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Fountains, histograms, and q–identities

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We solve the recursion $S_n = S_{n-1} - q^n S_{n-p}$, both, explicitly, and in the limit for $n \to \infty$, proving in this way a formula due to Merlini and Sprugnoli. It is also discussed how computer algebra could be applied.

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1 Fountains and histograms

Merlini and Sprugnoli [6] discuss *fountains* and *histograms*; for the reader's convenience, we review a few key issues here.

A *fountain with n coins* is an arrangement of n coins in rows such that each coin in a higher row touches exactly two coins in the next lower row.

A *p-histogram* is a sequence of columns in which the height of the $(j+1)$st column is at most $k+p$, if k is the height of column j; the first column has height r, with $1 \le r \le p$.

It can be shown that the enumeration of coins in a fountain is equivalent with the enumeration of 1-histograms. The paper [6] addresses the enumeration of p-histograms with respect to area (=number of cells). Let $f_n^{[p]}$ be the number p-histograms with area n and $F^{[p]}(q)$ the corresponding generating function $F^{[p]}(q) = \sum_n f_n^{[p]} q^n$. The authors of [6] use two different approaches: one produces the answer in the form

$$F^{[p]}(q) = \lim_{m\to\infty} \frac{D_m}{E_m},$$

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with some polynomials D_m, E_m defined in the next section, and the other gives it as

$$F^{[p]}(q) = \sum_{k\geq 0} \frac{(-1)^k q^{p\binom{k+1}{2}}}{(1-q)\dots(1-q^k)} \Big/ \sum_{k\geq 0} \frac{(-1)^k q^{k+p\binom{k}{2}}}{(1-q)\dots(1-q^k)}.$$

According to [5], it would be nice to have a direct argument that these two answers coincide. This is the subject of the present note.

2 Generalized Schur polynomials

The polynomials mentioned in the introduction are for fixed $p \geq 1$ defined as follows:

$$E_n = E_{n-1} - q^n E_{n-p}, \quad n \geq p, \qquad E_0 = \dots = E_{p-1} = 1,$$
$$D_n = D_{n-1} - q^n D_{n-p}, \quad n \geq p, \qquad D_i = 1 - \sum_{j=1}^{i} q^j,\ i = 0, \dots, p-1.$$

They can be compared with the classical Schur polynomials [8], which occur for $p = 2$ and $q = -1$. Then Merlini and Sprugnoli want a direct proof of the formulæ

$$E_\infty := \lim_{n\to\infty} E_n = \sum_{k\geq 0} \frac{(-1)^k q^{p\binom{k+1}{2}}}{(1-q)\dots(1-q^k)},$$
$$D_\infty := \lim_{n\to\infty} D_n = \sum_{k\geq 0} \frac{(-1)^k q^{k+p\binom{k}{2}}}{(1-q)\dots(1-q^k)}.$$

We will not only achieve that but actually derive *explicit* expressions for these polynomials!

It should be mentioned that Cigler [4] developed independently a combinatorial method to deal with recursions as ours, but also more general ones.

Let us study the generic recursion

$$S_n = S_{n-1} + t q^{n-p} S_{n-p},$$

with unspecified initial values $S_0, \dots, S_{p-1}$. For $p = 2$, these polynomials were studied by Andrews (and others) in the context of *Schur polynomials*, see [2].

We will use standard notation from q–calculus, see [1]:

$$(x)_n = (1-x)(1-xq)\dots(1-xq^{n-1}), \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}}.$$

It will be convenient to define $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for $n < 0$ or $k > n$.

Now we will proceed as in [1] and consider noncommutative variables x, η, such that $x\eta = q\eta x$; all other variables commute.

Lemma 1.

$$\left(x + x^p \eta\right)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{p\binom{n}{2} - pnk + p\binom{k+1}{2}} x^{k+p(n-k)} \eta^{n-k}.$$

Proof. We write

$$\left(x+x^p\eta\right)^n=\sum_{k=0}^{n}a_{n,k}x^{k+p(n-k)}\eta^{n-k},$$

and $\left(x+x^p\eta\right)^{n+1}=\left(x+x^p\eta\right)^n\left(x+x^p\eta\right)$ resp. as $\left(x+x^p\eta\right)^{n+1}=\left(x+x^p\eta\right)\left(x+x^p\eta\right)^n$, compare coefficients, and get the recursions

$$\begin{aligned}a_{n+1,k}&=a_{n,k-1}+a_{n,k}q^{k+p(n-k)},\\ a_{n+1,k}&=a_{n,k-1}q^{n+1-k}+a_{n,k}q^{p(n-k)}.\end{aligned}$$

From this we derive, taking differences,

$$a_{n,k}=\frac{1-q^{n+1-k}}{1-q^k}q^{-p(n-k)}a_{n,k-1}.$$

The result follows from iteration by noting that $a_{n,0}=q^{p\binom{n}{2}}$. □

Of course we also have

$$\left(x+tx^p\eta\right)^n=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}q^{p\binom{n}{2}-pnk+p\binom{k+1}{2}}x^{k+p(n-k)}t^{n-k}\eta^{n-k}.$$

Now we derive the generating function for

$$F(x)=\sum_{n\ge 0}S_nx^n;$$

the following procedure is inspired by [2]. Note that we can alternatively view η as an operator, defined by $\eta f(x)=f(qx)$. Cigler worked also much with this technique [3, 4]. We find

$$\sum_{n\ge p}S_nx^n=\sum_{n\ge p}S_{n-1}x^n+\sum_{n\ge p}tq^{n-p}S_{n-p}x^n=x\sum_{n\ge p-1}S_nx^n+tx^p\sum_{n\ge 0}\eta S_nx^n$$

or

$$F(x)-\sum_{n<p}S_nx^n=xF(x)-x\sum_{n<p-1}S_nx^n+tx^p\eta F(x),$$

and

$$F(x)=\frac{1}{1-x-tx^p\eta}\left(\sum_{i<p}S_ix^i-\sum_{i<p-1}S_ix^{i+1}\right).$$

Now we can apply our lemma and write

$$\begin{aligned}F(x)&=\sum_{n\ge 0}\left(x+tx^p\eta\right)^n\left(\sum_{i<p}S_ix^i-\sum_{i<p-1}S_ix^{i+1}\right)\\ &=\sum_{n\ge 0}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}q^{p\binom{n}{2}-pnk+p\binom{k+1}{2}}x^{k+p(n-k)}t^{n-k}\eta^{n-k}\left(\sum_{i<p}S_ix^i-\sum_{i<p-1}S_ix^{i+1}\right)\\ &=\sum_{n\ge 0}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}q^{p\binom{n}{2}-pnk+p\binom{k+1}{2}}x^{k+p(n-k)}t^{n-k}\left(\sum_{i<p}S_iq^{i(n-k)}x^i-\sum_{i<p-1}S_iq^{(i+1)(n-k)}x^{i+1}\right)\end{aligned}$$

$$= \sum_{n\ge0}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}q^{p\binom{k}{2}}x^{n-k+pk}t^k\Bigg(\sum_{i<p}S_iq^{ik}x^i-\sum_{i<p-1}S_iq^{(i+1)k}x^{i+1}\Bigg)$$
$$= \sum_{k,n\ge0}\begin{bmatrix}n+k\\k\end{bmatrix}q^{p\binom{k}{2}}x^{n+pk}t^k\Bigg(\sum_{i<p}S_iq^{ik}x^i-\sum_{i<p-1}S_iq^{(i+1)k}x^{i+1}\Bigg)$$
$$= \sum_{k\ge0}q^{p\binom{k}{2}}x^{pk}t^k\frac{1}{(x)_{k+1}}\Bigg(\sum_{i<p}S_iq^{ik}x^i-\sum_{i<p-1}S_iq^{(i+1)k}x^{i+1}\Bigg).$$

From this we find an explicit formula for S_n (the quantity S_{-1} has to be interpreted as 0):

$$S_n=\sum_{0\le i<p}(S_i-S_{i-1})\sum_{k\ge0}\begin{bmatrix}n-(p-1)k-i\\k\end{bmatrix}q^{p\binom{k}{2}+ik}t^k.$$

Now we specialize this to our instance. Here, $t=-q^p$, and thus

$$S_n=\sum_{0\le i<p}(S_i-S_{i-1})\sum_{k\ge0}\begin{bmatrix}n-(p-1)k-i\\k\end{bmatrix}q^{p\binom{k+1}{2}+ik}(-1)^k.$$

Therefore

$$E_n=\sum_{k\ge0}\begin{bmatrix}n-(p-1)k\\k\end{bmatrix}q^{p\binom{k+1}{2}}(-1)^k.$$

From this, the limit of E_n is immediate. For D_n we eventually get the following form

$$D_n=\sum_{k\ge0}\begin{bmatrix}n-(p-1)(k-1)\\k\end{bmatrix}q^{k+p\binom{k}{2}}(-1)^k,$$

from which the formula for D_∞ is immediate. To prove it, we need a simple lemma whose proof is just a routine calculation.

Lemma 2.

$$\begin{bmatrix}m-i\\k\end{bmatrix}q^{i(k+1)}=g(i)-g(i-1)\qquad\text{where}\qquad g(i)=-\begin{bmatrix}m-i\\k+1\end{bmatrix}q^{(i+1)(k+1)}.$$

□

Now we can plug into the general formula above and compute

$$D_n=E_n-\sum_{i=1}^{p-1}\sum_{k\ge0}\begin{bmatrix}n-(p-1)k-i\\k\end{bmatrix}q^{p\binom{k+1}{2}+i(k+1)}(-1)^k$$
$$=E_n-\sum_{k\ge0}(-1)^kq^{p\binom{k+1}{2}}\sum_{i=1}^{p-1}\begin{bmatrix}n-(p-1)k-i\\k\end{bmatrix}q^{i(k+1)}$$
$$=E_n-\sum_{k\ge0}(-1)^kq^{p\binom{k+1}{2}}\left\{q^{k+1}\begin{bmatrix}n-(p-1)k\\k+1\end{bmatrix}-q^{p(k+1)}\begin{bmatrix}n-(p-1)(k+1)\\k+1\end{bmatrix}\right\}$$
$$=1-\sum_{k\ge0}(-1)^kq^{p\binom{k+1}{2}}q^{k+1}\begin{bmatrix}n-(p-1)k\\k+1\end{bmatrix},$$

which is the announced formula after a simple change of variable. Note that in the penultimate step the telescoping property of the lemma has been used.

3 Computer algebra proofs

The polynomial families (E_n) and (D_n) give rise to the following study with respect to possible computer proofs. Let us take as input our sum representations of E_n and D_n:

$$
\begin{aligned}
E_n &= \sum_{k\geq 0} \begin{bmatrix} n-(p-1)k \\ k \end{bmatrix} q^{p\binom{k+1}{2}} (-1)^k, \\
D_n &= \sum_{k\geq 0} \begin{bmatrix} n-(p-1)(k-1) \\ k \end{bmatrix} q^{k+p\binom{k}{2}} (-1)^k.
\end{aligned} \tag{3.1}
$$

Then, if p is chosen as a specific positive integer, Riese's package qZeil [7] returns the recurrences $S_n = S_{n-1} - q^n S_{n-p}$ $(n \geq p)$ together with a certificate function Cert for independent verification. Despite the fact that for general "generic" integer parameter p there is no algorithm available, a general pattern can be easily guessed from running the algorithm for $p=1$, $p=2$, and $p=3$, say.

For example, let $F(n,k)$ be the kth summand in our sum representation (3.1) of E_n, then the recurrence for E_n can be refined to the following statement.

Theorem 3.1. *For $n \geq p$ and $\delta_k f(n,k) = f(n,k) - f(n,k-1)$, we have*

$$
F(n,k) - F(n-1,k) + q^n F(n-p,k) = \delta_k \mathsf{Cert}(n,k) F(n,k), \tag{3.2}
$$

where

$$
\mathsf{Cert}(n,k) = q^n \frac{(q^{n-p(k+1)+1})_p}{(q^{n-(p-1)(k+1)})_p}.
$$

Proof. After dividing both sides of (3.2) by $F(n,k)$ the proof reduces to checking equality of rational functions. Namely, note that

$$
\begin{aligned}
\frac{F(n-1,k)}{F(n,k)} &= \frac{1-q^{n-pk}}{1-q^{n-(p-1)k}}, \\
\frac{F(n,k-1)}{F(n,k)} &= -\frac{q^{pk}}{1-q^k} \frac{(q^{n-pk+1})_p}{(q^{n-(p-1)k+1})_{p-1}},
\end{aligned}
$$

and

$$
\frac{F(n-p,k)}{F(n,k)} = q^{-n} \mathsf{Cert}(n,k).
$$

□

Analogously, there is a refined version of the recurrence for D_n. The certificate in this case is

$$
\mathsf{Cert}(n,k) = q^n \frac{(q^{n-pk})_p}{(q^{n-(p-1)k})_p}.
$$

Summarizing, with the sum representation for E_n and D_n in hand, the corresponding recurrences follow immediately by summing both sides of the computer recurrences (3.2) over all $k \geq 0$.

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