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Analytic combinatorics for a certain well-ordered class of iterated exponential terms

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The aim of this paper is threefold: firstly, to explain a certain segment of ordinals in terms which are familiar to the analytic combinatorics community, secondly to state a great many of associated problems on resulting count functions and thirdly, to provide some weak asymptotic for the resulting count functions. We employ for simplicity Tauberian methods. The analytic combinatorics community is encouraged to provide (maybe in joint work) sharper results in future investigations.

Keywords: analytic combinatorics, Tauberian theorems orders of infinity, slow varying functions, ordinals

1 Introduction

It is usually difficult to attract the attention of mathematicians without background in logic to questions about ordinals. We hope to change this situation a bit by explaining a certain quite far reaching initial segment of these in terms of Hardy’s 1910 [6] orders of infinity.

Ordinals reflect the process of counting, thus they start like 0, 1, 2, 3, . . ., n, . . . Then the first limit element ω appears and counting continues with ω, ω + 1, ω + 2, ω + 3, . . ., ω + n, . . ., and the longer the process lasts the more obscure the ordinals become.

Let us now switch the scene to the following subclass E of Hardy’s order of infinity. Let E be the least set of functions f : N → N such that the constant zero function x → 0 is contained in E and such that with f and g also the function x → xf(x) + g(x) belongs to E.

Define f ≺ g via eventual domination, i.e. f ≺ g holds if there exists an n0 such that for all n ≥ n0 we have f(n) < g(n). Let k0 denote the constant function with value n and then notice k0 ≺ k1 ≺ k2 ≺ . . . ≺ kn ≺ . . . The first limit element with respect to ≺ is then obviously given by the identity function id, i.e. x → x. Moreover id ≺ id + k1 ≺ id + k2 ≺ . . . ≺ id + id · id ≺ id · id · id ≺ id · id · id · . . . As long as we stay within E all mysteriosity of the counting into the infinite disappears and we can consider the initial segment of ordinals provided by E as a natural mathematical structure for which no background in logic is necessary. To understand how E works one may verify that every polynomial function with non negative integer coefficients represents a function in E. (Note that e.g. k1 = id·k0 + k0.)

Hardy proved already in 1910 that E is linearly ordered with respect to ≺, hence every non zero function f in E has a unique ‘term’ representation f = idf1 + · · · + idfm where f1 ≥ . . . ≥ fm. If further the non zero function g has a corresponding representation g = idg1 + · · · + idgm where g1 ≥ . . . ≥ gm then we can decide f ≺ g using the corresponding exponents as follows; f ≺ g iff either m < n and for all i ≤ m we have fi = gi or there exists an k ≤ min{m, n} such that fk < gl and for all l < k we have fl = gl.

Usually it is assumed that proving the well-foundedness of E with respect to ≺ is difficult to see. As a sidestep let us show how to resolve this. We show that every nonempty subset of E has a ≺minimal element, or equivalently, there does not exist a strictly descending chain of elements in E, or equivalently for every function F : N → E there exists an n such that F(n) ≤ F(n + 1). A non logical argument uses

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either an appeal to Kruskal’s tree theorem or a compactness argument which is familiar from the proof of the Bolzano Weierstraß theorem.

Indeed, let \( id_1 := id \) and \( id_{n+1} \) be defined recursively as \( x \mapsto x^{id_n(x)} \). Let \( \mathcal{E}_n := \{ f \in \mathcal{E} : f \prec id_n \} \). Then \( \mathcal{E} = \bigcup_{n \in \mathbb{N}} \mathcal{E}_n \). We show by induction on \( m \) that for every function \( F : \mathbb{N} \to \mathcal{E}_m \) there exists an \( n \) such that \( F(n) \preceq F(n+1) \). Indeed this is clear for \( m = 1 \) since the natural numbers are well ordered. Now assume that there exists an infinitely descending chain \( F(0) \succ F(1) \succ F(2) \succ \ldots \) etc. in \( \mathcal{E}_{m+1} \). Then the corresponding lists of exponents of \( F(0), F(1), F(2) \) can be arranged in an infinite but finitely branching tree such that along any branch we obtain a strict descent of corresponding exponent functions. By compactness we would obtain an infinite strictly descending chain of functions appearing as exponents but this would give a strictly descending chain in \( \mathcal{E}_m \) which is excluded by induction hypothesis.

Thus we can use the structure \( \mathcal{E} \) for all sorts of transfinite recursion but in this paper we will not continue in exploring this further.

2 Some basic results

The elements of \( \mathcal{E} \) come along with various natural norm functions. The most canonical choice is given as follows: \( N(c_0) := 0 \) and \( N(id^f + g) := 1 + N(f) + N(g) \). (This is well defined as a moment reflection shows.) Then for every \( f \in \mathcal{E} \) and every natural number \( n \) there are finitely many \( g \prec f \) such that \( N(g) \leq n \). We may thus consider

\[
c_f(n) := \# \{ g \prec f : N(g) = n \}.
\]

For specific choices of \( f \) one re-obtains classical count functions, e.g. if \( f = id^d \) then \( c_f(n) \) is the number of partitions of \( n \) which has a well known and intriguing asymptotic.

A proof of this correspondence simply observe that every function \( id^{k_1} \cdots id^{k_m} \prec id^d \) with \( k_1 \geq \ldots \geq k_m \) corresponds uniquely to the partition \((i_1, \ldots, i_m = 1)\).

The author has learned that there has been recently a lot of progress in classifying \( c_f \) (in the context of Lie algebras) and thus we will not pursue this issue further. We just quote (besides the standard ones) the results of Petrogradsky [9]. Let \( id_0(f) := f \) and \( id_{m+1}(f) := id^{id_m}(f) \). Moreover let \( \ln^{(m)}(n) := n \) and \( \ln^{(m+1)}(n) := \ln(\ln^{(m)}(n)) \).

**Theorem 1**

1. \( c_{id^d}(n) \sim \frac{1}{d((d-1))^{d-1}} \).
2. \( c_{id^d}(n) \sim \exp(\frac{\pi}{4\sqrt{ln(n)}}) \).
3. Let \( \sigma := \left( 1 + \frac{1}{2} \right) (\frac{1}{(d-1)^{3}}\zeta(d+1) \right)^\frac{1}{d-1} \). Then \( \ln(c_{id^d}(n)) \sim \sigma \cdot \sqrt{\ln(n)} \).
4. There is an explicitly calculable constant \( C \) such that \( \ln(c_{id^{d+2}(k_2)}(n)) \sim C \frac{\ln(n)}{\sqrt{\ln(m)(\ln(n))}} \).

Moreover it is known from [10] that \( \lim_{n \to \infty} \frac{c^{(n+1)}(n)}{c^{(n)}(n)} = 1 \) for all \( f \in \mathcal{E} \).

There is a multiplicative norm which is canonically associated with \( N \). It is inferred by the indices of the enumeration function for the primes \((p_i)_{i \geq 1} \). Let \( I(k_0) := 1 \) and \( I(id^f + g) := p_{I(f)} \cdot I(g) \). The corresponding count function is

\[
c_f(n) := \# \{ g \prec f : I(g) \leq n \}.
\]

This norm is natural in a far as it provides a bijection between \( \mathcal{E} \) and the positive integers using the theorem on unique prime factor decomposition for positive integers. (The commutativity of addition is reflected by the commutativity of multiplication.) By elementary calculations with Dirichlet functions following the advice provided in Burris one can prove the following Theorem.

**Theorem 2**

1. There exists an explicitly calculable constant \( C \) such that \( c_{id^d}(n) \sim C(\ln(n))^d \).
2. \( \ln(c_{id^d}(n)) \sim \pi \cdot \sqrt{\frac{2}{3\ln(2)}} \ln(n) \).
3. \( \ln(c_{id^d}(n)) = \Theta(\ln(n)^\frac{d}{2}) \).
4. \( \ln(c_{id^{d+2}}(n)) = \Theta\left(\frac{\ln(n)}{\sqrt{\ln(m)(\ln(n))}} \right) \).

Moreover it is known that \( c_f(n) \) is slowly varying at infinity for each \( f \in \mathcal{E} \). We conjecture that the \( \Theta \) results can be sharpened to weak asymptotic similarly to Theorem 1 using \( \sigma := (1 + \frac{1}{2}) \left( \frac{1}{(d-1)\ln(2)} \right)^\frac{1}{d-1} \).
3 Exponential norms

The main emphasis of this paper is put on a norm which arises naturally in the context of logic, in almost every book on recursion theory. This exponential coding norm $E$ is defined by $E(k_n) := 1$ and $E(id^f + \cdots + id^{f_n}) := p_1^{E(f_1)} \cdots p_n^{E(f_n)}$ if $f_1 \geq \cdots \geq f_n$. Let

$$c^E_f(n) := \#\{g < f : E(g) \leq n\}.$$

An additive version of the exponential coding norm which leads to generalized Mahler partitions is as follows. Let the Mahler norm be defined by $M(0) := 0$ and $M(id^f + g) := 2M(f) + M(g)$. Moreover let

$$c^M_f(n) := \#\{g < f : M(g) \leq n\}.$$

Note that for $f = id^d$ the number $c^M_f(n)$ is the number of sequences $(i_0, \ldots, i_t)$ such that $i_0 \geq \cdots \geq i_t$ and $2^{i_0} + \cdots + 2^{i_t} \leq n$, hence a version of the Mahler partition function. For other values of $f$ one gets suitably generalized Mahler partitions. In particular we obtain the following standard partition identity for $M$ which can be used to obtain the asymptotic for the resulting count functions: \( \sum_{g=0}^{\infty} c^M_f(n) \cdot z^n = \prod_{i=1}^{\infty} \frac{1}{(1-2^i)^2(z)} \). The treatment of weak asymptotic for $c^M_f$ is very analogous to $c^E_f$ and we therefore stick to the functions $c^E_f$ from now on. (For better results the techniques of Dumas and Flajolet [5] seem appropriate here.)

Following Hardy and Ramanujan let $l_i := p_1 \cdots p_i$ where $l_0 := 1$. As a warm up exercise we indicate how the asymptotic for $c^E_f$ can be obtained by Karamata’s Tauberian theorem which seems to be tailor made for asymptotic on bounded partitions. (The proof is very similar to one found in [11].)

**Lemma 1** Let $hr_d(x) := c^E_{id^d}(x)$. Then

$$hr_d(x) = \sum_{e=1}^{d} \sum_{j_1 \leq \cdots \leq j_e} \sum_{l_1 \leq l_2 \leq \cdots \leq l_e} \#\{l_1^{j_1}, l_2^{j_2-l_1}, \ldots, l_e^{j_e-l_{e-1}} \leq x\}.$$

Proof. It suffices to show

$$\#\{f < id^d : E(f) \leq x\} = \sum_{e=1}^{d} \sum_{j_1 \leq \cdots \leq j_e} \sum_{l_1 \leq l_2 \leq \cdots \leq l_e} \#\{l_1^{j_1}, l_2^{j_2-l_1}, \ldots, l_e^{j_e-l_{e-1}} \leq x\}.$$

This is more or less obvious by grouping the factors appropriately together. (In some sense this is similar when one counts partitions and their conjugates. In terms of block diagrams this simply means that we are counting blocks at one time via columns and at the other time via rows.)

Let $L(s) := \sum_{n=1}^{\infty} \frac{n^{-s}}{\ln(n)}$. Theorem 3 (Hardy and Ramanujan [7]) $L(s) \sim \frac{1}{s \ln(2)}$ for $s \to 0^+$.

Recall that a (measurable) function $f : \mathbb{R} \to [0, \infty]$ is called slowly varying if $\lim_{t \to \infty} \frac{f(tx)}{f(t)} = 1$ for $x > 0$.

**Theorem 4** (Karamata’s Tauberian Theorem [2]) Let $U$ be a non decreasing right continuous function on the real numbers with $U(x) = 0$ for all $x < 0$. Let $LU(s) = \int_{0}^{\infty} \exp(-sx) \cdot dU(x)$. If $f : \mathbb{R} \to [0, \infty]$ varies slowly and $c \geq 0$, $\rho \geq 0$ the following are equivalent

1. $U(x) \sim cx^{\rho} f(x)$ for $x \to \infty$,
2. $LU(s) \sim cs^{-\rho} f(\frac{s}{2})$ as $s \to 0^+$.

As a nice application we obtain the following result.

**Theorem 5** $hr_d(x) \sim \frac{1}{(d!)^x} \prod_{i=1}^{\infty} \left(\frac{1}{\ln(n)}\right)^{i} \frac{1}{\ln(n_i)}(\frac{1}{\ln(n_i)})^d$ for $x \to \infty$.

Proof. Define natural numbers $a_n$ by the equation

$$\sum_{n=1}^{\infty} a_n n^{-s} = \sum_{e=1}^{d} \sum_{j_1 \leq \cdots \leq j_e} \sum_{l_1 \leq l_2 \leq \cdots \leq l_e} \#\{l_1^{j_1}, l_2^{j_2-l_1}, \ldots, l_e^{j_e-l_{e-1}} \leq x\}.$$
Then $\sum_{n \leq x} a_n = hr_f(x)$. Let $U(x) = \sum_{\ln(n) \leq x} a_n$. Then, as $s \to 0^+$,
\[
\frac{1}{d!} \prod_{e=1}^{d-1} \left( p_e - 1 \right) \prod_{e=1}^{d-2} l_e \left( \frac{1}{s \ln(x)} \right)^d 
\approx \sum_{e=1}^{d} \frac{1}{l_{j_1} \ldots l_{j_d} s \ln(l_{j_1}) \ldots \ln(l_{j_d})} \frac{1}{(l_{j_e} - l_{j_{e-1}}) s \ln((l_{j_e} - l_{j_{e-1}}) s)} 
\approx \sum_{e=1}^{d} \sum_{j_1 < \ldots < j_d < e} \frac{1}{d!} \sum (l_{j_1}^{l_1} - l_{j_2}^{l_2} - \ldots - l_{j_{e-1}}^{l_{e-1}})^{-s} 
\approx \sum_{e=1}^{d} \sum_{j_1 < \ldots < j_d < e} \sum \left( l_{j_1}^{l_1} - l_{j_2}^{l_2} - \ldots - l_{j_{e-1}}^{l_{e-1}} \right)^{-s} 
= \sum_{n=1}^{\infty} a_n n^{-s} 
= \int_0^\infty \exp(-sx) dU(x) = LU(s).
\]

The function $s \mapsto \frac{1}{\ln(x)}$ is slowly varying. Theorem 4 yields
\[
U(x) \sim \frac{1}{(d!)^2} \prod_{e=1}^{d-1} \left( p_e - 1 \right) \prod_{e=1}^{d-2} l_e \left( \frac{x}{\ln(x)} \right)^d
\]
for $x \to \infty$. Now $\sum_{n \leq x} a_n = U(\ln(x))$ and the result follows.

Now we consider count functions for functions $id^f$ where $f$ growth at least linearly. It turns out that tailor made Tauberian theorems are provided by Parameswaran [8].

**Theorem 6 (de Bruijn [3])** If $M$ is slowly varying, then there is a (asymptotically uniquely determined) slowly varying function $M^*$ such that $M^*(x \cdot M(x)) \cdot M(x) \to 1$ as $x \to \infty$ and $M(x \cdot M^*(x)) \cdot M^*(x) \to 1$ as $x \to \infty$.

**Theorem 7 (Parameswaran [8])** Suppose that the following conditions hold.
1. $L(u)$ and $P(u)$ are functions on the non negative reals such that $\int_0^R L(u) du$ and $\int_0^R P(u) du$ exist in the Lebesgue sense for every positive $R$.
2. $\exp(s \int_0^\infty e^{-mu} L(u) du) = s \int_0^\infty P(u) e^{-su} du$ for all positive $s$.
3. $\langle M, M^* \rangle$ form a pair of conjugate slowly varying functions.
4. $M$ is non decreasing.
5. $\int_0^u \frac{L(t)}{t} dt \sim M(u)$ as $u \to \infty$, and
6. $P(u)$ is non decreasing.
Then $\log P(u) \sim \frac{1}{M'(u)}$ as $u \to \infty$.

**Theorem 8** We have $c_{id^f}^E(n) = \# \{ (l_1, \ldots, l_m) : i_1 \geq \ldots \geq i_m & p_1^{l_1} \ldots p_m^{l_m} \leq n \}$. Moreover, $\ln(c_{id^f}^E(n)) = \Theta\left( \frac{(\ln(\ln(n)))^2}{\ln(\ln(\ln(n)))} \right)$.

**Proof.** We have
\[
c_{id^f}^E(n) 
\leq \# \{ (l_1, \ldots, l_m) : i_1 \geq \ldots \geq i_m & 2^{l_1} \ldots 2^{l_m} \leq n \} 
= \# \{ (l_1, \ldots, l_m) : i_1 \geq \ldots \geq i_m & l_1 + \ldots + l_m \leq \frac{\ln(n)}{\ln(2)} \}.
\]
Let
\[
Q(n) = \# \{ (l_1, \ldots, l_m) : i_1 \geq \ldots \geq i_m & l_1 + \ldots + l_m \leq n \}
\]
Theorem 10

Let $\theta(x)$ be the prime number theorem, which states that $\theta(x) = \sum_{p \leq x} \log(p)$.

Thus, Parameswaran [8] yields

$$C \geq \# \{l_i, \ldots, l_m \} \text{ such that } \phi(p_i) = \prod_{j=1}^{m} p_j^a_j \text{ for large } n.$$
The lower bound is obtained similarly. Indeed, we have

\[
\begin{align*}
\# \{ (g_1, \ldots, g_k) : id_m \succ g_1 \geq \cdots \geq g_k & \land p_1^{Eg_1} \cdots p_k^{Eg_k} \leq n \} \\
\geq & (g_1, \ldots, g_k) : id_m \succ g_1 \geq \cdots \geq g_k & \land (Kk\ln(k))^{Eg_1 + \cdots + Eg_k} \leq n \\
= & (g_1, \ldots, g_k) : id_m \succ g_1 \geq \cdots \geq g_k & \land Eg_1 + \cdots + Eg_k \leq \sqrt{\ln(n)} \\
= & (g_1, \ldots, g_k) : id_m \succ g_1 \geq \cdots \geq g_k & \land 2^{Eg_1 + \cdots + Eg_k} \leq \sqrt{\ln(n)} \\
\geq & \exp_{m-1} \left( C \cdot \frac{(\ln^{(m+1)}(n))^2}{\ln^{(m+2)}(n)} \right)
\end{align*}
\]

since \( m \geq 2 \). \( \square \)

The same proof yields the following refinement.

**Theorem 11** Then \( \ln^{(m)}(e^{id_{m+1}(c_n)}(n)) = \Theta(\ln^{(m+2)}(n)(\ln^{(m+1)}(n))^{d+1}) \).

Investigations on count functions have applications in logic. Let us state one application to the phase transition for the Ackermann function. Let \( F \) be a number-theoretic function and let \( \text{count}_E^F(F)(m) \) be the maximal possible number of \( g_1, \ldots, g_k \in E \) such that \( f \succ g_1 \succ \cdots \succ g_k \) and \( (\forall i \leq k)[E(g_i) \leq m + F(i)] \). This is well defined by a compactness argument for every function \( F \). Then for \( f = id_{m+2}, d \) fixed, and functions \( F \) with \( F(i) \geq 2^{4\sqrt{\ln(m)}(i)} \) for \( i \) large enough the function \( \text{count}_E^F(F) \) will eventually dominate every primitive recursive function. But for \( f = id_{m+2} \) and functions \( F \) with \( F(i) \leq \ln^{(m)}(i) \) (for \( i \) large enough) the function \( \text{count}_E^F(F) \) will be bounded by a double exponential function.

We close with some conjectures.

**Conjecture 1**

1. \( \ln^{(m)}(c^{E}_{id_{m+1}}(c_n)(n)) = \Theta(\ln^{(m)}(n)^2) \).

2. \( c^{E}_{F} \) is slowly varying for each \( F \) in \( E \).

3. \( n \mapsto \ln(c^{M}_{F}(n)) \) is slowly varying for each \( F \) in \( E \).

Following Burris’s philosophy on logical limit laws we conjecture that for the norm functions \( E \) and \( M \) there will be associated zero one laws for first order logic.

**References**


