. Let $S$ be the set of vertices $\{x_1, x_2, \ldots, x_{p+1}\}$. Each vertex $x_i$, with $1 \leq i \leq p+1$, is adjacent to $n-2$ vertices. Since $n-2 \geq p+1$, then each vertex $x_i$ ($1 \leq i \leq p+1$) of $S$ is adjacent to all colors other than $c_i$. So the set $S$ is a dominating system.

   We now show that, in any dominating proper coloring, vertices $x_i$ and $x_{i+p+1}$ must have the same color. For the subgraph induced by vertices $x_1, x_2, \ldots, x_{p+1}$, we have a clique and we can assume without loss of generality that these vertices are colored by $1, 2, \ldots, p+1$ respectively. If there exists a dominating vertex of color $j$, for some $j > p+1$, then this vertex is $x_{p+1+i}$ for some $i$ ($1 \leq i \leq p+1$). Vertex $x_{p+1+i}$ is not adjacent to $x_i$, but every other vertex is adjacent to $x_i$, so that $x_{p+1+i}$ cannot be a dominating vertex, a contradiction. Therefore $\varphi(C_n^p) = p+1$ for $n = 2p+2$.

3. Let $k = \min(n-p-1, p+1 + \left\lceil \frac{n-p-1}{3} \right\rceil)$.

   By Lemma 5 there exists a dominating proper $k$-coloring for $2p+3 \leq n \leq 3p$. Therefore $\varphi(C_n^p) \geq \min(n-p-1, p+1 + \left\lceil \frac{n-p-1}{3} \right\rceil)$. For example, in Figure 3, we give a dominating proper 6-coloring of $C_{11}^4$.

4. Let $k = p+1 + \left\lceil \frac{n-p-1}{3} \right\rceil$.

   For $3p+1 \leq n \leq 4p$, Lemma 3 gives a dominating proper $k$-coloring. This proves that $\varphi(C_n^p) \geq \min(n-p-1, p+1 + \left\lceil \frac{n-p-1}{3} \right\rceil)$. For example, Figure 4 shows a dominating proper 6-coloring of $C_{11}^3$.

   Next, we prove that $\varphi(C_n^p) \leq k$. Suppose there exists a dominating proper $k'$-coloring for $C_n^p$, with $k' \geq p+2 + \left\lceil \frac{n-p-1}{3} \right\rceil$, for the sake of contradiction. Let $W = \{w_1, w_2, \ldots, w_{k'}\}$ be a set of dominating vertices on $C_n$ (following the orientation of $C_n$, we meet $w_1, w_2, \ldots, w_{k'}$). We distinguish two cases.
The b-chromatic number of some power graphs

51

Fig. 3: Coloring of $C_{11}^4 (n - p - 1 = 6, p + 1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor = 7$ and $\phi(C_{11}^4) \geq 6$)

Fig. 4: Coloring of $C_{11}^3$

Case 1: for each $i$, with $1 \leq i \leq k'$, \(|(w_i, w_{i+1})| \leq p - 1$.

As $k' \geq p + 2 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$, by a straightforward modification of the proof of Lemma 5, we have $n < 2k'$. So, there exists at least one color $c$ not repeated in $C^n_p$ (i.e. there are not two distinct vertices with the same color $c$). Without loss of generality, suppose that $c$ appears on the vertex $x$, with $x \in V(C_{11})$. Therefore $x$ is a dominating vertex and each other dominating vertex is adjacent to $x$. Then, \(|w_1, x| \leq p$ and \(|x, w_{k'}| \leq p$. As for each $i$, with $1 \leq i \leq k'$, we have \(|(w_i, w_{i+1})| \leq p - 1$ and since on the cycle the next dominating vertex from $w_{k'}$ is $w_1$, then

\(|(w_{k'}, w_1)| \leq p - 1$,

where

\(|(w_{k'}, w_1)| = n - |w_1, x| - |x, w_{k'}| - 1$.

Therefore, we have

\[ n - |w_1, x| - |x, w_{k'}| - 1 \leq p - 1, \]

\[ n - 2p - 1 \leq p - 1, \]

\[ n \leq 3p, \]

which is a contradiction.

Case 2: There exists $r$, with $1 \leq r \leq k'$ and $r$ is taken modulo $k'$, such that \(|(w_r, w_{r+1})| \geq p$.

Let $X$ be the set of vertices of $[w_{r+1}, w_r]$ (see Figure 5). Let $X_C$ be the set of colors appearing in $X$. Let $\Gamma_X(x_i)$ be the set of neighbors of $x_i$ in $X$ and $\Gamma_{X_C}(x_i)$ the set of colors appearing in $\Gamma_X(x_i)$, with $1 \leq i \leq n$. Let $A = X_C \setminus (\Gamma_X(w_r) \cup \{c_{w_r}\})$. Let $B = X_C \setminus (\Gamma_X(w_{r+1} + 1) \cup \{c_{w_{r+1}}\})$. We discuss two subcases.
This completes the proof of Subcase 2: which is a contradiction.

Fig. 5: A dominating system on $C'_n$ and the set $X$

Subcase 1: $|X| \leq 2p + 2$. Since all dominating vertices belong to $X$, we have $|X| \geq k'$. Then, $|(w_r, w_{r+1})| \leq n - k'$ and $|X_c| = k'$. As the vertices of $\Gamma_X(w_r)$ form a clique, then $|\Gamma_X(w_r)| = |\Gamma_X(w_r)| = p$. So we have $|A| = |X_c| - |\Gamma_X(w_r)| - 1 = k' - p - 1$. In the same way, we deduce that $|B| = k' - p - 1$. As $|X| \leq 2p + 2$, we have $X \subseteq (\Gamma_X(w_r) \cup \Gamma_X(w_{r+1}) \cup \{w_r, w_{r+1}\})$ (see Figure 5a). So, $A \subseteq (\Gamma_X(w_{r+1}) \cup \{c_{w_{r+1}}\})$ and $B \subseteq (\Gamma_X(w_r) \cup \{c_{w_r}\})$. Let $q \in \{1, 2, \ldots, k'\}$. If $q \in A$ (resp. $q \in B$) then $q \notin (\Gamma_X(w_r) \cup \{c_{w_r}\})$ (resp. $q \notin (\Gamma_X(w_{r+1}) \cup \{c_{w_{r+1}}\})$ and so $q \notin B$ (resp. $q \notin A$). Therefore, $A \cap B = \emptyset$. As $w_r$ (resp. $w_{r+1}$) is a dominating vertex and $|(w_r, w_{r+1})| \geq p$, the colors of $A$ (resp. $B$) must be repeated in $(w_r, w_{r+1})$. Therefore,

$$|A| + |B| \leq |(w_r, w_{r+1})|,$$

$$2(k' - p - 1) \leq n - k',$$

$$3k' \leq n + 2p + 2,$$

$$3 \left\lfloor \frac{n - p - 1}{3} \right\rfloor \leq n - p - 4,$$

which is a contradiction.

Subcase 2: $|X| \geq 2p + 3$. As in Subcase 1, we have $|A| = k' - p - 1$ and $|B| = k' - p - 1$. Let $X' = X \setminus (\Gamma_X(w_r) \cup \Gamma_X(w_{r+1}) \cup \{w_r, w_{r+1}\})$ (see Figure 5b). So $|X'| \geq |A \cap B|$. Since $w_r$ (resp. $w_{r+1}$) is a dominating vertex and $|(w_r, w_{r+1})| \geq p$, the colors of $A$ (resp. $B$) must be repeated in $(w_r, w_{r+1})$. Then,

$$|A| + |B| - |A \cap B| \leq |(w_r, w_{r+1})| \leq n - 2p - 2 - |X'|,$$

$$2(k' - p - 1) - |A \cap B| \leq n - 2p - 2 - |A \cap B|,$$

$$2(p + 2 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor) \leq n,$$

which is a contradiction. Therefore there does not exist a dominating proper $k'$-coloring, with $k' \geq p + 2 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor$.

This completes the proof of $\varphi(C'_n) = p + 1 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor$.
5. As \( \Delta = 2p \), by Proposition \[ \Phi(C_p^n) \leq 2p + 1. \]

We then give a proper \((2p + 1)\)-coloring. It is constructed in two steps. First, we put \((2p + 1)\) different colors on the \((2p + 1)\) first vertices \((c_x := i\text{ for } 1 \leq i \leq 2p + 1)\). In the second step, we have two cases. If \( n = 4p + 1 \), we color the remaining vertices as follows: \( c_x := c_{x-2p-1} \) for \( 2p + 2 \leq i \leq n \). If \( n \geq 4p + 2 \), then the remaining vertices are colored as follows: \( c_x := c_{x-2p-1} \) for \( 2p + 2 \leq i \leq 4p + 2 \), and \( c_x := c_{3p-1} \) for \( 4p + 3 \leq i \leq n \). Then the distance between two vertices colored by the same color \( c \) is at least \( p + 1 \). So the coloring is proper. By an analogue proof of Lemma \[ \], one can prove that each vertex \( x_i \), with \( p + 1 \leq i \leq 3p + 1 \), is a dominating vertex. So this coloring is a dominating proper \((2p + 1)\)-coloring. This construction shows that \( \Phi(C_p^n) \geq 2p + 1 \). Therefore we have proved that \( \Phi(C_p^n) = 2p + 1 \). For example, Figure \[ \] gives a dominating proper 7-coloring \( C_{16}^3 \).

\[ \]

4 Open Problem

In section \[ \], we have obtained the exact values of \( \Phi(C_p^n) \), except in case \( 2p + 3 \leq n \leq 3p \) where we give a lower bound. We believe that \( \min(n - p - 1, p + 1 + \left\lfloor \frac{n-p-1}{2} \right\rfloor) \) is the exact value of \( \Phi(C_p^n) \) for \( 2p + 3 \leq n \leq 3p \).
Acknowledgments

The authors thank the referees for useful suggestions that led to this improved version.

References


