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We are looking for the maximum number of subsets of an n-element set not containing 4 distinct subsets satisfying A ⊂ B, C ⊂ B, C ⊂ D. It is proved that this number is at least the number of the $\left\lfloor \frac{n}{2} \right\rfloor$-element sets times $1 + \frac{2}{n}$, on the other hand an upper bound is given with 4 replaced by the value 2.

Keywords: extremal problems, families of subsets

Let $[n] = \{1,2,\ldots,n\}$ be a finite set, families $\mathcal{F}, \mathcal{G}$, etc. of its subsets will be investigated. $\binom{[n]}{k}$ denotes the family of all $k$-element subsets of $[n]$. Let $P$ be a poset. The goal of the present investigations is to determine the maximum size of a family $\mathcal{F} \subseteq 2^{[n]}$ which does not contain $P$ as a (non-necessarily induced) subposet. This maximum is denoted by $La(n,P)$. In some cases two posets, say $P_1, P_2$ could be excluded. The maximum number of subsets is denoted by $La(n, P_1, P_2)$ in this case.

The easiest example is the case when $P$ consist of two comparable elements. Then we are actually looking for the largest family without inclusion that is without two distinct members $F, G \in \mathcal{F}$ such that $F \subset G$. The well-known Sperner theorem ([4]) gives the answer, the maximum is $\binom{\left\lfloor \frac{n}{2} \right\rfloor}{n} - 1$.

We say that the distinct sets $A, B_1, \ldots, B_r$ form an $r$-fork if they satisfy $A \subset B_1, \ldots, B_r$. $A$ is called the handle, $B_i$s are called the prongs of the fork. On the other hand, the distinct sets $A, B_1, \ldots, B_r$ form an $r$-brush if they satisfy $B_1, \ldots, B_r \subset A$. The $r$-forks and the $r$-brush are denoted by $F(r), B(r)$, respectively. An old theorem solves the problem when the 2-fork and the 2-brush are excluded.

**Theorem 1** [3]

$$La(n, F(2), B(2)) = 2 \left( \frac{n-1}{\left\lfloor \frac{n-1}{2} \right\rfloor} \right).$$

The optimal construction is the family

$$\mathcal{F} = \left\{ F : F \in \binom{[n-1]}{\left\lfloor \frac{n-1}{2} \right\rfloor} \right\} \cup \left\{ F \cup \{n\} : F \in \binom{[n-1]}{\left\lfloor \frac{n-1}{2} \right\rfloor} \right\}.$$ 

We have proved the following theorem in a paper appearing soon.

**Theorem 2** [2] Let $n \geq 3$. If the family $\mathcal{F} \subseteq 2^{[n]}$ contains no four distinct sets $A, B, C, D$ such that $A \subset C, A \subset D, B \subset C, B \subset D$, then $|\mathcal{F}|$ cannot exceed the sum of the two largest binomial coefficients of order $n$, i.e., $|\mathcal{F}| \leq \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor} + \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor + 1}$. 

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Following the suggestion of J.R. Griggs, such a family could be called a butterfly-free meadow. The optimal construction here is obvious, one can take all the subsets of sizes $\lfloor n/2 \rfloor$ and $\lfloor n/2 \rfloor + 1$.

In all of these cases the maximum size of the family is exactly determined. This is not true when the $r$-fork is excluded. In a paper under preparation A. De Bonis and the present author proved the following theorem.

**Theorem 3** [1]

\[
\binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{r}{n} + O\left(\frac{1}{n^2}\right)\right) \leq \La(F(r + 1)) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + 2\frac{r}{n} + O\left(\frac{\log n}{n^{3/2}}\right)\right).
\]

A weaker version of the upper bound in this theorem was obtained in [5]: the constant in the second term was larger. There is still a gap between the lower and upper bounds in the second term: a factor 2. This however seems to be a serious difficulty. The best construction (lower bound) contains all sets in one level and a thinned next level.

Let the poset $N$ consist of 4 elements illustrated here with 4 distinct sets satisfying $A \subset B, C \subset B, C \subset D$. We were not able to determine $\La(n, N)$ for a long time. Recently, a new method jointly developed by J.R. Griggs, helped us to prove the following theorem.

**Theorem 4**

\[
\binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{2}{n} + o\left(\frac{1}{n}\right)\right) \leq \La(n, N) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{4}{n} + o\left(\frac{1}{n}\right)\right).
\]

**References**


